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Research Article

Separated Boundary Value Problems of Sequential Caputo and Hadamard Fractional Differential Equations

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In this paper, we discuss the existence and uniqueness of solutions for new classes of separated boundary value problems of Caputo-Hadamard and Hadamard-Caputo sequential fractional differential equations by using standard fixed point theorems. We demonstrate the application of the obtained results with the aid of examples.

1. Introduction

Fractional differential equations have been of increasing importance for the past decades due to their diverse applications in science and engineering such as biophysics, bioengineering, virology, control theory, signal and image processing, blood flow phenomena, etc.; see [1–6]. Many interesting results of the existence of solutions of various classes of fractional differential equations have been obtained; see [7–15] and the references therein.

Sequential fractional differential equations are also found to be of much interest [16, 17]. In fact, the concept of sequential fractional derivative is closely related to the nonsequential Riemann-Liouville derivatives, for details, see [3]. For some recent results on boundary value problems for sequential fractional differential equations; see [18–22] and references cited therein.

In this paper, we discuss existence and uniqueness of solutions for two sequential Caputo-Hadamard and Hadamard-Caputo fractional differential equations subject to separated boundary conditions as

$$\begin{aligned}
&{}^{C}D^{p}\left({}^{H}D^{q}x\right)(t) = f(t, x(t)), \quad t \in (a,b),\\
&\alpha_1 x(a) + \alpha_2 \left({}^{H}D^{q}x\right)(a) = 0,\\
&\beta_1 x(b) + \beta_2 \left({}^{H}D^{q}x\right)(b) = 0,
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
&{}^{H}D^{q}\left({}^{C}D^{p}x\right)(t) = f(t, x(t)), \quad t \in (a,b),\\
&\alpha_1 x(a) + \alpha_2 \left({}^{C}D^{p}x\right)(a) = 0,\\
&\beta_1 x(b) + \beta_2 \left({}^{C}D^{p}x\right)(b) = 0,
\end{aligned} \tag{2}$$

where ${}^{C}D^{p}$ and ${}^{H}D^{q}$ are the Caputo and Hadamard fractional derivatives of orders p and q, respectively, $0 < p, q \leq 1$, $f : [a,b]\times\mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, $a > 0$ and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$.

It can be observed that the sequential Caputo-Hadamard and Hadamard-Caputo fractional differential equations in (1) and (2) are different type when $p = 1$ and $q = 1$, since

$$\frac{d}{dt}\left(t\frac{d}{dt}x(t)\right) = t\frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} = f(t, x(t)), \tag{3}$$

and

$$t\frac{d}{dt}\left(\frac{d}{dt}x(t)\right)=t\frac{d^2x(t)}{dt^2}=f(t,x(t)), \quad (4)$$

for $t\in(a,b)$, respectively.

The rest of the paper is arranged as follows. In Section 2, we establish basic results that lay the foundation for defining a fixed point problem equivalent to the given problems (1) and (2). The main results, based on Banach's contraction mapping principle, Krasnoselskii's fixed point theorem, and nonlinear alternative of Leray-Schauder type, are obtained in Section 3. Illustrating examples are discussed in Section 4.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [4, 5] and present preliminary results needed in our proofs later.

Definition 1 (see [5]). For an at least n-times differentiable function $g:[a,\infty)\longrightarrow\mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^{C}D^{q}g(t)=\frac{1}{\Gamma(n-q)}\int_a^t (t-s)^{n-q-1}g^{(n)}(s)\,ds, \quad n-1<q<n,\ n=[q]+1, \quad (5)$$

where $[q]$ denotes the integer part of the real number q.

Definition 2 (see [5]). The Riemann-Liouville fractional integral of order q of a function $g:[a,\infty)\longrightarrow\mathbb{R}$ is defined as

$$^{RL}I^{q}g(t)=\frac{1}{\Gamma(q)}\int_a^t\frac{g(s)}{(t-s)^{1-q}}ds,\quad q>0, \quad (6)$$

provided the integral exists.

Definition 3 (see [5]). For an at least n-times differentiable function $g:[a,\infty)\longrightarrow\mathbb{R}$, the Caputo-type Hadamard derivative of fractional order q is defined as

$$^{H}D^{q}g(t)=\frac{1}{\Gamma(n-q)}\int_a^t\left(\log\frac{t}{s}\right)^{n-q-1}\delta^n g(s)\frac{ds}{s}, \quad n-1<q<n,\ n=[q]+1, \quad (7)$$

where $\delta=t(d/dt)$, $\log(\cdot)=\log_e(\cdot)$.

Definition 4 (see [5]). The Hadamard fractional integral of order q is defined as

$$^{H}I^{q}g(t)=\frac{1}{\Gamma(q)}\int_a^t\left(\log\frac{t}{s}\right)^{q-1}g(s)\frac{ds}{s},\quad q>0, \quad (8)$$

provided the integral exists.

Lemma 5 (see [5]). *For $q>0$, the general solution of the fractional differential equation $^{C}D^{q}u(t)=0$ is given by*

$$u(t)=c_0+c_1(t-a)+\cdots+c_{n-1}(t-a)^{n-1}, \quad (9)$$

where $c_i\in\mathbb{R}$, $i=0,1,2,\ldots,n-1$ $(n=[q]+1)$.

In view of Lemma 5, it follows that

$$^{RL}I^{q}\left(^{C}D^{q}u\right)(t)=u(t)+c_0+c_1(t-a)+\cdots+c_{n-1}(t-a)^{n-1}, \quad (10)$$

for some $c_i\in\mathbb{R}$, $i=0,1,2,\ldots,n-1$ $(n=[q]+1)$.

Lemma 6 (see [23]). *Let $u\in AC_\delta^n[a,b]$ or $C_\delta^n[a,b]$ and $q\in\mathbb{C}$, where $X_\delta^n[a,b]=\{g:[a,b]\longrightarrow\mathbb{C}:\delta^{n-1}g(t)\in X[a,b]\}$. Then, we have*

$$^{H}I^{q}\left(^{H}D^{q}\right)u(t)=u(t)-\sum_{k=0}^{n-1}c_k\left(\log\left(\frac{t}{a}\right)\right)^k, \quad (11)$$

where $c_i\in\mathbb{R}$, $i=0,1,2,\ldots,n-1$ $(n=[q]+1)$.

In order to define the solution of the boundary value problem (1), we consider the linear variant

$$\begin{aligned}&^{C}D^{p}\left(^{H}D^{q}x\right)(t)=y(t),\quad t\in(a,b),\\&\alpha_1x(a)+\alpha_2\left(^{H}D^{q}x\right)(a)=0,\\&\beta_1x(b)+\beta_2\left(^{H}D^{q}x\right)(b)=0,\end{aligned} \quad (12)$$

where $y\in C([a,b],\mathbb{R})$.

Lemma 7. *Let*

$$\Omega:=\beta_1\alpha_2-\alpha_1\left(\beta_1\frac{(\log(b/a))^q}{\Gamma(q+1)}+\beta_2\right)\neq0. \quad (13)$$

Then, the unique solution of the separated boundary value problem of sequential Caputo and Hadamard fractional differential equation (12) is given by the integral equation

$$\begin{aligned}x(t)=&\frac{\beta_1}{\Omega}\left(\alpha_1\frac{(\log(t/a))^q}{\Gamma(q+1)}-\alpha_2\right){}^{H}I^{q}\left(^{RL}I^{p}y\right)(b)\\&+\frac{\beta_2}{\Omega}\left(\alpha_1\frac{(\log(t/a))^q}{\Gamma(q+1)}-\alpha_2\right){}^{RL}I^{p}y(b)\\&+{}^{H}I^{q}\left(^{RL}I^{p}y\right)(t),\quad t\in[a,b].\end{aligned} \quad (14)$$

Proof. Taking the Riemann-Liouville fractional integral of order p to the first equation of (12), we get

$$\left(^{H}D^{q}x\right)(t)=c_1+{}^{RL}I^{p}y(t),\quad c_1\in\mathbb{R}. \quad (15)$$

Again taking the Hadamard fractional integral of order q to the above equation, we obtain

$$x(t)=c_2+c_1\frac{(\log(t/a))^q}{\Gamma(q+1)}+{}^{H}I^{q}\left(^{RL}I^{p}y\right)(t),\quad c_2\in\mathbb{R}. \quad (16)$$

Substituting $t=a$ in (15)-(16) and applying the first boundary condition of (12), it follows that

$$\alpha_2c_1+\alpha_1c_2=0. \quad (17)$$

For $t = b$ in equations (15)-(16) and using the second boundary condition of (12), it yields

$$c_1\left(\beta_1\frac{(\log(b/a))^q}{\Gamma(q+1)}+\beta_2\right)+\beta_1 c_2 = -\beta_1{}^HI^q\left({}^{RL}I^p y\right)(b)-\beta_2{}^{RL}I^p y(b). \quad (18)$$

Solving the linear system of (17) and (18) for finding two constants c_1, c_2, we get

$$c_1 = \frac{\alpha_1\beta_1}{\Omega}{}^HI^q\left({}^{RL}I^p y\right)(b)+\frac{\alpha_1\beta_2}{\Omega}{}^{RL}I^p y(b) \quad (19)$$

and

$$c_2 = -\frac{\beta_1\alpha_2}{\Omega}{}^HI^q\left({}^{RL}I^p y\right)(b)-\frac{\alpha_2\beta_2}{\Omega}{}^{RL}I^p y(b). \quad (20)$$

Substituting constants c_1 and c_2 in (16), we get the integral equation (14). The converse follows by direct computation. The proof is completed. □

In the same way, we can prove the following lemma, which concerns a linear variant of problem (2):

$$\begin{gathered}{}^HD^q\left({}^CD^p x\right)(t)=z(t),\quad t\in(a,b),\\ \alpha_1 x(a)+\alpha_2\left({}^CD^p x\right)(a)=0,\\ \beta_1 x(b)+\beta_2\left({}^CD^p x\right)(b)=0,\end{gathered} \quad (21)$$

where $z \in C([a,b],\mathbb{R})$.

Lemma 8. *Let*

$$\Omega^* := \beta_1\alpha_2-\alpha_1\left(\beta_1\frac{(b-a)^p}{\Gamma(p+1)}+\beta_2\right)\neq 0. \quad (22)$$

Then, the unique solution of the separated boundary value problem of sequential Caputo and Hadamard fractional differential equation (21) is given by the integral equation

$$\begin{aligned}x(t) &= \frac{\beta_1}{\Omega^*}\left(\alpha_1\frac{(t-a)^p}{\Gamma(p+1)}-\alpha_2\right){}^{RL}I^p\left({}^HI^q z\right)(b)\\ &+\frac{\beta_2}{\Omega^*}\left(\alpha_1\frac{(t-a)^p}{\Gamma(p+1)}-\alpha_2\right){}^HI^q z(b)\\ &+{}^{RL}I^p\left({}^HI^q z\right)(t),\quad t\in[a,b].\end{aligned} \quad (23)$$

3. Main Results

We set some abbreviate notations for sequential Riemann-Liouville and Hadamard fractional integrals of a function with two variables as

$$\begin{aligned}{}^HI^q\left({}^{RL}I^p(f_x)\right)(\phi) &= \frac{1}{\Gamma(q)\Gamma(p)}\\ &\cdot\int_a^\phi\int_a^s\left(\log\frac{\phi}{s}\right)^{q-1}(s-r)^{p-1}f(r,x(r))\,dr\frac{ds}{s},\end{aligned} \quad (24)$$

and

$$\begin{aligned}{}^{RL}I^p\left({}^HI^q(f_x)\right)(\phi) &= \frac{1}{\Gamma(p)\Gamma(q)}\\ &\cdot\int_a^\phi\int_a^s(\phi-s)^{p-1}\left(\log\frac{s}{r}\right)^{q-1}f(r,x(r))\frac{dr}{r}ds,\end{aligned} \quad (25)$$

where $\phi \in \{t, b\}$. Also we use this one for a single Riemann-Liouville and Hadamard fractional integrals of orders p and q, respectively.

In this section, we will use fixed point theorems to prove the existence and uniqueness of solution for problems (1) and (2). To accomplish our purpose, we define the Banach space $\mathscr{C} = C([a,b],\mathbb{R})$, of all continuous functions on $[a,b]$ to $\mathbb{R}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [a,b]\}$. In addition, we define the operator $\mathscr{K} : \mathscr{C} \longrightarrow \mathscr{C}$ by

$$\begin{aligned}\mathscr{K}x(t) &= \frac{\beta_1}{\Omega}\left(\alpha_1\frac{(\log(t/a))^q}{\Gamma(q+1)}-\alpha_2\right){}^HI^q\left({}^{RL}I^p(f_x)\right)(b)\\ &+\frac{\beta_2}{\Omega}\left(\alpha_1\frac{(\log(t/a))^q}{\Gamma(q+1)}-\alpha_2\right){}^{RL}I^p(f_x)(b)\\ &+{}^HI^q\left({}^{RL}I^p(f_x)\right)(t),\end{aligned} \quad (26)$$

where $\Omega \neq 0$ is defined by (13) and $f_x(t) = f(t, x(t))$. Note that the separated boundary value problem (1) has solutions if and only if $x = \mathscr{K}x$ has fixed points.

For computational convenience we put

$$\begin{aligned}\Omega_1 &= \frac{|\beta_1|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)}+|\alpha_2|\right){}^HI^q\left({}^{RL}I^p(1)\right)(b)\\ &+\frac{|\beta_2|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)}+|\alpha_2|\right){}^{RL}I^p(1)(b)\\ &+{}^HI^q\left({}^{RL}I^p(1)\right)(b).\end{aligned} \quad (27)$$

To prove the existence theorems of problem (2), we define the operator $\mathscr{A} : \mathscr{C} \longrightarrow \mathscr{C}$ by

$$\begin{aligned}\mathscr{A}x(t) &= \frac{\beta_1}{\Omega^*}\left(\alpha_1\frac{(t-a)^p}{\Gamma(p+1)}-\alpha_2\right){}^{RL}I^p\left({}^HI^q(f_x)\right)(b)\\ &+\frac{\beta_2}{\Omega^*}\left(\alpha_1\frac{(t-a)^p}{\Gamma(p+1)}-\alpha_2\right){}^HI^q(f_x)(b)\\ &+{}^{RL}I^p\left({}^HI^q(f_x)\right)(t).\end{aligned} \quad (28)$$

Now, we prove the existence and uniqueness result for problem (1). For problem (2) the proof is similar and omitted.

Theorem 9. *Suppose that*

(H_1) *there exists a function $\psi(t) > 0$, $t \in [a,b]$, such that*

$$|f(t,x) - f(t,y)| \le \psi(t)|x-y| \quad \text{for all } t \in [a,b] \text{ and } x,y \in \mathbb{R}. \tag{29}$$

If $\psi^*\Omega_1 < 1$, where $\psi^* = \sup\{\psi(t) : t \in [a,b]\}$, then the separated boundary value problem (1) has a unique solution on $[a,b]$.

Proof. Firstly, we define a ball B_r as $B_r = \{x \in \mathscr{C} : \|x\| \le r\}$, where the constant r satisfies

$$r \ge \frac{M\Omega_1}{1-\psi^*\Omega_1}, \tag{30}$$

where $M = \sup\{f(t,0) : t \in [a,b]\}$. Next, we will show that $\mathscr{K}B_r \subset B_r$. For any $x \in B_r$ and using the triangle inequality $|f_x| \le |f_x - f_0| + |f_0|$, we have

$$\begin{aligned}
|\mathscr{K}x(t)| &\le \frac{|\beta_1|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right) \\
&\quad \cdot {}^{H}I^{q}\left({}^{RL}I^{p}(|f_x|)\right)(b) \\
&\quad + \frac{|\beta_2|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right){}^{RL}I^{p}(|f_x|)(b) \\
&\quad + {}^{H}I^{q}\left({}^{RL}I^{p}(|f_x|)\right)(t) \\
&\le \frac{|\beta_1|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right) \\
&\quad \cdot {}^{H}I^{q}\left({}^{RL}I^{p}(|f_x - f_0| + |f_0|)\right)(b) \\
&\quad + \frac{|\beta_2|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right) \\
&\quad \cdot {}^{RL}I^{p}(|f_x - f_0| + |f_0|)(b) \\
&\quad + {}^{H}I^{q}\left({}^{RL}I^{p}(|f_x - f_0| + |f_0|)\right)(b) \\
&\le \frac{|\beta_1|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right) \\
&\quad \cdot {}^{H}I^{q}\left({}^{RL}I^{p}(\psi^* r + M)\right)(b) \\
&\quad + \frac{|\beta_2|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right){}^{RL}I^{p}(\psi^* r + M) \\
&\quad \cdot (b) + {}^{H}I^{q}\left({}^{RL}I^{p}(\psi^* r + M)\right)(b) = \psi^*\Omega_1 r + M\Omega_1 \\
&\le r,
\end{aligned} \tag{31}$$

which implies that $\mathscr{K}B_r \subset B_r$. Let $x, y \in B_r$, then

$$\begin{aligned}
|\mathscr{K}x(t) - \mathscr{K}y(t)| &\le \frac{|\beta_1|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right) \\
&\quad \cdot {}^{H}I^{q}\left({}^{RL}I^{p}\left(|f_x - f_y|\right)\right)(b) \\
&\quad + \frac{|\beta_2|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right){}^{RL}I^{p}\left(|f_x - f_y|\right) \\
&\quad \cdot (b) + {}^{H}I^{q}\left({}^{RL}I^{p}\left(|f_x - f_y|\right)\right)(t) \\
&\le \frac{|\beta_1|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right)\psi^*\|x-y\| \\
&\quad \cdot {}^{H}I^{q}\left({}^{RL}I^{p}(1)\right)(b) \\
&\quad + \frac{|\beta_2|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right)\psi^*\|x-y\| \\
&\quad \cdot {}^{RL}I^{p}(1)(b) + \psi^*\|x-y\|\,{}^{H}I^{q}\left({}^{RL}I^{p}(1)\right)(t) \\
&= \psi^*\Omega_1\|x-y\|,
\end{aligned} \tag{32}$$

which yields that $\|\mathscr{K}x - \mathscr{K}y\| \le \psi^*\Omega_1\|x-y\|$. Since $\psi^*\Omega_1 < 1$, we deduce that the operator $\mathscr{K}$ is a contraction. By Banach contraction mapping principle the operator $\mathscr{K}$ has a unique fixed point, which leads that problem (1) has a unique solution on $[a,b]$. □

Theorem 10. *Let (H_1) in Theorem 9 holds. If $\psi^*\Omega_1^* < 1$, where*

$$\begin{aligned}
\Omega_1^* &= \frac{|\beta_1|}{|\Omega^*|}\left(|\alpha_1|\frac{(b-a)^p}{\Gamma(p+1)} + |\alpha_2|\right){}^{RL}I^{p}\left({}^{H}I^{q}(1)\right)(b) \\
&\quad + \frac{|\beta_2|}{|\Omega^*|}\left(|\alpha_1|\frac{(b-a)^p}{\Gamma(p+1)} + |\alpha_2|\right){}^{H}I^{q}(1)(b) \\
&\quad + {}^{RL}I^{p}\left({}^{H}I^{q}(1)\right)(b),
\end{aligned} \tag{33}$$

then the separated boundary value problem (2) has a unique solution on $[a,b]$.

Our second existence result is based on Krasnoselskii's fixed point theorem.

Theorem 11 ((Krasnoselskii's fixed point theorem) [24]). *Let Q be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be operators such that*

(a) *$Ax + By \in Q$ where $x, y \in Q$;*

(b) *A is compact and continuous;*

(c) *B is a contraction mapping.*

Then there exists $z \in Q$ such that $z = Az + Bz$.

Theorem 12. *Let $f : [a,b]\times\mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying (H_1) in Theorem 9. In addition, assume that*

(H_2) *$|f(t,x)| \le \varphi(t)$, $\forall(t,x) \in [a,b]\times\mathbb{R}$ and $\varphi \in C([a,b],\mathbb{R}^+)$.*

If

$$\psi^*\left[{}^{H}I^{q}\left({}^{RL}I^{p}(1)\right)(b)\right]<1, \tag{34}$$

then the separated boundary value problem (1) has at least one solution on $[a,b]$.

Proof. Let $B_\rho=\{x\in\mathscr{C}:\|x\|\le\rho\}$, where a constant ρ satisfying $\rho\ge\varphi^*\Omega_1$ and $\varphi^*=\sup\{\varphi(t):t\in[a,b]\}$. We decompose the operator $\mathscr{K}$ into two operators $\mathscr{K}_1$ and $\mathscr{K}_2$ on B_ρ with

$$\begin{aligned}\mathscr{K}_1x(t)&=\frac{\beta_1}{\Omega}\left(\alpha_1\frac{(\log(t/a))^q}{\Gamma(q+1)}-\alpha_2\right){}^{H}I^{q}\left({}^{RL}I^{p}(f_x)\right)(b)\\&\quad+\frac{\beta_2}{\Omega}\left(\alpha_1\frac{(\log(t/a))^q}{\Gamma(q+1)}-\alpha_2\right){}^{RL}I^{p}(f_x)(b),\\&\qquad t\in[a,b],\\\mathscr{K}_2x(t)&={}^{H}I^{q}\left({}^{RL}I^{p}(f_x)\right)(t),\quad t\in[a,b].\end{aligned} \tag{35}$$

Note that the ball B_ρ is a closed, bounded, and convex subset of the Banach space $\mathscr{C}$.

Now, we will show that $\mathscr{K}_1x+\mathscr{K}_2y\in B_\rho$ for satisfying condition (a) of Theorem 11. Setting $x,y\in B_\rho$, then we have

$$\begin{aligned}&\left|\mathscr{K}_1x(t)+\mathscr{K}_2y(t)\right|\\&\le\frac{|\beta_1|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)}+|\alpha_2|\right)\\&\cdot{}^{H}I^{q}\left({}^{RL}I^{p}(|f_x|)\right)(b)\\&+\frac{|\beta_2|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)}+|\alpha_2|\right){}^{RL}I^{p}(|f_x|)(b)\\&+{}^{H}I^{q\,RL}I^{p}(|f_y|)(t)\\&\le\varphi^*\left(\frac{|\alpha_1\beta_1|}{|\Omega|}\frac{(\log(b/a))^q}{\Gamma(q+1)}+\frac{|\beta_1\alpha_2|}{|\Omega|}\right){}^{H}I^{q\,RL}I^{p}(1)\\&\cdot(b)+\varphi^*\left(\frac{|\alpha_1\beta_2|}{|\Omega|}\frac{(\log(b/a))^q}{\Gamma(q+1)}+\frac{|\alpha_2\beta_2|}{|\Omega|}\right){}^{RL}I^{p}(1)\\&\cdot(b)+\varphi^{*}\,{}^{H}I^{q}\left({}^{RL}I^{p}(1)\right)(b)=\varphi^*\Omega_1\le\rho.\end{aligned} \tag{36}$$

This means that $\mathscr{K}_1x+\mathscr{K}_2y\in B_\rho$. To prove that $\mathscr{K}_2$ is a contraction mapping, for $x,y\in B_\rho$, we have

$$\begin{aligned}\left\|\mathscr{K}_2x-\mathscr{K}_2y\right\|&\le{}^{H}I^{q}\left({}^{RL}I^{p}\left(|f_x-f_y|\right)\right)(b)\\&\le\psi^*\left[{}^{H}I^{q}\left({}^{RL}I^{p}(1)\right)(b)\right]\|x-y\|\end{aligned} \tag{37}$$

by condition (H_1), which is a contraction, by (34). Therefore, the condition (c) of Theorem 11 is satisfied. Next we will show that the operator $\mathscr{K}_1$ is compact and continuous. By using the continuity of the function f on $[a,b]\times\mathbb{R}$, we can conclude that the operator $\mathscr{K}_1$ is continuous. For $x\in B_\rho$, it follows that

$$\left\|\mathscr{K}_1x\right\|\le\varphi^*\Omega_2, \tag{38}$$

where

$$\begin{aligned}&\Omega_2\\&=\frac{|\beta_1|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)}+|\alpha_2|\right){}^{H}I^{q}\left({}^{RL}I^{p}(1)\right)(b)\\&\quad+\frac{|\beta_2|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)}+|\alpha_2|\right){}^{RL}I^{p}(1)(b),\end{aligned} \tag{39}$$

which implies that the set $\mathscr{K}_1B_\rho$ is uniformly bounded. Now we are going to prove that $\mathscr{K}_1B_\rho$ is equicontinuous. For τ_1, $\tau_2\in[a,b]$ such that $\tau_1<\tau_2$ and for $x\in B_\rho$, we have

$$\begin{aligned}&\left|\mathscr{K}_1x(\tau_2)-\mathscr{K}_1x(\tau_1)\right|\\&\le\frac{|\alpha_1\beta_1|}{|\Omega|\Gamma(q+1)}\left|\left(\log\left(\frac{\tau_2}{a}\right)\right)^q-\left(\log\left(\frac{\tau_1}{a}\right)^q\right)\right|\\&\cdot{}^{H}I^{q}\left({}^{RL}I^{p}(f_x)\right)(b)\\&+\frac{|\alpha_1\beta_2|}{|\Omega|\Gamma(q+1)}\left|\left(\log\left(\frac{\tau_2}{a}\right)\right)^q-\left(\log\left(\frac{\tau_1}{a}\right)^q\right)\right|\\&\cdot{}^{RL}I^{p}(f_x)(b)\\&\le\varphi^*\Omega_2\left|\left(\log\left(\frac{\tau_2}{a}\right)\right)^q-\left(\log\left(\frac{\tau_1}{a}\right)^q\right)\right|,\end{aligned} \tag{40}$$

which is independent of x and also tends to zero as $\tau_1\longrightarrow\tau_2$. Hence the set $\mathscr{K}_1B_\rho$ is equicontinuous. Therefore the set $\mathscr{K}_1B_\rho$ is relatively compact. By applying the Arzelá-Ascoli theorem, the operator $\mathscr{K}_1$ is compact on B_ρ. Therefore the operators $\mathscr{K}_1$ and $\mathscr{K}_2$ satisfy the assumptions of Theorem 11. By the conclusion of Theorem 11, we get that the separated boundary value problem (1) has at least one solution on $[a,b]$. This completes the proof. □

Theorem 13. *Assume that* (H_1) *and* (H_2) *are fulfilled. If* $\psi^*[{}^{RL}I^{p}({}^{H}I^{q}(1))(b)]<1$, *then the separated boundary value problem (2) has at least one solution on* $[a,b]$.

The above theorem can be proved by applying Krasnoselskii's fixed point theorem to the operator $\mathscr{A}$ defined in (28).

Remark 14. If the operators $\mathscr{K}_1$ and $\mathscr{K}_2$ are interchanged, then we have the existence results as follows:

(i) If $\psi^*\Omega_2<1$, then problem (1) has at least one solution on $[a,b]$.

(ii) If $\psi^*\Omega_2^*<1$, then problem (2) has at least one solution on $[a,b]$, where

$$\begin{aligned}\Omega_2^*&=\frac{|\beta_1|}{|\Omega^*|}\left(|\alpha_1|\frac{(b-a)^p}{\Gamma(p+1)}+|\alpha_2|\right){}^{RL}I^{p}\left({}^{H}I^{q}(1)\right)(b)\\&\quad+\frac{|\beta_2|}{|\Omega^*|}\left(|\alpha_1|\frac{(b-a)^p}{\Gamma(p+1)}+|\beta_2|\right){}^{H}I^{q}(1)(b).\end{aligned} \tag{41}$$

However, in application to existence theory, the computation of values ${}^{H}I^{q}({}^{RL}I^{p}(1))(b)$ and ${}^{RL}I^{p}({}^{H}I^{q}(1))(b)$ is easier than Ω_2 and Ω_2^*, respectively.

The third existence result will be proved by applying Leray-Schauder nonlinear alternative.

Theorem 15 ((nonlinear alternative for single valued maps) [25]). *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $\mathcal{D} : \overline{U} \longrightarrow C$ is a continuous; compact (that is, $\mathcal{D}(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *$\mathcal{D}$ has a fixed point in $\overline{U}$ or*

(ii) *there is a $x \in \partial U$ (the boundary of U in C) and $\nu \in (0,1)$ with $x = \nu\mathcal{D}(x)$.*

Let us state and prove the existence theorem.

Theorem 16. *Suppose that*

(H_3) there exist a continuous nondecreasing function $\xi : [0,\infty) \longrightarrow (0,\infty)$ and a function $\eta \in C([a,b],\mathbb{R}^+)$ such that

$$\left|f(t,x)\right| \le \eta(t)\,\xi(|x|) \quad \text{for each } (t,x) \in [a,b]\times\mathbb{R}; \tag{42}$$

(H_4) there exists a constant $K > 0$ such that

$$\frac{K}{\|\eta\|\,\xi(K)\,\Omega_1} > 1. \tag{43}$$

Then the separated boundary value problem (1) has at least one solution on $[a,b]$.

Proof. Let the operator $\mathcal{K}$ be defined in (26). Let us prove that the operator $\mathcal{K}$ maps bounded sets (balls) into bounded sets in $\mathcal{C}$. For a constant $\lambda > 0$, we define a bounded ball $B_\lambda = \{x \in \mathcal{C} : \|x\| \le \lambda\}$. Then for $t \in [a,b]$, one has

$$\begin{aligned}
|\mathcal{K}x(t)| &\le \frac{|\beta_1|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right) \\
&\cdot {}^{H}I^{q}\left({}^{RL}I^{p}(|f_x|)\right)(b) \\
&+ \frac{|\beta_2|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right){}^{RL}I^{p}(|f_x|)(b) \\
&+ {}^{H}I^{q}\left({}^{RL}I^{p}(|f_x|)\right)(t) \le \|\eta\|\,\xi(|x|) \\
&\cdot \frac{|\beta_1|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right){}^{H}I^{q}\left({}^{RL}I^{p}(1)\right)(b) \\
&+ \|\eta\|\,\xi(|x|)\frac{|\beta_2|}{|\Omega|}\left(|\alpha_1|\frac{(\log(b/a))^q}{\Gamma(q+1)} + |\alpha_2|\right) \\
&\cdot {}^{RL}I^{p}(1)(b) + \|\eta\|\,\xi(|x|)\,{}^{H}I^{q}\left({}^{RL}I^{p}(1)\right)(b) \le \|\eta\| \\
&\cdot \xi(\lambda)\,\Omega_1,
\end{aligned} \tag{44}$$

which implies that

$$\|\mathcal{K}x\| \le \|\eta\|\,\xi(\lambda)\,\Omega_1. \tag{45}$$

After that we will show that the operator $\mathcal{K}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let τ_1, τ_2 be any two points in $[a,b]$ such that $\tau_1 < \tau_2$. Then for $x \in B_\lambda$, we have

$$\begin{aligned}
&\left|(\mathcal{K}x)(\tau_2) - (\mathcal{K}x)(\tau_1)\right| \\
&\le \frac{|\alpha_1\beta_1|}{|\Omega|\,\Gamma(q+1)}\left|\left(\log\left(\frac{\tau_2}{a}\right)\right)^q - \left(\log\left(\frac{\tau_1}{a}\right)\right)^q\right| \\
&\cdot {}^{H}I^{q}\left({}^{RL}I^{p}(|f_x|)\right)(b) \\
&+ \frac{|\alpha_1\beta_2|}{|\Omega|\,\Gamma(q+1)}\left|\left(\log\left(\frac{\tau_2}{a}\right)\right)^q - \left(\log\left(\frac{\tau_1}{a}\right)\right)^q\right| \\
&\cdot {}^{RL}I^{p}(|f_x|)(b) \\
&+ \left|{}^{H}I^{q}\left({}^{RL}I^{p}(f_x)\right)(\tau_2) - {}^{H}I^{q}\left({}^{RL}I^{p}(f_x)\right)(\tau_1)\right| \\
&\le \frac{\|\eta\|\,\xi(\lambda)\,|\alpha_1\beta_1|}{|\Omega|\,\Gamma(q+1)}\left|\left(\log\left(\frac{\tau_2}{a}\right)\right)^q - \left(\log\left(\frac{\tau_1}{a}\right)\right)^q\right| \\
&\cdot {}^{H}I^{q}\left({}^{RL}I^{p}(1)\right)(b) \\
&+ \frac{\|\eta\|\,\xi(\lambda)\,|\alpha_1\beta_2|}{|\Omega|\,\Gamma(q+1)}\left|\left(\log\left(\frac{\tau_2}{a}\right)\right)^q - \left(\log\left(\frac{\tau_1}{a}\right)\right)^q\right| \\
&\cdot {}^{RL}I^{p}(1)(b) + \|\eta\|\,\xi(\lambda) \\
&\cdot \left|{}^{H}I^{q}\left({}^{RL}I^{p}(1)\right)(\tau_2) - {}^{H}I^{q}\left({}^{RL}I^{p}(1)\right)(\tau_1)\right|.
\end{aligned} \tag{46}$$

As $\tau_1 \longrightarrow \tau_2$, the right-hand side of the above inequality tends to zero independently of $x \in B_\rho$. Hence, by applying the Arzelá-Ascoli theorem, the operator $\mathcal{K} : \mathcal{C} \longrightarrow \mathcal{C}$ is completely continuous.

The result will be followed from the Leray-Schauder nonlinear alternative if we prove the boundedness of the set of the solutions to equation $x = \nu\mathcal{K}x$ for $\nu \in (0,1)$. Let x be a solution of the operator equation $x = \mathcal{K}x$. Then, for $t \in [a,b]$, by directly computation, we have

$$|x(t)| \le \|\eta\|\,\xi(\|x\|)\,\Omega_1, \tag{47}$$

which leads to

$$\frac{\|x\|}{\|\eta\|\,\xi(\|x\|)\,\Omega_1} \le 1. \tag{48}$$

From the assumption (H_4), there exists a positive constant K such that $\|x\| \ne K$. Let us set

$$U = \{x \in \mathcal{C} : \|x\| < K\}. \tag{49}$$

It is easy to see that the operator $\mathcal{K} : \overline{U} \longrightarrow \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \nu\mathcal{K}x$ for some $\nu \in (0,1)$. Therefore, by the nonlinear alternative of Leray-Schauder type (Theorem 15), we deduce that the operator $\mathcal{K}$ has a fixed point $x \in \overline{U}$ which is a solution of problem (1). The proof is completed. □

Theorem 17. *Assume that the condition* (H_3) *in Theorem 16 is satisfied. If a positive constant* K_1 *satisfying*

$$\frac{K_1}{\|\eta\| \xi(K_1) \Omega_1^*} > 1, \tag{50}$$

then the separated boundary value problem (2) has at least one solution on $[a, b]$.

The next two special cases can be obtained by setting $\eta(t) = 1$, $t \in [a, b]$ and $\xi(y) = Ey + G$, $y \in [0, \infty)$ with two constants $E \geq 0$, $G > 0$.

Corollary 18. *Let* $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function satisfying* $|f(t, x)| \leq E|x| + G$, *for all* $x \in \mathbb{R}$. *Then*

(i) *if* $E\Omega_1 < 1$, *then the separated boundary value problem (1) has at least one solution on* $[a, b]$;

(ii) *if* $E\Omega_1^* < 1$, *then the separated boundary value problem (2) has at least one solution on* $[a, b]$.

4. Examples

In this section, we present some examples to illustrate our results.

Example 1. Consider the following sequential Caputo-Hadamard fractional differential equations with separated boundary conditions

$$\begin{gathered} {}^{C}D^{1/2}\left({}^{H}D^{1/3}x\right)(t) = f(t, x(t)), \\ t \in \left(\frac{1}{2}, \frac{5}{2}\right), \\ \frac{1}{4}x\left(\frac{1}{2}\right) + \frac{3}{5}\left({}^{H}D^{1/3}x\right)\left(\frac{1}{2}\right) = 0, \\ \frac{5}{8}x\left(\frac{5}{2}\right) + \frac{7}{9}\left({}^{H}D^{1/3}x\right)\left(\frac{5}{2}\right) = 0. \end{gathered} \tag{51}$$

Here $p = 1/2$, $q = 1/3$, $a = 1/2$, $b = 5/2$, $\alpha_1 = 1/4$, $\alpha_2 = 3/5$, $\beta_1 = 5/8$, and $\beta_2 = 7/9$. From given information, we find that $\Omega = -0.0244992447$, ${}^{H}I^{1/3}({}^{RL}I^{1/2}(1))(5/2) = 1.622871815$, and ${}^{RL}I^{1/2}(1)(5/2) = 1.595769121$ which yield $\Omega_1 = 87.06444876$.

(i) Let $f : [1/2, 5/2] \times \mathbb{R} \longrightarrow \mathbb{R}$ with

$$f(t, x) = \frac{\cos^2 \pi t}{2((t - 1/2) + 90)}\left(\frac{x^2 + |x|}{|x| + 1}\right) + \frac{1}{2}. \tag{52}$$

It follows that

$$\begin{aligned} \left|f(t, x) - f(t, y)\right| &\leq \frac{\cos^2 \pi t}{((t - 1/2) + 90)}\left|x - y\right| \\ &:= \psi(t)\left|x - y\right|. \end{aligned} \tag{53}$$

Then condition (H_1) is satisfied with $\psi^* = 1/90$. Thus $\psi^*\Omega_1 = 0.9673827640 < 1$. Hence, by Theorem 9, problem (51) with (52) has a unique solution on $[1/2, 5/2]$.

(ii) Given $f : [1/2, 5/2] \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(t, x) = \frac{\cos^2 \pi t}{2((t - 1/2) + 2)}\left(\frac{x^2 + |x|}{|x| + 1}\right) + \frac{1}{2}. \tag{54}$$

Observe that the function f defined in (54) satisfies (H_1) with $\psi^* = 1/2$. But the Theorem 9 can not be applied to this case because the value of $\psi^*\Omega_1 = 43.53222438 > 1$. However, by the benefit of Theorem 12, we have $\psi^*[{}^{H}I^{1/3}({}^{RL}I^{1/2}(1))(5/2)] = 0.8114359075 < 1$. By the conclusion of Theorem 12, problem (51) with (54) has at least one solution on $[1/2, 5/2]$.

Example 2. Consider the following sequential Hadamard-Caputo fractional differential equations with separated boundary conditions

$$\begin{gathered} {}^{H}D^{3/4}\left({}^{C}D^{2/5}x\right)(t) = g(t, x(t)), \\ t \in \left(\frac{1}{8}, \frac{7}{8}\right), \\ \frac{3}{11}x\left(\frac{1}{8}\right) + \frac{\pi}{4}\left({}^{C}D^{2/5}x\right)\left(\frac{1}{8}\right) = 0, \\ \frac{\sqrt{2}}{9}x\left(\frac{7}{8}\right) + \frac{2}{13}\left({}^{C}D^{2/5}x\right)\left(\frac{7}{8}\right) = 0. \end{gathered} \tag{55}$$

Here $q = 3/4$, $p = 2/5$, $a = 1/8$, $b = 7/8$, $\alpha_1 = 3/11$, $\alpha_2 = \pi/4$, $\beta_1 = \sqrt{2}/9$, and $\beta_2 = 2/13$. From above information, we can find that $\Omega^* = 0.03840540910$, ${}^{RL}I^{2/5}({}^{H}I^{3/4}(1))(7/8) = 1.526044488$, and ${}^{H}I^{3/4}(1)(7/8) = 1.792656288$ which can be computed the value of $\Omega_1^* = 15.74791264$.

(i) The function $g : [1/8, 7/8] \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$g(t, x) = \frac{8\sin^4 \pi t}{263 + 8t}\left(\frac{x^6}{x^4 + 1} + 1\right). \tag{56}$$

Setting $\eta(t) = (8\sin^4 \pi t/(263 + 8t))$ and $\xi(y) = y^2 + 1$, we see that the condition (H_3) of Theorem 16 is satisfied with the above function $g(t, x)$. In addition, we can find that $\|\eta\| = 1/33$. Then there exists a constant K such that $K \in (0.7350333746, 1.360482441)$ satisfying inequality (50). Therefore, applying Theorem 17, problem (55) with (56) has at least one solution on $[1/8, 7/8]$.

(ii) Let $g : [1/8, 7/8] \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(t, x) = \frac{e^{-(t-1/8)^2}}{16}\left(\frac{x^8}{|x|^7 + 1}\right) + \frac{3}{4(1 + t^2)}. \tag{57}$$

It is easy to see that the function $g(t, x)$ defined in (57) can be expressed as $|g(t, x)| \leq (1/16)|x| + (3/4)$. Then $(1/16)\Omega_1^* = 0.9842445400 < 1$. Using (ii) of the Corollary 18, the problem (55) with (57) has at least one solution on $[1/8, 7/8]$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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