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Separating Minimal, Intuitionist, and Classical Logic

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Classical, two-valued propositional logic contains intuitionist logic. Intuitionist logic in turn contains minimal logic. Standard formulations of the classical system, however, tend to make it difficult to determine whether a given classical thesis is purely classical, is classical and intuitionist, or belongs in all three systems.

The present paper offers formulations of classical implication-negation logic that make separation of its intuitionist and minimal components very easy. Section 1 deals with some preliminaries. Section 2 gives a classical axiom base with no dependent axioms, which has proper subaxiomatics giving intuitionist and minimal logic. The final section offers a natural deduction style counterpart of the axiomatic system.

1 ***Preliminaries*** Our point of departure is the standard intuitionist axiomatic used by Horn in [2]. Since Horn proves that this base has the separation property, it is clear that the axioms in implication and negation are sufficient for all intuitionist theses in these connectives. There are four such axioms: *CpCqp, CCpCqrCCpqCpr, CCpNqCqNp,* and *CNpCpq.* The rules of inference are modus ponens and substitution for variables. A minimal logic base is obtained from this intuitionist one simply by omitting the axiom *CNpCpq.*

For present purposes, rather than take implication and negation as primitive, it is better to take implication and a constant proposition 0, and define negation. This can be done because the minimal *C–N* system given by

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D1 $0 =_{df} NCqq$

and the axioms

A1 *CpCqp*
A2 *CCpCqrCCpqCpr*
A3 *CCpNqCqNp*

is deductively equivalent to the *C*–0 system given by

D2 $N\alpha =_{df} C\alpha 0$

with just Axioms A1 and A2.

To prove this deductive equivalence we show first that A1 and A2 give *Cpp* and *CCsrCCpqCCrpCsq*; these two theses suffice to show replaceability of equivalents (for proof see [1], Theorem 21). So in both the *C–N* and the *C*–0 systems the replaceability rule holds. Next we show that *C*0*CCqq*0, *CCCqq*00, and *CCpCq*0*CqCp*0 are theses of the *C*–0 system; by means of D2 these theses give, respectively, *C*0*NCqq*, *CNCqq*0, and *CCpNqCqNp*. Thus the whole *C–N* system is contained in the *C*–0 one. Finally, we show that *CNpCpNCqq* and *CCpNCqqNp* are *C–N* theses; by means of D1 these theses give, respectively, *CNpCp*0 and *CCp*0*Np*. Thus the whole *C*–0 system is contained in the *C–N* system. The required derivations are as follows. (C. A. Meredith's condensed detachment operator is used abbreviatively: '*Dm.n.*' denotes the most general formula that can be obtained by applying modus ponens with *m*, or some substitution in it, as major premiss, and *n*, or some substitution in it, as minor premiss.)

1.	*CpCqp*	Axiom
2.	*CCpCqrCCpqCpr*	Axiom
3.	*CCpNqCqNp*	Axiom
4.	*Cpp*	*DD*2.1.1
5.	*CCqrCCpqCpr*	*DD*2.*D*1.2.1
6.	*CCpCqrCqCpr*	*DD*2.*DD*5.5.2.*D*1.1
7.	*CCpqCCqrCpr*	*D*6.5
8.	*CCsrCCpqCCrpCsq*	*DD*5.*DD*5.*DD*6.5.5.5.7
9.	*C*0*CCqq*0	1 *p*/0 *q*/*Cqq*
10.	*CCCppqq*	*DD*6.4.4
11.	*CCCqq*00	10 *p*/*q* *q*/0
12.	*CCpCq*0*CqCp*0	6 *r*/0
13.	*CNpCCqqNp*	1 *p*/*Np* *q*/*Cqq*
14.	*CCCqqNpCpNCqq*	3 *p*/*Cqq* *q*/*p*
15.	*CNpCpNCqq*	*DD*7.13.14
16.	*CCpNCqqCCqqNp*	3 *q*/*Cqq*
17.	*CCpNCqqNp*	*DD*6.16.4.

2 ***Axiomatic system*** The nucleus of our axiomatic system is the minimal *C*–0 system given by D2 together with A1 and A2. (If the constant 0 and D2 are omitted, we have the Hilbert positive implicational logic.) To this nucleus we add first

A4 *C*0*p*

and second

A5 *CCCp*0*pp*.

The first addition gives the intuitionist *C*–0 system; this follows from the proof given by Wajsberg that the resultant system is a deductive equivalent of the intuitionist *C*–*N* fragment (see [3], Section 5). The second addition gives a classical base that has no dependent axioms; this is proved below.

Since replaceability has already been proven for the minimal *C*–0 system, and *C*0*NCqq* and *CNCqq*0 have been shown to be minimal *C*–0 theses, completeness of the four axiom base for classical logic can be established simply by deriving from it the three Łukasiewicz *C*–*N* axioms. These are *CCpqCCqrCpr*, *CpCNpq*, and *CCNppp*. The first axiom is Thesis 7 above; the third follows from A5 by D2; the remaining axiom follows from *CpCCp*0*q* by D2. The derivation of this latter thesis is

18.	*C*0*p*	Axiom
19.	*CCqrCsCpqCsCpr*	*DD*5.5.5
20.	*CpCCp*0*q*	*DD*19.18.*D*6.4.

The only nonintuitionist thesis among the four axioms is A5; its independence therefore is clear. To prove the independence of the remaining axioms the three following matrices are used. With 1 as the only designated value and 3 as the value of the constant proposition, each of these matrices verifies modus ponens and the definition.

C	1	2	3	*N*
*1	1	3	3	3
2	3	3	1	1
3	1	1	1	1

M1

C	1	2	3	*N*
*1	1	3	3	3
2	1	3	1	1
3	1	1	1	1

M2

C	1	2	3	*N*
*1	1	2	3	3
2	1	1	3	3
3	1	2	1	1

M3

M1 verifies all the axioms except the first, which fails for *p*/1, *q*/2; M2 verifies all the axioms except the second, which fails for *p*/2, *q*/3, *r*/2; and M3 verifies all except the third, which fails for *p*/2.

3 Natural deduction counterpart of the axiomatic system Basic to our natural deduction style system is the concept of hypotheses leading to a conclusion. The hypotheses $\alpha_1, \ldots, \alpha_n$ are said to *yield* the conclusion β (written $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$) if and only if there is a finite sequence of formulas $\alpha_{n+1}, \ldots, \alpha_m$ such that $\alpha_m = \beta$, and for each $\alpha_i (n < i \leqslant m)$ one of the three following is true: α_i is identical with some $\alpha_j (1 \leqslant j \leqslant n)$; α_i follows from one or more formulas α_k, $\alpha_1 (1 \leqslant k < i; 1 \leqslant 1 < i)$ by primitive inference; α_i is a substitution instance of a thesis. The definition of course is not complete until the primitive inferences have been enumerated, and some means has been specified for obtaining theses. In the minimal system the only primitive inferences will be, modus ponens

MP From $C\alpha\beta$ and α, β may be inferred

and the inferences given by the definition of N

DEF $N\alpha =_{df} C\alpha 0$.

Theses result from the rule of conditional proof

CON If $\alpha_1, \ldots, \alpha_n \Rightarrow \beta$, then $C\alpha_1 \ldots C\alpha_n\beta$ is a thesis ($n \geqslant 0$).

Treating already proven formulas as a distinguished subclass of formulas whose substitution instances can be adjoined to any derivation has two advantages. First, we can dispense with Fitch-style subproofs and the concomitant apparatus for keeping track of the status of hypotheses; a single use of the rule of conditional proof in our system dismisses all hypotheses. Second, there are no complications with respect to substitution; the only time substitution can occur is when a thesis is adjoined to a derivation, and the possibility of doing a substitution in the variable of a hypothesis does not arise. To illustrate use of the system we prove the thesis *CCpqCNqNp* which gives the derived minimal inference modus tollens. In addition to the abbreviations noted above, we use 'HYP' for assumption of a hypothesis and 'REP' for its repetition. A thesis is marked with '⊢' on its first appearance; the notation '⊢m' is used when the already proven thesis m, or a substitution instance thereof, is adjoined to a derivation. 'Λ' is the null hypothesis.

1.	*Cpq*	HYP
2.	*Cqr*	HYP
3.	*p*	HYP
4.	*q*	MP 1 3
5.	*r*	MP 2 4
6.	⊢*CCpqCCqrCpr*	CON
7.	Λ	HYP
8.	*CCpqCCq*0*Cp*0	⊢6
9.	*CCpqCNqNp*	DEF 8
10.	⊢*CCpqCNqNp*	CON.

Another worthwhile derived minimal inference is given by the thesis *CNCpqNq*.

11.	*p*	HYP
12.	*q*	HYP
13.	*p*	REP
14.	⊢*CpCqp*	CON
15.	*NCpq*	HYP
16.	*CCpq*0	DEF 15
17.	*CCqCpqCCCpq*0*Cq*0	⊢6
18.	*CqCpq*	⊢14
19.	*CCCpq*0*Cq*0	MP 17 18
20.	*Cq*0	MP 19 16
21.	*Nq*	DEF 20
22.	⊢*CNCpqNq*	CON.

That our natural deduction system gives only minimal theses follows from the fact that the Deduction Theorem is known to hold for the system given by A1 and A2 with modus ponens and substitution for variables as the rules of inference. That it gives all such theses follows from the fact that both the axioms can easily be proven (A1 appears as 14 above). To get from this system to the intuitionist and classical systems, the stock of primitive inferences must be increased. Following the augmentation pattern of the previous section, we add the inference from the constant false proposition

FAL From 0, α may be inferred

and the *consequentia mirabilis*

MIR From $CN\alpha\alpha$, α may be inferred.

By conditional proof, the first of these easily gives $C0p$, and the second $CCNppp$, and thus we have the intuitionist and classical systems.

In the classical system the rule of indirect proof

IND If $N\alpha \Rightarrow \beta$ and $N\alpha \Rightarrow N\beta$ then α is a thesis

can be proven as follows. First, we have

23.	Np	HYP
24.	p	HYP
25.	$Cp0$	DEF 23
26.	0	MP 25 24
27.	q	FAL 26
28.	$\vdash CNpCpq$	CON.

Now assume the hypotheses of the rule, and assume further that β and $N\beta$ are reached in m steps. Then

k.	$N\alpha$	HYP
.		
.		
l.	β	
.		
.		
m.	$N\beta$	
i.	$CN\beta C\beta\alpha$	$\vdash$28
ii.	$C\beta\alpha$	MP i m
iii.	α	MP ii l
iv.	$\vdash CN\alpha\alpha$	CON
v.	Λ	HYP
vi.	$CN\alpha\alpha$	$\vdash$iv
vii.	α	MIR vi
viii.	$\vdash\alpha$	CON.

A very easily used classical system results from the fact that the minimal inference given by $CNCpqNq$ is complemented by the classical inference given by $CNCpqp$.

29.	$NCpq$	HYP
30.	$CNCpqCCpqp$	$\vdash$28

31.	*CCpqp*	MP 30 29
32.	*CCNpCpqCCCpqpCNpp*	$\vdash$6
33.	*CNpCpq*	$\vdash$28
34.	*CCCpqpCNpp*	MP 32 33
35.	*CNpp*	MP 34 31
36.	*p*	MIR 35
37.	$\vdash$*CNCpqp*	CON.

Marking the inference given by this thesis '*Sa m*' (for "selection of antecedent"), the inference given by its minimal companion '*Sc m*' (for "selection of consequent"), and modus tollens 'MT', proof of Peirce's law gives a good illustration of the system.

39.	*NCCCpqpp*	HYP
40.	*CCpqp*	Sa 39
41.	*Np*	Sc 39
42.	*NCpq*	MT 40 41
43.	*p*	Sa 42
44.	$\vdash$*CCCpqpp*	IND.

While the technique shown is always appropriate for purely classical theses, it does not always give the shortest proof. A better proof for *CCCpqrCNrp*, for instance, than that starting from the hypothesis *NCCCpqrCNrp,* is the following.

45.	*CCpqr*	HYP
46.	*Nr*	HYP
47.	*NCpq*	MT 45 46
48.	*p*	Sa 47
49.	$\vdash$*CCCpqrCNrp*	CON.

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