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SEPARATION AXIOMS FOR INTERVAL TOPOLOGIES

MARCEL ERNÉ

ABSTRACT. In Theorem 1 of this note, results of Kogan [2], Kolibiar [3], Matsushima [4] and Wolk [7] concerning interval topologies are presented under a common point of view, and further characterizations of the T_2 axiom are obtained. A sufficient order-theoretical condition for regularity of interval topologies is established in Theorem 2. In lattices, this condition turns out to be equivalent both to the T_2 and to the T_3 axiom. Hence, a Hausdorff interval topology of a lattice is already regular. However, an example of a poset is given where the interval topology is T_2 but not T_3.

1. Introduction. It is well known that the interval topology of any chain satisfies each of the separation axioms $T_0, \ldots, T_5$ (cf. [6, p. 67]). This no longer holds if chains are replaced with arbitrary lattices or posets. Although the interval topology of any poset is T_1, there are even complete lattices for which the interval topology is not T_2. A necessary and sufficient condition for the interval topology to be T_2 has been given by M. Kolibiar [3]. In this note, we mainly study under which conditions the interval topology may be regular. Specifically, in the case of lattices we shall find that for interval topologies, the T_2 axiom and the T_3 axiom are equivalent.

2. Basic notations. Let X be an arbitrary poset, the partial ordering of which is indicated by the symbol $\leqslant$. For $Y \subset X$, $Y^\dagger$ and Y^* denote the set of all lower and upper bounds of Y, respectively. The sets

$$x^\dagger := \{x\}^\dagger = \{y \in X: y \leqslant x\} \quad \text{and} \quad x^* := \{x\}^* = \{y \in X: x \leqslant y\}$$

are referred to as *closed rays*. Every closed ray, every set of the form

$$[x, z] = x^* \cap z^\dagger = \{y \in X: x \leqslant y \leqslant z\},$$

and the entire set X are called (*closed*) *intervals*. The *interval topology* $\mathfrak{T}$ on X is the smallest topology in which all intervals are closed sets. Thus the interval topology (considered as the collection of all open sets) is generated by the set-complements of all closed rays. In particular, each singleton

$$\{x\} = [x, x]$$

is closed in $\mathfrak{T}$, and one has

LEMMA 1. *The interval topology of any poset is* T_1.

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3. The T_2 axiom. For a subset W, let $N(W)$ denote the set of all elements in X which are neither upper nor lower bounds of W, i.e.

$$N(W) := X \setminus (W^\dagger \cup W^*).$$

We shall write $N(x)$ for $N(\{x\})$ and $N(x,y)$ for $N(\{x,y\})$. Thus $N(x)$ consists of all elements not comparable with x. A set $Y \subset X$ is *finitely separable* if there is a finite subset V of Y such that every element of Y is comparable with some element of V. For the sake of brevity, we put

$$\downarrow Y := \{x \in X: x \leqslant y \text{ for some } y \in Y\} = \bigcup \{y^\dagger: y \in Y\},$$

$$\uparrow Y := \{x \in X: y \leqslant x \text{ for some } y \in Y\} = \bigcup \{y^*: y \in Y\}.$$

Hence a set $Y \subset X$ is finitely separable iff $Y \subset \downarrow V \cup \uparrow V$ for some finite $V \subset Y$. The following assertion is clear by the definition of the interval topology.

LEMMA 2. *The sets $X \setminus (\downarrow Y \cup \uparrow Z)$, where Y and Z run over all finite subsets of X, form a base of the interval topology.*

As an easy consequence, one obtains

LEMMA 3. *The interval topology $\mathfrak{T}$ of any poset X is T_2 iff for all $x \neq y$, there are finite sets Y, Z with $Y \cap \{x,y\}^* = \varnothing$, $Z \cap \{x,y\}^\dagger = \varnothing$, $X = \downarrow Y \cup \uparrow Z$.*

PROOF. If $\mathfrak{T}$ is T_2 then for $x \neq y$, we can find finite sets Y_x, Z_x, Y_y, Z_y such that $x \in U := X \setminus (\downarrow Y_x \cup \uparrow Z_x)$, $y \in V := X \setminus (\downarrow Y_y \cup \uparrow Z_y)$, $U \cap V = \varnothing$. Thus $Y := Y_x \cup Y_y$ and $Z := Z_x \cup Z_y$ have the required properties.

Conversely, let Y, Z be finite sets such that $Y \cap \{x,y\}^* = \varnothing$, $Z \cap \{x,y\}^\dagger = \varnothing$, $X = \downarrow Y \cup \uparrow Z$. Then $U := X \setminus (\downarrow(Y \setminus x^*) \cup \uparrow(Z \setminus x^\dagger))$ and $V := X \setminus (\downarrow(Y \setminus y^*) \cup \uparrow(Z \setminus y^\dagger))$ are disjoint open neighbourhoods of x and y, respectively.

LEMMA 4. *For any subset W of X containing at least two elements, $N(W)$ is finitely separable iff there are finite sets Y, Z with $Y \cap W^* = \varnothing$, $Z \cap W^\dagger = \varnothing$, $\downarrow Y \cup \uparrow Z = X$.*

PROOF. Suppose $V \subset N(W) \subset \downarrow V \cup \uparrow V$, $|V| < \infty$. Since $|W| \geqslant 2$, we can choose $y, z \in W$ with $z \nleqslant y$. The sets $Y := V \cup \{y\}$ and $Z := V \cup \{z\}$ have the required properties. Conversely, assume $Y \cap W^* = \varnothing$, $Z \cap W^\dagger = \varnothing$, $\downarrow Y \cup \uparrow Z = X$ for some finite sets Y, Z. Then $V := N(W) \cap (Y \cup Z)$ is a finite subset of $N(W)$, and it is easily verified that $N(W) \subset \downarrow V \cup \uparrow V$.

COROLLARY 1. *If $N(W)$ is finitely separable and $|W| \geqslant 2$ then for all sets Y containing W, $N(Y)$ is also finitely separable.*

PROOF. $W \subset Y$ implies $Y^\dagger \subset W^\dagger$ and $Y^* \subset W^*$, so we can apply Lemma 4.

By an *antichain*, we mean a set of pairwise incomparable elements. Adjoining a least element 0 and a greatest element 1 to an infinite antichain, a complete lattice is obtained in which $N(0)$ and $N(1)$ are finitely separable (being empty) while $N(0, 1)$ is not.

THEOREM 1. *Consider the following statements for a partially ordered set X*:

(a) *X contains no infinite antichain.*

(a′) *Every subset of X is finitely separable.*

(b) *For all $x \in X$, the set $N(x)$ is finitely separable.*

(b′) *For every subset W of X, $N(W)$ is finitely separable.*

(c) *For all $x \neq y$, the set $N(x, y)$ is finitely separable.*

(c′) *For every subset W of X containing more than one element, $N(W)$ is finitely separable.*

(c″) *The interval topology of X is T_2.*

One has the following implications:

$$(a) \Leftrightarrow (a') \Rightarrow (b) \Leftrightarrow (b') \Rightarrow (c) \Leftrightarrow (c') \Leftrightarrow (c'').$$

PROOF. The equivalence of (a) and (a′) is clear. In view of Lemma 3, Lemma 4, and Corollary 1, it only remains to show the implication (b) $\Rightarrow$ (c): Assume $x < z < y$ or $y < z < x$ for some z. Then we find a finite set $V \subset N(z)$ with $N(z) \subset \downarrow V \cup \uparrow V$, and $W := V \cup \{z\}$ is a finite subset of $N(x, y)$ with $\downarrow W \cup \uparrow W = X$. In particular, $N(x, y)$ is finitely separable. In all other cases, we obtain $N(x, y) = N(x) \cup N(y)$, and then hypothesis (b) ensures again that $N(x, y)$ is finitely separable.

The implication (a) $\Rightarrow$ (c″) is due to E. S. Wolk [7], and (b) $\Rightarrow$ (c″) has been shown by Y. Matsushima [4] who also disproved the implication (b) $\Rightarrow$ (a) by a counterexample. The equivalence of (c) and (c″) has been stated without proof by M. Kolibiar in [3]. Moreover, Kolibiar has constructed an example where (c″) but not (b) holds. Finally, as F. S. Northam [5] has remarked, (c″) implies that every *open interval* $]x, y[= \{z: x < z < y\}$ be finitely separable.

4. **The T_3 axiom.** The investigation of regularity for interval topologies involves some more effort. First of all, we reduce the separation of points from closed sets to subbasic closed sets.

LEMMA 5. *The interval topology $\mathfrak{T}$ of a poset X is regular* (*i.e.* T_3) *iff the following condition and its dual hold*:

(∗) *For all $x, y \in X$ with $x \nleq y$, there exist disjoint $\mathfrak{T}$-open sets U, V with $x \in U$, $y^{\dagger} \subset V$.*

PROOF. The necessity of this condition is clear. To show sufficiency, let A be an arbitrary $\mathfrak{T}$-closed set with $x \notin A$. Then A can be written as an intersection of sets A_i each of which is a finite union of closed rays, say

$$A_i = \downarrow Y_i \cup \uparrow Z_i$$

where Y_i and Z_i are finite subsets of X ($i \in I$). $x \notin A$ implies $x \notin A_i$ for some $i \in I$, and it follows $x \nleq y$ for all $y \in Y_i$, $z \nleq x$ for all $z \in Z_i$. Hence we find open sets U_y, V_y with $x \in U_y$, $y^{\dagger} \subset V_y$, $U_y \cap V_y = \varnothing$ ($y \in Y_i$), and open sets U'_z, V'_z with $x \in U'_z$, $z^{*} \subset V'_z$, $U'_z \cap V'_z = \varnothing$ ($z \in Z_i$). $U := \bigcap\{U_y: y \in Y_i\} \cap \bigcap\{U'_z: z \in Z_i\}$ is a neighbourhood of x, and $V := \bigcup\{V_y: y \in Y_i\} \cup \bigcup\{V'_z: z \in Z_i\}$ is an open set disjoint from U and containing A_i. In particular, $A \subset V$, and we have separated x and A by disjoint open sets U, V.

Remember that the interval topology is T_2 iff the sets

$$N(x,y) = X \setminus ((x^\dagger \cap y^\dagger) \cup (x^* \cap y^*)) \qquad (x \neq y)$$

are finitely separable. Now defining

$$M(x,y) := X \setminus ((x^\dagger \cap y^\dagger) \cup x^*),$$
$$P(x,y) := X \setminus ((x^* \cap y^*) \cup y^\dagger),$$

we can show a similar result concerning the T_3 axiom:

THEOREM 2. *For the interval topology of a poset X to be regular it is sufficient that for all $x, y \in X$ with $x \nleq y$, the sets $M(x,y)$ and $P(x,y)$ are finitely separable. In lattices, this condition is also necessary.*

PROOF. Suppose $x \nleq y$. We have to separate x and $y^\dagger$ by open sets. (Dual arguments show that y and x^* can be separated.) By hypothesis, there is a finite set W with $W \subset M(x,y) \subset \downarrow W \cup \uparrow W$. Define

$$Y := W \cup \{y\}, \qquad Z := W \setminus x^\dagger, \qquad Z' := (W \setminus y^\dagger) \cup \{x\}.$$

Then Y, Z, Z' are finite sets such that

(1) $W \subset Y$,
(2) $W \cap x^\dagger \cap y^\dagger = \varnothing$, $W = (W \setminus x^\dagger) \cup (W \setminus y^\dagger) \subset Z \cup Z'$,
(3) $x \notin \downarrow Y$ (since $x \nleq y$ and $W \cap x^* = \varnothing$),
(4) $x \notin \uparrow Z$ (since $Z \cap x^\dagger = \varnothing$),
(5) $y^\dagger \cap \uparrow Z' = \varnothing$ (since $y^\dagger \cap Z' = \varnothing$),
(6) $X \setminus M(x,y) = (x^\dagger \cap y^\dagger) \cup x^* \subset \downarrow Y \cup \uparrow Z'$,
(7) $M(x,y) \subset \downarrow W \cup \uparrow W \subset \downarrow Y \cup \uparrow Z \cup \uparrow Z'$ (by 1 and 2).

Hence,

$$x \in U := X \setminus (\downarrow Y \cup \uparrow Z) \in \mathfrak{T} \quad \text{(by 3 and 4)},$$
$$y^\dagger \subset V := X \setminus \uparrow Z' \in \mathfrak{T} \quad \text{(by 5)},$$
$$U \cap V = \varnothing \quad \text{(by 6 and 7)}.$$

In a lattice with join $\bigvee$ and meet $\bigwedge$, one has

$$N(x,y) = X \setminus ((x \wedge y)^\dagger \cup (x \vee y)^*),$$
$$M(x,y) = X \setminus ((x \wedge y)^\dagger \cup x^*),$$
$$P(x,y) = X \setminus (y^\dagger \cup (x \vee y)^*).$$

Furthermore, $x \neq y$ implies $x \wedge y < x \vee y$, and $x \nleq y$ implies $x \wedge y < x$ and $y < x \vee y$. Accordingly, the second statement in Theorem 2 can be sharpened to

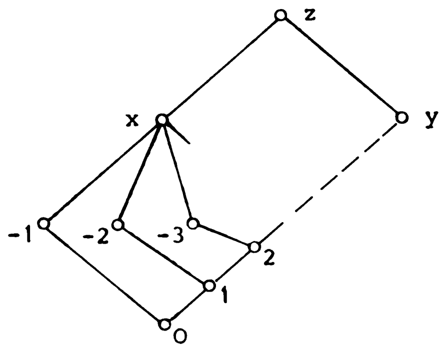
THEOREM 3. *In a lattice, the following statements are equivalent*:
(a) *For all x, y with $y < x$, the set $X \setminus (y^\dagger \cup x^*)$ is finitely separable.*
(b) *The interval topology is* T_2.
(c) *The interval topology is* T_3 (*regular*).

The equivalence of (a) and (b) is due to S. A. Kogan [2].

In *lattices*, Matsushima's condition (Theorem 1, b) implies that for all x, y with $x \nleq y$, $M(x,y)$ and $P(x,y)$ are finitely separable. This implication no longer holds in arbitrary posets, and not even in semilattices.

EXAMPLE 1. Adjoin three new elements x, y, z to the set $\mathbf{Z}$ of all integers and define a partial ordering on $X := \mathbf{Z} \cup \{x, y, z\}$ by setting

$$x^\dagger := \mathbf{Z} \cup \{x\}, \qquad y^\dagger := \{a \in \mathbf{Z}: a > 0\} \cup \{y\}, \qquad z^\dagger := X,$$
$$a^\dagger := \{b \in \mathbf{Z}: 0 < b < a\} \qquad (a \in \mathbf{Z}, a > 0),$$
$$b^\dagger := \{a \in \mathbf{Z}: 0 < a < -b\} \cup \{b\} \qquad (b \in \mathbf{Z}, b < 0).$$



Then X becomes a join-semilattice in which all points except x are isolated in the interval topology. In particular, the separation axioms are trivially fulfilled. Moreover, an easy verification shows that for all $w \in X$, $N(w)$ is finitely separable. However, $M(x, y)$ is an infinite antichain and therefore not finitely separable. This example also shows that in general, the condition stated in Theorem 2 is not necessary for regularity.

Finally, let us construct a counterexample disproving the conjecture that Matsushima's condition might imply regularity in general. Thereby, we shall see that in arbitrary posets, the T_3 axiom is strictly stonger than the T_2 axiom.

EXAMPLE 2. For any integer $j > 1$, let $p(j)$ denote the least prime divisor of j. Consider a set X constituted by three sequences (a_n), (b_n), (c_n), and define a partial ordering on X by setting

$$b_j < a_i \quad \Leftrightarrow \quad i \neq j \quad \text{and} \quad (i = 1 \text{ or } j = 1 \text{ or } p(j) < i),$$
$$c_k < b_j \quad \Leftrightarrow \quad k \neq 1 \quad \text{and} \quad j \neq 1 \text{ and } p(j) > k,$$
$$c_k < a_i \quad \Leftrightarrow \quad k \neq 1 \quad \text{and} \quad (i = 1 \text{ or } k < i),$$
$$c_k < c_n \quad \Leftrightarrow \quad n = 1 \quad \text{or} \quad 1 < k < n,$$

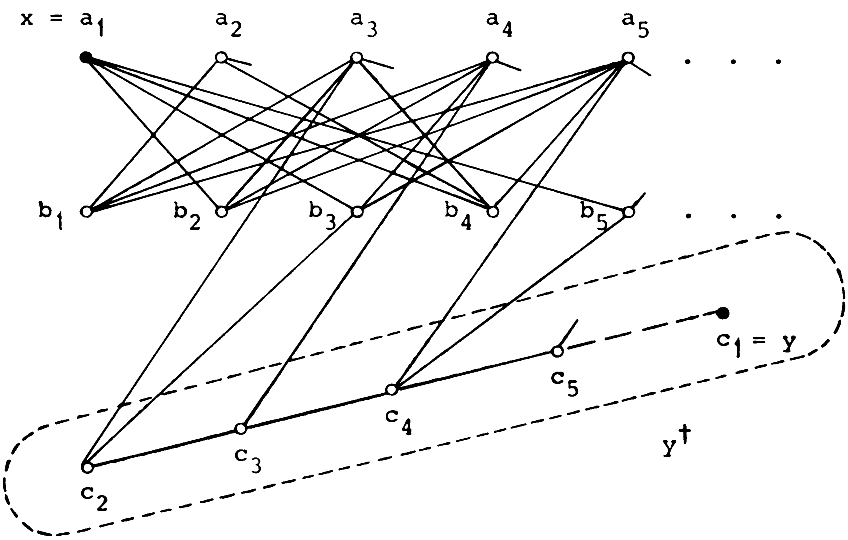
while all other pairs of distinct elements are assumed to be incomparable. A straightforward computation shows that for all $w \in X$, the set $N(w)$ is finitely separable, and in particular, the interval topology is T_2. However, for $x := a_1$ and $y := c_1$, we shall see that x and the closed ray $y^\dagger$ cannot be separated by open sets, disproving regularity. In fact, a neighbourhood base for the point x is made up by the sets

$$U_{im} := \{b_j : j > m, p(j) > i\} \cup \{x\} \qquad (m > i),$$

and for each prime number $p = k + 1 > 2$, a neighbourhood base of c_k is constituted by the sets

$$V_{kn} = \{b_j : j > n, p(j) = p\} \cup \{c_k\} \qquad (n \geqslant p).$$

All other points c_k are isolated in the interval topology. Assume there would exist disjoint open sets U, V such that $x \in U$, $y^\dagger \subset V$. Then we find some $m \geqslant i$ with $x \in U_{im} \subset U$. Choose a prime number $p = k + 1 > m$. Then $c_k \in y^\dagger \subset V$ implies $c_k \in V_{kn} \subset V$ for some $n \geqslant p$. But for $j := p^n$, we obtain $b_j \in U_{im} \cap V_{kn} = \varnothing$, a contradiction.



Concluding Remark. By a theorem of O. Frink [1], the interval topology of a lattice is compact iff the lattice is complete. Observing that a compact T_2 space is always normal (cf. [6, p. 25]), we infer that in *complete* lattices, the separation axioms T_2, T_3 and T_4 are trivially equivalent for the interval topology. It remains open whether this is true in arbitrary lattices.

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Institut für Mathematik, Universität Hannover, Hannover, Federal Republic of Germany