Separation of Variables and the Computation of Fourier Transforms on Finite Groups

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1. The Fourier Transform on a Semisimple Algebra

We define the Fourier transform on a semisimple algebra A:

Definition 1. *Let $\{a_i\}_{i\in I}$ be a basis for A and let $f = \sum_{i\in I} f(a_i)a_i$.*

*(i) Let ρ be a matrix representation of A. Then the **Fourier transform of** f **at** ρ, denoted $\hat{f}(\rho)$, is the matrix sum*

$$\hat{f}(\rho) = \sum_{i\in I} f(a_i)\rho(a_i).$$

*(ii) Let R be a set of matrix representations of A. Then the **Fourier transform of** f **on** R is the direct sum*

$$\mathcal{F}_R(f) = \bigoplus_{\rho\in R} \hat{f}(\rho) \in \bigoplus_{\rho\in R} M_{\dim\rho}(\mathbb{C})$$

of Fourier transforms of f at the representations in R.

Our main interest will be in the case in which $A = \mathbb{C}[G]$, the group algebra for G a finite group. The group algebra $\mathbb{C}[G]$ is the space of all formal complex linear combinations of group elements under the product

$$\left(\sum_{s\in G} f(s)s\right)\left(\sum_{t\in G} h(t)t\right) = \sum_{s,t\in G} f(s)h(t)st.$$

The Fourier transform of $f = \sum_{s\in G} f(s)s$ at a matrix representation ρ of $\mathbb{C}[G]$ is

$$\sum_{s\in G} f(s)\rho(s).$$

This is equivalent to the d_ρ^2 individual Fourier transforms at the corresponding matrix elements

$$\hat{f}(\rho_{ij}) = \sum_{s\in G} f(s)\rho_{ij}(s).$$

When we compute the Fourier transform for a complete set of inequivalent irreducible representations R of G we refer to the calculation as the **computation of a Fourier transform on** G (with respect to R).

2. Complexity

Given a set of matrix representations $\rho_1, \ldots, \rho_N$ of a group G of dimensions $d_1, \ldots, d_N$ respectively, a direct computation of the Fourier transform would require at most $2|G|\sum_i d_i^2$ arithmetic operations (where an arithmetic operation is a complex addition or multiplication). **Fast Fourier transforms** (FFTs) are algorithms for computing Fourier transforms that improve on this upper bound. A priori, the number of operations needed to compute the Fourier transform may depend on the specific representations used.

Definition 2. *Let G be a finite group, R any set of matrix representations of G.*

*(i) The **arithmetic complexity** of a Fourier transform on R, denoted $T_G(R)$, is the minimum number of arithmetic operations needed to compute the Fourier transform of f on R via a straight-line program for an arbitrary complex-valued function f defined on G.*

*(ii) The **reduced complexity**, denoted $t_G(R)$, is defined by*

$$t_G(R) = \frac{1}{|G|}T_G(R).$$

Rewriting the above, we have

$$|G| - 1 \le T_G(R) \le |G|^2.$$

3. Example: The Classical DFT and FFT

For $G = C_n$ a cyclic group, all irreducible representations are 1-dimensional. Let $\zeta_j(k) = e^{2\pi ijk/n}$. Then the set of representations $\{\zeta_j | \ 0 \le j \le n-1\}$ forms a complete set of inequivalent irreducible representations and the corresponding Fourier transform on C_n at $\zeta_j(k)$ is the usual discrete Fourier transform:

$$\sum_{k=0}^{n-1} f(k)e^{2\pi ijk/n}$$

Through the use of a combination of approaches, it is known that $T_{C_n} \le O(|C_n|\log|C_n|)$.

4. Factoring Through Subgroup Chains; Schur's Lemma

We compute the Fourier transform efficiently by taking advantage of subalgebra structure: consider a chain of subalgebras:

$$\mathbb{C}[G] = \mathbb{C}[G_n] \hookleftarrow \mathbb{C}[G_{n-1}] \hookleftarrow \cdots \hookleftarrow \mathbb{C}[G_1] = \mathbb{C}$$

The idea of using the coset decompositions of elements in the group to relate a Fourier transform on G to Fourier transforms on a subgroup H generalizes naturally to the group algebra. To explain: let H be a subgroup of G and let $Y \subset G$ be a set of coset representatives for G/H. Thus $G = \sqcup_{y\in Y} yH$. Then

$$\hat{f}(\rho) = \sum_{s\in G} f(s)\rho(s) = \sum_{y\in Y}\rho(y)\sum_{h\in H} f_y(h)\rho(h),$$

and so becomes a sum of Fourier transforms on H.

We also use Schur's lemma to find sparse structure in certain matrix representations:

Schur's Lemma 1. *Let K be a subgroup of G and ρ a K-adapted representation of G such that $\rho = \eta_1 \oplus\cdots\oplus \eta_1 \oplus\cdots\oplus \eta_r \oplus\cdots\oplus \eta_r$, where n_i the inequivalent irreducible representations of K and η_i occurs with multiplicity m_i. Then the centralizer of $\rho(K)$ is*

$$(M_{m_1}(\mathbb{C}) \otimes I_{d_{\eta_1}}) \oplus \cdots \oplus (M_{m_r}(\mathbb{C}) \otimes I_{d_{\eta_r}}).$$

We take advantage of both subgroup and centralizer algebra structure to compute the Fourier transform efficiently.

5. Bratteli Diagrams and the Path Algebra

We provide an isomorphism between the chain of group algebras and the chain of path algebras associated to the Bratelli diagram $\mathcal{B}$ of the group algebra chain:

$$\mathbb{C}[\mathcal{B}] = \mathbb{C}[\mathcal{B}_n] \hookleftarrow \mathbb{C}[\mathcal{B}_{n-1}] \hookleftarrow \cdots \hookleftarrow \mathbb{C}[\mathcal{B}_1] = \mathbb{C},$$

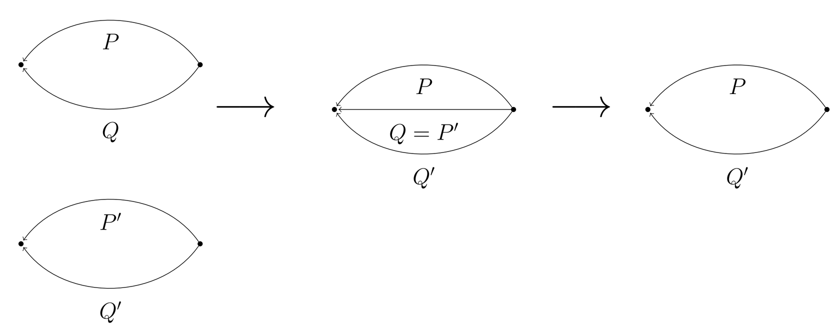
where $\mathbb{C}[\mathcal{B}_i]$ is is the $\mathbb{C}$-vector space with basis given by pairs of paths of length i in $\mathcal{B}$ which start at the root of $\mathcal{B}$ and end at the same vertex in level i of $\mathcal{B}$.

The path algebra $\mathbb{C}[\mathcal{B}_i]$ is an algebra under the multiplication

$$\sum_{(P,Q)} a_{PQ}(P,Q) * \sum_{(P',Q')} b_{P'Q'}(P',Q') = \sum\left(\sum_Q a_{PQ}b_{QQ'}\right)(P,Q'),$$

i.e., linearly extending the multiplication $(P,Q)*(P',Q') = \delta_{QP'}(P,Q')$.

Pictorially, the multiplication corresponds to:

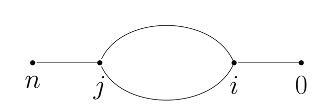


Further, $\mathbb{C}[\mathcal{B}_i]$ injects into $\mathbb{C}[\mathcal{B}_{i+1}]$ by mapping any pair of paths $(P,Q) \in \mathbb{C}[\mathcal{B}_i]$ to the sum $\sum_f (f\circ P, f\circ Q)$, where the sum is over all arrows f such that the tail of f is the head of P (equivalently, of Q), and $\circ$ denotes concatenation of paths.

6. Translating to Quiver Operations

We use the isomorphism with the path algebra to represent multiplication of elements in the algebra chain as gluing and summing operations on quivers. After factoring each element in the algebra through the algebra chain, we have products of elements in subchains. Each element, h, in these products then corresponds to a quiver $Q_{j,i}$ for j the index of the smallest subalgebra $\mathbb{C}[G_j]$ containing h, and i the index of the largest subalgebra $\mathbb{C}[G_i]$ centralizing h:

$Q_{j,i}$



We 'glue together' the quivers Q_{ji} for each element in the product by translating multiplication in the path algebra into operations on quivers. The complexity of the Fourier transform is reduced to counting morphisms of quivers into the Bratteli diagram of the algebra chain.

7. Example: the Symmetric Group

A complete set of coset representatives for S_n/S_{n-1} is

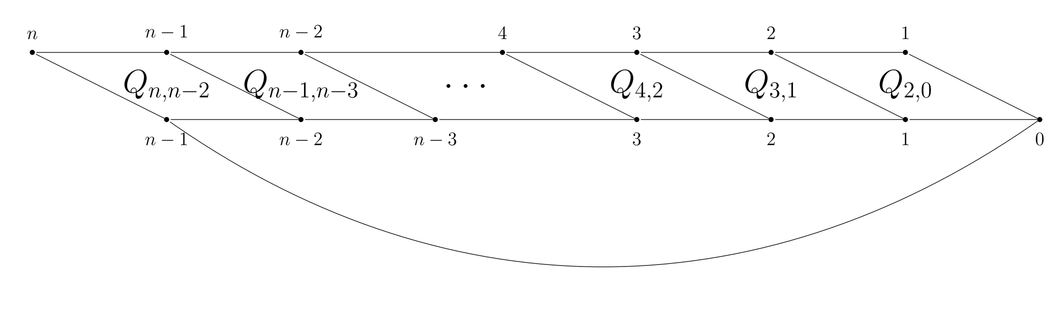
$$\{t_2\cdots t_n, t_3\cdots t_n, \ldots, t_n, e\},$$

where t_i is the transposition $(i-1\ i)$. Then the Fourier transform of f on S_n reduces to computing sums of the form

$$\sum_{i=1}^{n} t_{i+1}\cdots t_n F_i,$$

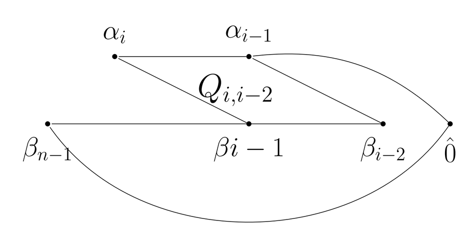
for F_i an element of $\mathbb{C}[S_{n-1}]$. Note that each t_i lies in $\mathbb{C}[S_i]$ and the centralizer of $\mathbb{C}[S_{i-2}]$. Thus each t_i corresponds to the quiver $Q_{i,i-2}$, and gluing together we build the quiver G below.

G



The complexity of the Fourier transform then comes down to counting the number of morphisms of the quivers G_i below into the Bratteli diagram for the symmetric group (ie, Young's diagram).

G_i



Combinatorial results give:

$$t_{S_n}(R) \le \frac{3n(n-1)}{4}$$

References

[1] D. Maslen, D. Rockmore, and S. Wolff, *Separation of Variables and the Computation of Fourier Transforms on Finite Groups*, in preparation.