Separation of Variables

—— *New Trends* ——

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The review is based on the author's papers since 1985 in which a new approach to the separation of variables (SoV) has being developed. It is argued that SoV, understood generally enough, could be the most universal tool to solve integrable models of the classical and quantum mechanics. It is shown that the standard construction of the action-angle variables from the poles of the Baker-Akhiezer function can be interpreted as a variant of SoV, and moreover, for many particular models it has a direct quantum counterpart. The list of the models discussed includes XXX and XYZ magnets, Gaudin model, Nonlinear Schrödinger equation, $SL(3)$-invariant magnetic chain. New results for the 3-particle quantum Calogero-Moser system are reported.

§ 1. Introduction

The separation of variables (SoV), at least, in its most elementary forms such as SoV in cartesian, spherical or ellipsoidal coordinates, is an indispensable part of the basic mathematical/physical curriculum. Briefly, SoV can be characterized as a reduction of a multidimensional problem to a set of one-dimensional ones. Originated from the works of D'Alembert and Fourier (wave theory) and Jacobi (Hamiltonian mechanics), the SoV for the long time was the only known method of "exact" solution of problems of mathematical physics. However, in the last decades the new techniques, including Inverse Scattering Method (ISM) as well as its quantum version (QISM) together with Bethe Ansatz, seemed to oust the SoV out of fashion.

The aim of the present review is to draw attention to the recent progress in understanding SoV and its relations to ISM and QISM. I am going to argue that SoV is far yet from being outdated and, even more, has good chances to remain as the most universal method of solving completely integrable (classical and quantum) models. There are two basic observations which give support to that claim. First: for the classical integrable systems subject to ISM the standard construction of the action-angle variables using the poles of the Baker-Akhiezer function is in fact equivalent to a separation of variables. And second: in many cases it is possible to find the precise quantum analog of this construction.

This point of view has being gradually clarified since my first publications[1),2)] of 1985 which were deeply influenced by I. V. Komarov (St. Petersburg State University) who used to insist that SoV and ISM are closely related and to underline that the notion of SoV should not be restricted to the purely coordinate change of variables as it was frequently done. Generally speaking, SoV can be produced by a complicated canonical transformation involving both coordinates and momenta. In the quantum case the canonical transformation should be replaced with a unitary operator. The papers of Gutzwiller on 3, 4-particle Toda lattice[3] and of Komarov on Goryachev-Chaplygin top[4] had provided the first examples of how such a unitary operator could be guessed using known classical SoV. In Refs. 1) and 2) the results of Komarov and Gutzwiller were reproduced using the machinery of QISM.[5] The algebraic techniques of QISM (R-matrix method) opened the way to methodical construction of quantum SoV for the whole classes of integrable models generated by different R-matrices (solutions to the Yang-Baxter equation). Though the realization of this program is far yet from completion, the list of the models, to which the new approach has been applied successfully, grows steadily and presently includes:

- XXX magnetic chain[6),7)]
- XXX Gaudin model,[8)] see also Refs. 9)~16).
- XYZ magnetic chain (classical case)[17)]
- Goryachev-Chaplygin top,[4),1)] see also Ref. 18).
- Toda lattice,[2)] including relativistic case[19)] and boundary conditions;[20)] see also Ref. 21).
- Nonlinear Schrödinger equation (infinite volume)[22)]
- sinh-Gordon model (infinite volume)[23)]
- $SL(3)$ magnetic chain[24),25)]
- 3-particle Calogero-Moser model, see § 7 of the present paper.

The early stages of the work were summarized in the reviews,[6),7)] where the term "Functional Bethe Ansatz" was used instead SoV, the latter one seeming now to be the more accurate. I tried to avoid in the present review the detailed discussions of particular models which one can find in Refs. 6) and 7) but, instead, to summarize the development of the field using the unifying concept of the Baker-Akhiezer function. The importance of choosing a proper normalization of B-A function is stressed. More attention is paid to the particular models which were not discussed in the previous reviews (Nonlinear Schrödinger equation in the infinite volume, classical XYZ magnet). New results are reported concerning SoV in the quantum 3-particle Calogero-Moser model (elliptic and trigonometric).

§ 2. SoV: general notions

2.1. *Basic definitions*

Let us start with the classical case. Consider a Hamiltonian mechanical system having a finite number D of degrees of freedom and integrable in Liouville's sense. It means[26)] that one is given a $2D$-dimensional symplectic manifold (phase space) and D independent hamiltonians H_j commuting with respect to the Poisson bracket

$$\{H_j, H_k\}=0, \qquad j, k=1, \cdots, D. \tag{2·1}$$

A system of canonical variables $(\boldsymbol{p}, \boldsymbol{x})$

$$\{x_j, x_k\}=\{p_j, p_k\}=0\,, \qquad \{p_j, x_k\}=\delta_{jk} \tag{2·2}$$

will be called *separated* if there exist D relations of the form

$$\Phi_j(x_j, p_j, H_1, H_2, \cdots, H_D)=0\,, \quad j=1, 2, \cdots, D \tag{2·3}$$

binding together each pair (p_j, x_j) and the hamiltonians H_n. Note that fixing the values of hamiltonians $H_n=h_n$ one obtains from (2·3) an explicit factorization of the Liouville tori into the one-dimensional ovals given by equations

$$\Phi_j(x_j, p_j, h_1, h_2, \cdots, h_D)=0\,, \qquad j=1, 2, \cdots, D\,. \tag{2·4}$$

The *action* function $S(\boldsymbol{h}, \boldsymbol{x})$ is defined[26] as the generating function of the canonical transformation from $(\boldsymbol{p}, \boldsymbol{x})$ to the *action-angle* variables $(\boldsymbol{I}, \boldsymbol{\varphi})$ that is

$$\boldsymbol{h}=\boldsymbol{h}(\boldsymbol{I})\,, \qquad p_j=\frac{\partial S(\boldsymbol{h}(\boldsymbol{I}), \boldsymbol{x})}{\partial x_j}\,, \qquad \varphi_j=\frac{\partial S(\boldsymbol{h}(\boldsymbol{I}), \boldsymbol{x})}{\partial I_j} \tag{2·5}$$

and satisfies the Hamilton-Jacobi equation

$$H_j\left(\frac{\partial S(\boldsymbol{h}(\boldsymbol{I}), \boldsymbol{x})}{\partial \boldsymbol{x}}, \boldsymbol{x}\right)=h_j(\boldsymbol{I}) \tag{2·6}$$

for each H_j considered as a function of $(\boldsymbol{p}, \boldsymbol{x})$. The relations (2·4) allow immediately to split the complete solution to *partial differential equation* (2·6) into the sum of terms

$$S(\boldsymbol{h}, x_1, \cdots, x_D)=S_1(\boldsymbol{h}, x_1)+\cdots+S_D(\boldsymbol{h}, x_D) \tag{2·7}$$

satisfying each the *ordinary differential equation*

$$\Phi_j(x_j, \partial_{x_j}S_j, h_1, h_2, \cdots, h_D)=0 \tag{2·8}$$

(we consider $\boldsymbol{h}$ in $S_j(x_j)\equiv S_j(\boldsymbol{h}, x_j)$ as fixed parameters).

The last result justifies apparently the term "separation of variables". It is important to warn the reader that many authors restrict the notion of SoV only with the situations when the separated variables $(\boldsymbol{p}, \boldsymbol{x})$ are obtained from some original canonical variables, say $(\boldsymbol{P}, \boldsymbol{Q})$, by purely *coordinate* transform

$$x_j=x_j(\boldsymbol{Q})\,, \qquad p_j=\sum_k \frac{\partial Q_k}{\partial x_j} P_k\,. \tag{2·9}$$

That condition, though reasonable for a certain class of problems, would be, however, too restrictive for our purposes excluding almost all the examples we are going to consider.

Let us examine now the quantum case. The condition (2·1) is replaced by the commutativity of quantum operators

$$[H_j, H_k]=0\,, \qquad j, k=1, \cdots, D \tag{2·10}$$

the relations (2·3) retaining the same form

$$\Phi_j(x_j, p_j, H_1, H_2, \cdots, H_D)=0 \tag{2·11}$$

with the only difference that now we have to fix some ordering of the noncommutative operators x_j, p_j, H_n. Let us assume that the operators in (2·11) are ordered exactly as they are enlisted that is x always precedes p and p precedes H (the relative ordering of H_n is of no importance since they are commutative (2·10)).

It is convenient to work in the x-representation that is to realize the quantum states as the functions $\Psi(\boldsymbol{x})$ from some Hilbert space $\mathcal{H}$. The canonical operators x_j and p_j

$$[x_j, x_k]=[p_j, p_k]=0\,, \quad [p_j, x_k]=-i\hbar\delta_{jk} \tag{2·12}$$

can be realized respectively as the multiplication and differentiation $p_j=-i\hbar\partial_{x_j}$ operators.

Let Ψ be a common eigenfunction

$$H_j\Psi=h_j\Psi \tag{2·13}$$

of the commuting hamiltonians. Then, under assumptions made about the operator ordering and the realization of $(\boldsymbol{p}, \boldsymbol{x})$ it follows from (2·11) that $\Psi(\boldsymbol{x})$ satisfies the equations

$$\Phi_j(x_j, -i\hbar\partial_{x_j}, h_1, h_2, \cdots, h_D)\Psi=0\,, \tag{2·14}$$

which suggests immediately the factorization of Ψ

$$\Psi(x_1, \cdots, x_D)=\prod_{j=1}^{D}\psi_j(x_j) \tag{2·15}$$

into functions $\psi_j(x_j)$ of one variable only satisfying the equation

$$\Phi_j(x_j, -i\hbar\partial_{x_j}, h_1, h_2, \cdots, h_D)\psi_j(x_j)=0\,. \tag{2·16}$$

In complete analogy with the classical case, the original multidimensional spectral problem (2·13) is reduced to the set of the one-dimensional multiparameter spectral problems (2·16). For determining the admissible values of D spectral parameters h_j one has thus the system of D equations (2·16) supplememented with appropriate boundary conditions determined by the Hilbert space $\mathcal{H}$. The function Φ_j in (2·14) can be thought of as a symbol of a pseudodifferential operator. In particular, (2·16) becomes an ordinary differential equation if Φ_j is a polynomial in p or a finite-difference equation if Φ_j is a trigonometric polynomial in p.

Semiclassically,

$$\Psi(\boldsymbol{x})=e^{(i/\hbar)S(\boldsymbol{x})}\,, \qquad \psi_j(x_j)=e^{(i/\hbar)S_j(x_j)}\,,$$

which provides the correspondence of the formulas (2·7) and (2·15).

2.2. *Magic recipe*

Let us return now to the classical case and discuss the question how to find a SoV for a given integrable system. In the XIXth and the beginning of the present century for a number of models of classical mechanics, such as Neumann model or various cases of the rigid body motion, the SoV was found by guess or some more or less *ad hoc* methods.

Here we shall discuss the latest and seemingly the most powerful method based on the Baker-Akhiezer function. Suppose, as it is always done in the Inverse Scattering Method (ISM), that our commutative hamiltonians H_j can be obtained as the spectral invariants of some matrix $L(u)$ of dimensions $N\times N$, called L or *Lax operator*, whose elements are functions on the phase space and depend also on an additional parameter u called *spectral parameter*. It means that H_j can be expressed in terms of the coefficients $t_n(u)$ of the *characteristic polynomial* $W(z, u)$ of the matrix $L(u)$

$$W(z, u)=\det(z-L(u))=\sum_{n=0}^{N}(-1)^n t_n(u)z^{N-n},$$

$$t_0(u)=1, \qquad t_n(u)=\mathrm{tr}\overset{n}{\wedge}L(u), \qquad t_N(u)=\det L(u). \tag{2·17}$$

The *characteristic equation*

$$W(z, u)=0 \tag{2·18}$$

defines the eigenvalue $z(u)$ of $L(u)$ as a function on the corresponding N-sheeted Riemannian surface of parameter u. The *Baker-Akhiezer function* $\Omega(u)$ is defined then as the eigenvector of $L(u)$

$$L(u)\Omega(u)=z(u)\Omega(u) \tag{2·19}$$

corresponding to the eigenvalue $z(u)$.

Since an eigenvector is defined up to a scalar factor, to exclude the ambiguity in the definition of $\Omega(u)$ one has to *fix a normalization* of $\Omega(u)$ imposing a linear constraint

$$\sum_{n=1}^{N}\alpha_n(u)\Omega_n(u)=1, \tag{2·20}$$

where $\alpha_n(u)$ may, generally speaking, depend also on the dynamical variables. A normalization being fixed, $\Omega(u)$ becomes a meromorphic function on the Riemannian surface (2·18).

The poles x_j of the Baker-Akhiezer function play an important role in ISM.[27] In particular, the time evolution of x_j for the hamiltonian flow with any of the commuting hamiltonians H_n can be expressed explicitly in terms of the Riemannian theta-functions corresponding to the spectral curve (2·18). Moreover, it was observed that for many models the variables x_j Poisson commute and, together with the corresponding eigenvalues $z_j\equiv z(x_j)$ of $L(x_j)$, or some functions p_j of z_j, provide a set of separated canonical variables for the Hamiltonians H_n. It should be mentioned that, though the seminal papers[28] contain all the necessary results, the possibility of their interpretation in terms of SoV was not recognized at that time.

One of the reasons why the poles of $\Omega(u)$ could provide a SoV is easy to understand. Since $z_j=z(x_j)$ is an eigenvalue of $L(x_j)$ the pair (z_j, x_j) should lie on the spectral curve (2·18)

$$W(z_j, x_j)=0. \tag{2·21}$$

It remains to observe that, if z_j is a function of p_j, Eq. (2·21) fits exactly the form (2·3) since the coefficients $t_n(x_j)$ of the characteristic polynomial (2·17) contain nothing except x_j and the hamiltonians H_k.

However, the relations (2·3) alone are not enough to produce SoV. It is necessary that, in addition, the number of the poles x_j be exactly the number of degrees of freedom D and that the variables $(p_j(z_j), x_j)$ be canonical (2·2). Those properties are by no means obvious and the last one is rather difficult to verify. To perform the calculation, it is necessary to transform the above definitions of (z_j, x_j) into a more convenient form. Let $\Omega^{(j)}=\mathrm{res}_{u=x_j}\Omega(u)$. From (2·19) and (2·20) there follow, respectively, the eigenvalue equation and the normalization condition for $\Omega^{(j)}$

$$L(x_j)\Omega^{(j)}=z_j\Omega^{(j)}\,, \qquad \sum_{n=1}^{N}\alpha_n(x_j)\Omega_n^{(j)}=0\,. \tag{2·22}$$

Let $\boldsymbol{\alpha}(u)$ denote the one-row matrix $(\alpha_1(u),\cdots,\alpha_N(u))$. The existence of a non-zero solution $\Omega^{(j)}$ to the problem (2·22) is equivalent to the condition

$$\mathrm{rank}\begin{pmatrix}\boldsymbol{\alpha}(x_j)\\ L(x_j)-z_j\mathbf{1}\end{pmatrix}=N-1 \tag{2·23}$$

which, in turn, can be expressed generically as vanishing of any of two minors of order N, for instance

$$\begin{vmatrix} \alpha_1(x) & \alpha_2(x) & \alpha_3(x) & \cdots & \alpha_N(x)\\ L_{21}(x) & L_{22}(x)-z & L_{23}(x) & \cdots & L_{2N}(x)\\ L_{31}(x) & L_{32}(x) & L_{33}(x)-z & \cdots & L_{3N}(x)\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ L_{N1}(x) & L_{N2}(x) & L_{N3}(x) & \cdots & L_{NN}(x)-z \end{vmatrix}=0\,, \tag{2·24a}$$

$$\begin{vmatrix} \alpha_1(x) & \alpha_2(x) & \alpha_3(x) & \cdots & \alpha_N(x)\\ L_{11}(x)-z & L_{12}(x) & L_{13}(x) & \cdots & L_{1N}(x)\\ L_{31}(x) & L_{32}(x) & L_{33}(x)-z & \cdots & L_{3N}(x)\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ L_{N1}(x) & L_{N2}(x) & L_{N3}(x) & \cdots & L_{NN}(x)-z \end{vmatrix}=0\,. \tag{2·24b}$$

The pairs (z_j, x_j) are obtained then as the roots of the system of Eqs. (2·24) which allows, in principle, to count them and to calculate the Poisson brackets between them.

In the sections devoted to the nonlinear Schrödinger equation and XYZ model we shall need the formulas for the $N=2$ case. Equations (2·24) become then quite simple:

$$\begin{vmatrix} \alpha_1(x) & \alpha_2(x)\\ L_{21}(x) & L_{22}(x)-z \end{vmatrix}=0\,, \qquad \begin{vmatrix} \alpha_1(x) & \alpha_2(x)\\ L_{11}(x)-z & L_{12}(x) \end{vmatrix}=0\,. \tag{2·25}$$

Eliminating z one obtains the equation for x_j

$$B(x_j)=0\,, \qquad B=\alpha_1^2L_{12}-\alpha_1\alpha_2(L_{11}-L_{22})-\alpha_2^2L_{21}\,. \tag{2·26}$$

The eigenvalue z_j is obtained then from

$$z_j=\left(L_{11}-\frac{\alpha_1}{\alpha_2}L_{12}\right)_{u=x_j}=\left(L_{22}-\frac{\alpha_2}{\alpha_1}L_{21}\right)_{u=x_j} \qquad (2\cdot27)$$

(for the sake of brevity we have omitted the argument u in $L(u)$ and $\alpha(u)$ in the last two formulas).

Generally speaking, there is no guarantee that one obtains from (2·24) the canonical P.b. (2·2) for some $p_j(z_j)$. Amazingly, it turns out to be true for a fairly large class of integrable models, though the fundamental reasons responsible for such effectiveness of the magic recipe: "*Take the poles of the properly normalized Baker-Akhiezer function and the corresponding eigenvalues of the Lax operator and you obtain a SoV*", are still unclear. The key words in the above recipe are "the properly normalized". The choice of the proper normalization $\boldsymbol{\alpha}(u)$ of $\Omega(u)$ can be quite nontrivial (see below the discussion of the XYZ magnet) and for some integrable models the problem remains unsolved.[29]

2.3. *r-matrix formalism*

Given a particular L operator and a normalization of Ω, one is able, in principle, to calculate from (2·24) the Poisson brackets for (z_j, x_j) though it could be a formidable task. There are, however, techniques which simplify the calculation and allow to verify SoV for whole families of L operators instead of handling them individually.

According to a remarkable theorem proved by Babelon and Viallet,[30] the commutativity of the spectral invariants $t_n(u)$ (2·17) of the matrix $L(u)$

$$\{t_m(u), t_n(v)\}=0 \qquad (2\cdot28)$$

is equivalent to the existence of a matrix $r_{12}(u, v)$ of order $N^2\times N^2$ such that the Poisson brackets between the components of L are represented in the commutator form

$$\{\overset{1}{L}(u), \overset{2}{L}(v)\}=[r_{12}(u, v), \overset{1}{L}(u)]-[r_{21}(v, u), \overset{2}{L}(v)], \qquad (2\cdot29)$$

where the standard notation is introduced:[7] $L^1\equiv L\otimes\mathbf{1}$, $L^2\equiv\mathbf{1}\otimes L$, $r_{21}(u, v)=\mathcal{P}r_{12}(u, v)\mathcal{P}$, and $\mathcal{P}$ is the permutation operator: $\mathcal{P}x\otimes y=y\otimes x$, $\forall x, y\in \boldsymbol{C}^N$.

Generally speaking, the matrix $r_{12}(u, v)$ is a function of dynamical variables. So far, very little is known of such r-matrices, apart of few particular examples.[31)~34)] The best studied one is the case of purely numeric (c-number) r-matrices satisfying the *classical Yang-Baxter equation*

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)+r_{23}(u_2, u_3)]-[r_{13}(u_1, u_3), r_{32}(u_3, u_2)]=0\,, \qquad (2\cdot30)$$

which ensures the Jacobi identity for the Poisson bracket (2·29), and especially, the case of *unitary* numeric r-matrices, satisfying, in addition, the relation

$$r_{12}(u_1, u_2)=-r_{21}(u_2, u_1) \qquad (2\cdot31)$$

and depending on the difference u_1-u_2. For such r-matrices the relation (2·29) takes the form

$$\{\overset{1}{L}(u), \overset{2}{L}(v)\} = [r_{12}(u-v), \overset{1}{L}(u) + \overset{2}{L}(v)] \tag{2·32}$$

and (2·30), respectively,

$$[r_{12}(u), r_{13}(u+v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0 . \tag{2·33}$$

To a unitary numeric r-matrix one can associate not only the Poisson algebra (2·32) whose right-hand side is linear in L but also the algebra

$$\{\overset{1}{L}(u), \overset{2}{L}(v)\} = [r_{12}(u-v), \overset{1}{L}(u)\overset{2}{L}(v)] \tag{2·34}$$

with the quadratic r.h.s. Formally, (2·34) can be put into the form (2·29) with the *dynamic* r-matrix $\tilde{r}_{12}(u, v) = (1/2)(r_{12}(u-v)\overset{2}{L}(v) + \overset{2}{L}(v)r_{12}(u-v))$, $\tilde{r}_{21}(u, v) = -(1/2)(r_{12}(u-v)\overset{1}{L}(u) + \overset{1}{L}(u)r_{12}(u-v))$, but the very special structure of $\tilde{r}_{12}$ allows to consider the formula (2·34) rather as a modification of (2·32). Obviously, (2·32) can be obtained from (2·34) if one substitutes $L := 1 + \varepsilon L + O(\varepsilon^2)$, $r := \varepsilon r$ and let $\varepsilon \to 0$.

Another example of the quadratic P.b. algebra associated to a unitary numeric r-matrix is the algebra[35)]

$$\{\overset{1}{L}(u), \overset{2}{L}(v)\} = [r_{12}(u-v), \overset{1}{L}(u)\overset{2}{L}(v)] + \overset{1}{L}(u)r_{12}(u+v)\overset{2}{L}(v) - \overset{2}{L}(v)r_{12}(u+v)\overset{1}{L}(u) . \tag{2·35}$$

There exists a profound algebraic theory of the unitary numeric r-matrices[36)] which allows to classify r-matrices in families labelled by Lie algebras. A particularly important example is given for any semisimple Lie algebra $\mathcal{G}$ by the formula

$$r_{12}(u) = \frac{\rho}{u}\sum_{\alpha} I_\alpha \otimes I_\alpha , \tag{2·36}$$

where ρ is a numeric constant and $I_\alpha \in \mathcal{G}$ is an orthonormal basis with respect to the Killing form. The result does not depend on the choice of the basis. Taking then various finite-dimensional representations of $\mathcal{G}$ for the generators I_α one can obtain from (2·36) the family of r-matrices related to $\mathcal{G}$.

In particular, for $\mathcal{G} = gl(N)$ and the fundamental vector representation, one has

$$r_{12}(u) = \frac{\rho}{u}\mathcal{P} . \tag{2·37}$$

The last example deserves special attention since, so far, it is the only series of r-matrices for which a general SoV construction is obtained.

§ 3. *GL*(*N*)-type models

3.1. *Classical case*

It turns out that in case of the $GL(N)$-invariant r-matrix (2·37) the normalization of the Baker-Akhiezer function $\Omega(u)$ corresponding to any constant numeric vector $\boldsymbol{\alpha}$ in (2·20) produces SoV. The simplest choice of $\boldsymbol{\alpha}$ is

$$a_1(u)=\cdots=a_{N-1}(u)=0\,,\qquad a_N=1 \Longleftrightarrow \Omega_N(u)=1\,,\qquad \Omega_j=0\,. \tag{3·1}$$

The corresponding separated coordinates x_j are defined as the poles of $\Omega(u)$, and the canonically conjugated momenta (2·12) are $p_j=-\rho^{-1}z_j$ for the linear P.b. (2·32) and $p_j=-\rho^{-1}\ln z_j$ for the quadratic P.b. (2·34). For the linear P.b. case the above results were obtained in Ref. 37). The quadratic P.b. case was studied in Refs. 1) and 24) for $N=2, 3$ and generalized to arbitrary N in Ref. 38).

In case of the normalization (3·1), Eqs. (2·24) for (z, x) simplify a bit

$$\begin{vmatrix} L_{21}(x) & L_{22}(x)-z & L_{23}(x) & \cdots & L_{2,N-1}(x) \\ L_{31}(x) & L_{32}(x) & L_{33}(x)-z & \cdots & L_{3,N-1}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{N-1,1}(x) & L_{N-1,2}(x) & L_{N-1,3}(x) & \cdots & L_{N-1,N-1}(x)-z \\ L_{N1}(x) & L_{N2}(x) & L_{N3}(x) & \cdots & L_{N,N-1}(x) \end{vmatrix}=0\,, \tag{3·2a}$$

$$\begin{vmatrix} L_{11}(x)-z & L_{12}(x) & L_{13}(x) & \cdots & L_{1,N-1}(x) \\ L_{31}(x) & L_{32}(x) & L_{33}(x)-z & \cdots & L_{3,N-1}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{N-1,1}(x) & L_{N-1,2}(x) & L_{N-1,3}(x) & \cdots & L_{N-1,N-1}(x)-z \\ L_{N1}(x) & L_{N2}(x) & L_{N3}(x) & \cdots & L_{N,N-1}(x) \end{vmatrix}=0\,. \tag{3·2b}$$

Equations (3·2) themselves can be used already for calculation of P.b. between x_j and z_j, see Refs. 37) and 38). However, they are not convenient for quantization because of the operator ordering problem. So, we take one more step and eliminate z_j from (3·2). The result is one equation

$$B(x_j)=0 \tag{3·3}$$

for x_j where $B(u)$ is a certain polynomial of degree $N(N-1)/2$ in components of $L(u)$. The corresponding eigenvalue z_j is obtained then as the value

$$z_j=A(x_j) \tag{3·4}$$

of certain function $A(u)$ expressed rationally in components of $L(u)$.

For instance, for $N=2$

$$B(u)=L_{21}(u)\,,\qquad A(u)=L_{11}(u)\,. \tag{3·5}$$

Note that for $u=x_j$ the matrix $L(u)$ becomes triangular

$$L(x_j)=\begin{pmatrix} z_j & L_{12}(x_j) \\ 0 & L_{22}(x_j) \end{pmatrix}, \tag{3·6}$$

which explains why its eigenvalue z_j is given by $A(x_j)$.

For $N=3$ one obtains

$$B(u)=L_{31}(u)\begin{vmatrix} L_{11} & L_{12} \\ L_{31} & L_{32} \end{vmatrix}(u)+L_{32}(u)\begin{vmatrix} L_{21} & L_{22} \\ L_{31} & L_{32} \end{vmatrix}(u)\,. \tag{3·7}$$

There are two possible ways to choose $A(u)$

$$A_1(u)=\frac{\begin{vmatrix}L_{11} & L_{12}\\ L_{31} & L_{32}\end{vmatrix}(u)}{L_{32}(u)}, \qquad A_2(u)=-\frac{\begin{vmatrix}L_{21} & L_{22}\\ L_{31} & L_{32}\end{vmatrix}(u)}{L_{31}(u)}, \tag{3·8}$$

which are equivalent modulo $B(x_j)=0 \iff A_1(x_j)=A_2(x_j)$

$$z_j=A_1(x_j)=A_2(x_j)\,. \tag{3·9}$$

The expressions for $A(u)$ and $B(u)$ for the general N are given in Ref. 38). We have to warn the reader that due to a different choice of normalization of $\Omega(u)$ the formulas (3·5), (3·7), (3·8) differ from those in Refs. 7), 25) and 38).

Equations (3·3) and (3·4), like (3·2), can also be used for calculating the P.b. between x_j and z_j. Their advantage for the sake of quantization is that $B(u)$ Poisson commute

$$\{B(u), B(v)\}=0\,, \tag{3·10}$$

which entrains immediately the commutativity of x_j (see the next subsection).

The correct P.b. between x_j and z_j are not enough to establish SoV. The last condition (usually, easy to verify) is the correct number of variables x_j which should be equal to the number D of degrees of freedom. In some degenerate cases, the number of x_j could be less than D and some additional variables should be added (see example of Calogero-Moser model in § 7).

To conclude this subsection, let us stress that so far no generalization is known of the above results to the r-matrices corresponding to the Lie algebras other than the A_n series. The difficulty is that the simplest normalization (3·1) does not work more: the function $B(u)$ has too many zeroes (more than the number D of degrees of freedom) and they do not commute.[29] Hopefully, some other normalization (2·20) will work which remains a challenging problem.

3.2. *Quantization*

In the quantum case the components L_{mn} of the $N\times N$ matrix $L(u)$ become quantum operators, and the Poisson brackets should be replaced by some commutation relations satisfying the correspondence principle $[\ ,\ ]=-i\hbar\{\ ,\ \}$. A nice feature of the Poisson algebras (2·32) and (2·34) is that (in contrast with the general case (2·29) of dynamical r-matrices) it is well known how to quantize them.

In the linear case (2·32) the quantization is straightforward

$$[\overset{1}{L}(u), \overset{2}{L}(v)]=[\mathfrak{r}_{12}(u-v), \overset{1}{L}(u)+\overset{2}{L}(v)]\,, \quad \mathfrak{r}(u)=-i\hbar r(u)\,, \tag{3·11}$$

in the quadratic case (2·34) it is more tricky. The algebra (2·34) is replaced by

$$R_{12}(u-v)\overset{1}{L}(u)\overset{2}{L}(v)=\overset{2}{L}(v)\overset{1}{L}(u)R_{12}(u-v)\,, \tag{3·12}$$

where $R(u)$ satisfies the *quantum Yang-Baxter equation*

$$R_{12}(u)R_{13}(u+v)R_{23}(v)=R_{23}(v)R_{13}(u+v)R_{12}(u) \tag{3·13}$$

and is related to $r(u)$ through the semiclassical expansion

$$R(u)=1+i\hbar r(u)+O(\hbar^2)\,. \tag{3·14}$$

In the $GL(N)$ case (2·37) the quantum $\mathfrak{r}$ and R matrices are

$$\mathfrak{r}(u)=-\eta\frac{\mathcal{P}}{u}\,,\quad R(u)=1+\eta\frac{\mathcal{P}}{u}\,,\quad \eta=i\hbar\rho\,. \tag{3·15}$$

The relations (3·12) define the associative algebra $\mathcal{Y}[gl(N)]$ called yangian of $gl(N)$.

The quantum integrals of motions $t_n(u)$ for the yangian are obtained by appropriate deformation of the classical formulas (2·17)

$$t_n(u)=\mathrm{tr}L(u)\wedge L(u+\eta)\wedge\cdots\wedge L(u+(n-1)\eta)\,, \tag{3·16}$$

$$[t_m(u),\,t_n(u)]=0\,. \tag{3·17}$$

The quantity $t_N(u)$ (*the quantum determinant* of $L(u)$) produces central elements

$$[t_N(u),\,L(v)]=0 \tag{3·18}$$

of the yangian. Naturally, on the irreducible representations $t_N(u)$ is a number-valued function.

The quantum analog of the construction of the SoV based on Eqs. (3·3) and (3·4) is found presently only for $N=2$ and $N=3$. The formulas given below are taken from Refs. 7), 25) up to small variations due to the different choice of normalization of $\Omega(u)$. In the $GL(2)$ case the operator-valued functions $A(u)$ and $B(u)$ are defined by the same formulas (3·5) as in the classical case. By virtue of (3·12) the operator family $B(u)$ turns out to be commutative

$$[B(u),\,B(v)]=0\,, \tag{3·19}$$

which allows to define the operators x_n as the commuting "operator roots" of Eq. (3·3) (for the mathematical details see Ref. 7)). To give a sense to the formula (3·4) in the quantum case, it is necessary to fix the operator ordering. Assume that the x's in (3·4) are ordered to the left that is

$$A(x_j)=\sum_k x_j^k A_k \qquad \text{for} \qquad A(u)=\sum_k u^k A_k\,. \tag{3·20}$$

Then, using the commutation relations (3·12) it is possible to verify the relations

$$[x_j,\,x_k]=[z_j,\,z_k]=0\,, \qquad z_j x_k=(x_k+\eta\delta_{jk})z_j\,, \tag{3·21}$$

which suggest the realization of the operators z_j as the shift operators in an appropriate Hilbert space $\mathcal{H}\ni\Psi(\boldsymbol{x})$ of functions on the common spectrum of the operators x_j:

$$z_j\Psi(\boldsymbol{x})=\zeta(x_j)\Psi(\cdots,\,x_j+\eta,\,\cdots)\,. \tag{3·22}$$

The choice of the function $\zeta(x)$ in (3·22) is dictated by the properties of the Hilbert space $\mathcal{H}$ depending on the concrete model (see examples below). Note that there is certain liberty in choosing $\zeta(x_j)$ due to the canonical transformations

$$\Psi(\boldsymbol{x})\longrightarrow\prod_j\omega(x_j)\Psi(\boldsymbol{x}) \iff \zeta(x_j)\longrightarrow\frac{\omega(x_j+\eta)}{\omega(x_j)}\zeta(x_j)\,. \tag{3·23}$$

The SoV follows then from the relations

$$z_j^2 - z_j t_1(x_j) + t_2(x_j) = 0\,, \tag{3·24}$$

which generalize the classical characteristic equation (2·17) and fit the form (2·11) required for the quantum SoV (note the operator ordering!). Denoting by $\tau_n(u)$ the eigenvalues of the commuting operators $t_n(u)$ one obtains for the corresponding separated equation (2·16) the finite-difference equation of order 2:

$$\zeta(x_j+\eta)\zeta(x_j)\phi_j(x_j+2\eta) - \zeta(x_j)\tau_1(x_j+\eta)\phi_j(x_j+\eta) + \tau_2(x_j)\phi_j(x_j) = 0$$

or in more symmetric form

$$\tau_1(x_j)\phi_j(x_j) = \Delta_-(x_j)\phi_j(x_j-\eta) + \Delta_+(x_j)\phi_j(x_j+\eta) = 0\,, \tag{3·25}$$

where

$$\Delta_+(u) = \zeta(u)\,, \quad \Delta_-(u) = \frac{t_2(u-\eta)}{\zeta(u-\eta)}\,.$$

In the $GL(3)$ case the quantum $B(u)$ and $A(u)$ are obtained as deformations of the classical formulas (3·7) and (3·8):

$$B(u) = L_{31}(u-\eta)[L_{32}(u)L_{11}(u-\eta) - L_{31}(u)L_{12}(u-\eta)] + L_{32}(u-\eta)[L_{32}(u)L_{21}(u-\eta) - L_{31}(u)L_{22}(u-\eta)]\,, \tag{3·26}$$

$$A_1(u) = L_{32}^{-1}(u)[L_{32}(u)L_{11}(u-\eta) - L_{31}(u)L_{12}(u-\eta))]\,,$$

$$A_2(u) = -L_{31}^{-1}(u)[L_{32}(u)L_{21}(u-\eta) - L_{31}(u)L_{22}(u-\eta))]\,, \tag{3·27}$$

$$z_j = A_1(x_j) = A_2(x_j)\,. \tag{3·28}$$

The rest is similar to $GL(2)$ case. The quantum characteristic equation is

$$z_j^3 - z_j^2 t_1(x_j) + z_j t_2(x_j) - t_3(x_j) = 0\,, \tag{3·29}$$

and the separated equation is now a third-order finite-difference equation.[25)]

There is little doubt that the above constructions can be generalized to the arbitrary values of N, though the complicated structure of $B(u)$[38)] prevents rapid progress.

The analogous results for the linear commutation relations (3·11) can be obtained from the formulas for the quadratic relations (3·12) in the limit $L := 1 + \varepsilon L + O(\varepsilon^2)$, $\eta := \varepsilon\eta$, $\varepsilon \to 0$. However, though it is clear enough that the expansion in ε of the formulas (3·16) for the quantum integrals of motion for the yangian should produce, in principle, some commuting hamiltonians for the algebra (3·11), obtaining explicit formulas for them when N is arbitrary remains still unsolved problem, to say nothing about expressions for $A(u)$ and $B(u)$. In case of the r-matrix of the form (2·36) Feigin and Frenkel[39)] has proved for any Lie algebra $\mathcal{G}$ that the quantum commuting operators do exist which are deformations of the spectral invariants of the classical L matrix though their method of proof does not produce any effective formulas. Nevertheless, the simplest integrals of motion are easy to produce. Note that $t_1(u) = \mathrm{tr}L(u)$ is a trivial central element of the algebra (3·11), so it can be safely put to be 0, which corresponds to considering $sl(N)$ instead of $gl(N)$. The first nontrivial invariant is quadratic in $L(u)$ and coincides with the classical expression

$$\mathrm{t}_2(u)=\frac{1}{2}\mathrm{tr}L^2(u)\,, \qquad [\mathrm{t}_2(u), \mathrm{t}_2(v)]=0 \tag{3·30}$$

(for the general simple Lie algebra $\mathcal{G}$ one should use the corresponding Killing form).

The above quadratic invariant is enough to serve the $sl(2)$ case. The definitions (3·5) of $A(u)$ and $B(u)$ and, respectively, (3·3) and (3·4) of x_j and z_j remain the same as in the quadratic case. The commutation relations (3·21) and the quantum characteristic equation (3·24) are replaced, respectively, by

$$[x_j, x_k]=[z_j, z_k]=0\,, \qquad [z_j, x_k]=z_j\delta_{jk} \tag{3·31}$$

and

$$z_j^2-\mathrm{t}_2(x_j)=0 \tag{3·32}$$

(here we put $\eta=1$ for simplicity). Using then the realization

$$z_j=\partial_{x_j}+\zeta(x_j) \tag{3·33}$$

for z_j one obtains for the separated spectral problem the second order differential equation[8)]

$$\psi''+2\zeta\psi'+(\zeta^2+\zeta')\psi=\tau_2\psi\,. \tag{3·34}$$

It is easy to anticipate that for $GL(N)$ the separated equation should become an N-th order differential equation, though the calculation still waits to be done.

For discussion of SoV for the quadratic algebra (2·35) in the $sl(2)$ case (open Toda chain with boundary conditions) see Ref. 20).

The above scheme of quantum SoV is general enough to serve the variety of quantum integrable models obtained by taking concrete representations of the yangian (3·12) or the algebra (3·11). Below we shall illustrate on few examples of $GL(2)$-type models the diversity of possibilities arising. The main problem of adjusting the general scheme to a concrete model is to find the spectrum of the commuting operators $B(u)$ and, consequently, of x_j and to describe the functional space in which the one-dimensional spectral problem (3·25) or (3·34) has to be solved.

§ 4. XXX magnetic chain

The general finite-dimensional irreducible representation of the yangian $\mathcal{Y}[sl(2)]$ is realized in the tensor product of D finite-dimensional irreducible representations of the Lie algebra $sl(2)$ (classically, D is the number of degrees of freedom)

$$[S_m^3, S_n^\pm]=\pm S_n^\pm\delta_{mn}\,, \quad [S_m^+, S_n^-]=2S_n^3\delta_{mn}\,, \quad m, n=1, \cdots, D$$

$$(S_n^3)^2+\frac{1}{2}(S_n^+S_n^-+S_n^-S_n^+)=l_n(l_n+1)\,, \quad l_n\in\left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\right\} \tag{4·1}$$

and can be written in the factorized form (*monodromy matrix*)

$$L_{\mathrm{XXX}}(u)=K\mathcal{L}_D(u-\delta_D)\cdots\mathcal{L}_2(u-\delta_2)\mathcal{L}_1(u-\delta_1)\,, \tag{4·2}$$

where

$$\mathcal{L}_n(u)=u\mathbf{1}+\eta S_n\,,\qquad S_n=\begin{pmatrix} S_n^3 & S_n^- \\ S_n^+ & -S_n^3\end{pmatrix}, \tag{4·3}$$

and K is a constant numeric matrix. It is customary to use the notation $T(u)$ instead of $L(u)$ like e.g. in Ref. 7), but here we had to sacrifice it to preserve the coherence of notation. The representation given by (4·2) has dimension $\prod_{n=1}^{D}(2l_n+1)$ and is parametrized by: 2×2 matrix K, number D of degrees of freedom, spins l_n and shifts δ_n. The Casimir operator (quantum determinant (3·16)) $t_2(u)$ takes the value

$$t_2(u)=\det K\prod_{n=1}^{D}(u-\delta_n-l_n\eta)(u-\delta_n+l_n\eta+\eta)\,. \tag{4·4}$$

The corresponding quantum integrable model is called *inhomogeneous XXX magnet.*

The parameters of the representation being in the generic position, the representation (4·2) of the yangian turns out to be irreducible, and the spectrum of the operators x_j defined from (3·3) for $B(u)$ given by (3·5) turns out to be the finite set[7)]

$$\mathrm{spec}[x_j]=\Lambda_j=\{\lambda_j^-,\lambda_j^-+\eta,\cdots,\lambda_j^+-\eta,\lambda_j^+\}\,,\qquad \lambda_j^\pm=\delta_j\pm l_j\eta\,. \tag{4·5}$$

For the separated finite-difference equations (3·25) to be well defined on the finite set Λ_j, the coefficients $\Delta_\pm(x)$ must satisfy the boundary conditions

$$\Delta_\pm(\delta_n\pm l_n\eta)=0\,.$$

The most convenient choice of $\Delta_\pm(u)$ is

$$\Delta_\pm(u)=\kappa_\pm\prod_{n=1}^{D}(u-\delta_n\mp l_n\eta)\,,\qquad \kappa_+\kappa_-=\det K\,. \tag{4·6}$$

The $sl(2)$ Gaudin model is the degenerate case of XXX magnet obtained in the limit $\eta\to 0$. The corresponding L operator is produced renormalizing L operator (4·2), putting $K:=1+\eta\mathcal{K}$, $\mathrm{tr}\mathcal{K}=0$, and expanding in η:

$$\frac{L_{\mathrm{XXX}}(u)}{\prod_n(u-\delta_n)}=1+\eta L_{\mathrm{Gaudin}}(u)+O(\eta^2)\,.$$

The L operator

$$L_{\mathrm{Gaudin}}(u)=\mathcal{K}+\sum_{n=1}^{D}\frac{S_n}{u-\delta_n} \tag{4·7}$$

satisfies the linear commutation relations (3·11) with the $\mathfrak{r}$ matrix (3·15) for $\eta=1$. The spectral invariant $\mathrm{t}_2(u)$ (3·30) produces the commuting hamiltonians which are quadratic in spin operators

$$\mathrm{t}_2(u)=\frac{1}{2}\mathrm{tr}\mathcal{K}^2+\sum_{n=1}^{D}\frac{H_n}{u-\delta_n}+\sum_{n=1}^{D}\frac{l_n(l_n+1)}{(u-\delta_n)^2}\,, \tag{4·8}$$

$$H_n = \mathrm{tr}\mathcal{K}S_n + \sum_{\substack{m=1\\m\neq n}}^{D} \frac{\mathrm{tr}S_mS_n}{\delta_n - \delta_m}. \tag{4·9}$$

The normalization (4·6) of $\Delta_\pm(u)$ in (3·25) corresponds to the normalization

$$\zeta(u) = c - \sum_{n=1}^{D} \frac{l_n}{u-\delta_n}, \qquad c^2 = -\det\mathcal{K} \tag{4·10}$$

of $\zeta(u)$ in (3·33), (3·34).

In the limit $\eta \to 0$ the $2l_j+1$ points of the spectrum Λ_j of the operator x_j merge into one point $x_j = \delta_j$ of multiplicity $2l_j+1$. The space of functions on Λ_j is understood, respectively, as the ring of polynomials in x_j factorized over the ideal $(x_j-\delta_j)^{2l_j+1}=0$. The spectrum of the hamiltonians (4·9) is given then by the values of H_n in (4·8) for which the differential equation (3·34) with $\zeta(u)$ given by (4·10) in each of the points $x=\delta_j|_{j=1}^{D}$ has a regular solution $\psi_j(x) = 1 + \sum_{k=1}^{\infty}(x-\delta_j)^k \psi_j^{(k)}$, see Ref. 8).

If one realizes the spin operators S_n^α (4·1) as the differential operators

$$S_n^3 = y_n\partial_{y_n} - l_n, \qquad S_n^+ = y_n, \qquad S_n^- = y_n\partial_{y_n}^2 - 2l_n\partial_{y_n} \tag{4·11}$$

then the equation $B(x_j) = L_{21}(x_j) = 0$ defines a "purely coordinate", in the sense (2·9), change of variables $\{y_n\} \to \{x_j\}$. The separated coordinates x_j can be described as generalized ellipsoidal coordinates, see Ref. 9). In fact, all the models allowing SoV in generalized ellipsoidal coordinates, such as Neumann model[11] and its generalizations[15),14] or Euler-Manakov top,[13] can be considered as degenerate cases of Gaudin model.[10] For the generalization of Gaudin model to $osp(1|2)$ Lie superalgebra see Ref. 16). An application of Gaudin model with $L(u)$ having a second order pole to the atomic physics (Coulomb three-body problem) is considered in Ref. 12).

The particular simplicity of the Gaudin model makes it attractive for rigorous mathematical analysis. Much attention was devoted last years to the study of the connection between Gaudin model and Knizhnik-Zamolodchikov (KZ) equations for the correlators in conformal field theory. The fact that the eigenvalue problem for the Gaudin hamiltonians (4·9) coincides with the KZ equations on the critical level can be exploited both to produce integral representations for the solutions to KZ equations[40] and a new derivation of the formula for the norm of the Gaudin eigenfunctions.[41] In the recent paper[42] the representation theory for the affine Lie algebras is applied to derive, in particular, Bethe equations for the Gaudin model corresponding to arbitrary simple Lie algebra and to reveal thus the algebraic meaning of Bethe ansatz. Hopefully, the methods developed in Ref. 42) will be useful also in understanding the algebraic roots of SoV.

The XXX model, as well as the $sl(2)$ Gaudin model, presents a convenient possibility to compare the results of SoV method with those obtained by the Algebraic Bethe Ansatz (ABA).[5] Since the subject is discussed in detail in Refs. 6)~8) we present here only the summary of the analysis. The SoV and ABA methods lead to the same equation (3·25), resp. (3·34), though its interpretations differ. In SoV method the equation is solved on the finite set $\Lambda = \cup_j \Lambda_j$ whereas in ABA $\psi(u)$ is supposed to be a polynomial whose zeroes v_m parametrize the Bethe vector Ψ_v

$$\Psi_v=\prod_{m=1}^{M} L_{21}(v_m)|\,0\rangle\,, \qquad L_{12}(u)|\,0\rangle=0\,. \tag{4·12}$$

In the SoV approach there is one-to-one correspondence between the solutions of the problem (3·25) on Λ and the eigenvectors of the commuting quantum hamiltonians which is not the case in the ABA approach where the so-called "completeness problem" arises. The SoV method provides the basis for the rigorous analysis of the completeness problem and allows to formulate the criterion of completeness. In case of the Gaudin model the criterion sounds as follows.[8] Let $Q(u)$ be a polynomial solution to the differential equation (3·34). If $Q(\delta_j)\neq 0$, $\forall j=1,\cdots,D$ then the corresponding Bethe vector (4·12) is nonzero. If, however, $Q(\delta_j)=0$ for some j then the corresponding Bethe vector is zero, and the set of Bethe eigenvectors is incomplete. Moreover, the nonzero eigenvector corresponding to such $Q(u)$ exists if and only if the linearly independent solution of the differential equation is regular in the same point $x=\delta_j$.

The power of SoV is revealed most obviously in the cases when the representation of the yangian does not possess the highest vector $|\,0\rangle$ such that $L_{12}|\,0\rangle=0$ and hence ABA cannot be applied. These cases correspond to the infinite dimensional representations of $sl(2)$ for the operators S_n^α (4·3). The corresponding separated wave functions $\psi(x_j)$ are not more polynomials, and the separated equations (3·25), (3·34) should be accompanied with some square integrability conditions. Depending on the real form of $sl(2)$ in question there is plently of analytical possibilities.

For the Goryachev-Chaplygin top the spectrum of x_j is real and discrete, and the shift η in (3·25) is real, see Refs. 4), 1) and also Ref. 18) for generalizations.

In case of the Toda lattice the spectrum of x_j is real and the shifts η in (3·25) are imaginary.[2] See also Ref. 19) for the relativistic version and Ref. 20) for the lattice with boundary conditions.

A version of noncompact XXX magnet applied recently[43] to describe the QCD in the asymptotic high energy regime also does not have the highest weight vector, so the SoV is the natural approach to try.

For the analogous effects in the Neumann model, see Ref. 11).

§ 5. Infinite volume limit

So far, we discussed only the integrable models with a finite number D of degrees of freedom. The passage to $D=\infty$ can be made in two ways: either by taking the continuum limit or the infinite volume limit. In the continuum limit the representation (4·2) of the monodromy matrix $L(u)$ as a product of local $\mathcal{L}$ operators $\mathcal{L}_n(u)$ is replaced by the representation in the form of the ordered exponential[5),44)]

$$L_{\xi_-}^{\xi_+}(u)=:\overrightarrow{\exp}\int_{\xi_-}^{\xi_+}\mathfrak{L}(u,\xi)d\xi:\,. \tag{5·1}$$

For example, for the nonlinear Schrödinger equation, described in terms of the canonical fields $\Psi(\xi)$, $\Psi^*(\xi)$

$$[\Psi(\xi),\Psi(\xi')]=[\Psi^*(\xi),\Psi^*(\xi')]=0\,, \qquad [\Psi(\xi),\Psi^*(\xi')]=\delta(\xi-\xi') \tag{5·2}$$

acting in the Fock space $\Psi(\xi)|0\rangle=0$, the infinitesimal L operator $\mathfrak{L}(u,\xi)$ is given by

$$\mathfrak{L}(u,x)=\begin{pmatrix} -iu/2 & \sqrt{c}\,\Psi^*(\xi) \\ \sqrt{c}\,\Psi(\xi) & iu/2 \end{pmatrix}. \tag{5·3}$$

The corresponding quantum monodromy matrix $L_{\xi_-}^{\xi_+}(u)$ given by (5·1), where : : stands for the normal ordering, satisfies the commutation relations (3·12) with the R matrix of XXX type (3·15) and $\eta=-ic$. The quantum determinant $t_2(u)$ of $L_{\xi_-}^{\xi_+}(u)$ is equal to $e^{-cV/2}$, and the trace $t_1(u)$ of $L_{\xi_-}^{\xi_+}(u)$ generates the commuting hamiltonians, in particular

$$H=\int_{\xi_-}^{\xi_+}(\Psi_\xi^*\Psi_\xi+c\Psi^*\Psi^*\Psi\Psi)d\xi\,. \tag{5·4}$$

We assume that the fields $\Psi(\xi)$, $\Psi^*(\xi)$ are periodic in ξ with the period $V=\xi_+-\xi_-$ and the coupling constant c is positive $c>0$ (repulsive case).

It is convenient to choose the following normalization $\boldsymbol{\alpha}(u)$ of the Baker-Akhiezer function (2·20)

$$\alpha_1(u)=1\,,\qquad \alpha_2(u)=i\,.$$

The reason for such a choice is that the corresponding operator family $B(u)$ (2·26)

$$B(u)=-iL_{11}(u)+L_{12}(u)+L_{21}(u)+iL_{22}(u) \tag{5·5}$$

has the symmetry $B(u)^*=B(\bar{u})$ and its zeroes x_j are self-adjoint operators.

It is easy to find that the separated equation (3·25) takes the form

$$\tau_1(x)\phi(x)=e^{-ixV/2}\phi(x+ic)+e^{ixV/2}\phi(x-ic)\,. \tag{5·6}$$

Since $B(u)$, like $L_{\xi_-}^{\xi_+}(u)$, is not a polynomial but a holomorphic function of $u\in\boldsymbol{C}$, it has an infinite discrete set of zeroes x_j having asymptotics $x_j=2\pi j/V+O(j^{-1})$ as $j\to\infty$. The necessity to handle the functions of infinite number of variables complicates greatly the justification of the standard SoV construction even in the classical case.

The situation, however, simplifies drastically in the infinite volume limit (we consider the zero density case described again by the Fock representation for $\Psi(\xi)$, $\Psi^*(\xi)$). The definition of the monodromy matrix $L(u)$ for $V=\infty$ needs some caution. If one tries to pass to the limit $\xi_\pm\to\pm\infty$ directly in the expression (5·1) one finds immediately that, in order to get a finite expression, it is necessary to cancel the asymptotic behaviour of $L_{\xi_-}^{\xi_+}(u)$ introducing the exponential factors

$$L_{-\infty}^{+\infty}(u)=\lim_{\xi_\pm\to\pm\infty}e^{iu\sigma_3\xi_+/2}L_{\xi_-}^{\xi_+}(u)e^{-iu\sigma_3\xi_-/2}\,,\qquad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{5·7}$$

which, however, destroy completely the nice commutation relations (3·12). The solution of the problem is to factorize first the finite-volume monodromy matrix $L_{\xi_-}^{\xi_+}(u)=L_{\xi_0}^{\xi_+}(u)L_{\xi_-}^{\xi_0}(u)$ and to permute the factors: $L_{\xi_0}^{\xi_0}(u)=L_{\xi_-}^{\xi_0}(u)L_{\xi_0}^{\xi_+}(u)$. This operation does not change the quantity $t_1(u)=\mathrm{tr}L(u)$ generating the integrals of motion.

The matrix $L_{\xi_0}^{\xi_0}(u)$, however, behaves in the limit $\xi_\pm \to \pm\infty$ much better:

$$L_{\xi_0}^{\xi_0}(u) = L_{-\infty}^{\xi_0} e^{-iuV\sigma_3/2} L_{\xi_0}^{+\infty} \sim e^{\mp iuV/2} L_{-\infty}^{\xi_0} P_\pm L_{\xi_0}^{+\infty} ,$$

where

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and the upper (lower) sign corresponds, respectively, to $\operatorname{Im} u > 0$ (<0).

The scalar factor $e^{\mp iuV/2}$ can be cancelled out of $L_{\xi_0}^{\xi_0}(u)$ since it does not affect the relation (3·12). Hence, we can take for the monodromy matrix in the infinite volume the matrix

$$L(u) = L_{-\infty}^{\xi_0} P_\pm L_{\xi_0}^{+\infty} . \tag{5·8}$$

The matrix $L(u)$ is analytical in the complex plane of u except the cut along the real axis. Its quantum determinant $t_2(u)$ is zero.

It is interesting to compare the above results with those of Kyoto group on the XXZ magnetic chain.[45] In both cases the infinite volume case is considered, and a sort of monodromy matrix $L(u)$ satisfying (3·12) is constructed. In the XXZ case, however, all components of $L(u)$ are integrals of motion whereas in the nonlinear Schrödinger case only $\operatorname{tr} L(u)$ is one. Moreover, in the XXZ case $L(u)$ is analytic in $\boldsymbol{C}$ whereas in the NLS case $L(u)$ has a cut along the real axis. The above distinctions are probably due to the different nature of the vacuum state: antiferromagnetic for XXZ and ferromagnetic for NLS which in the last case requires introducing the asymptotic exponents $e^{\pm iu\sigma_3\xi_\pm/2}$.

Let us consider now how the whole SoV construction is modified in the infinite volume limit. As $\xi_\pm \to \pm\infty$ the zeroes x_j of $B(u)$ accumulate to a continuous distribution

$$\frac{2\pi}{V} j \to \nu , \qquad x_j \sim \nu + \frac{2\pi}{V} q(\nu)$$

with the density $q(\nu)$. The Hilbert space $\mathcal{H}$ of quantum states is realized then as the space of the linear functionals $W[q(\nu)]$ of $q(\nu)$, $\nu \in \boldsymbol{R}$ which are square integrable

$$\| W \|^2 = \int | W[q(\nu)] |^2 \delta m$$

with respect to the measure δm which, fortunately, happens to be Gaussian and is characterized uniquely by the correlator (covariance kernel)

$$\langle q(\mu) q(\nu) \rangle_\nu = \frac{1}{4\pi^2} \ln\left(1 + \frac{c^2}{(\mu - \nu)^2}\right).$$

The representation of $B(u)$ as an infinite product $B(u) \sim \prod_j (u - x_j)$ is replaced by the Cauchy integral

$$B(u) = \mp \frac{i}{2} \exp\left\{ -\int_{-\infty}^{+\infty} \frac{d\nu}{u - \nu} q(\nu) \right\} , \tag{5·9}$$

where the upper (lower) sign corresponds, respectively, to Im $u>0$ (<0).

Though the quantity $t_1(u)$ is represented in $\mathcal{H}$ as a complicated variational operator, its eigenfunctions nevertheless can be found exactly and have rather simple structure which is natural to consider as a continual analog of SoV. The separated equation (5·6) is replaced respectively by a boundary problem for analytical functions having a cut along the real axis. Unfortunately, the complications due to the cut make a brief explanation impossible and for the details we refer the reader to the original papers.[22)]

For the nonlinear Schrödinger equation the above SoV procedure can be justified rigorously by comparison with the results known from the algebraic Bethe Ansatz.[22)] The analogous construction for the relativistic sinh-Gordon model, though not so rigorous, can also be performed[23)] and leads to quite reasonable results concerning the spectrum of the model. It would be interesting to generalize the results of Ref. 23) to the relativistic Toda field theories.

§ 6. Classical XYZ magnet

The XYZ magnet provides an example of quite nontrivial normalization of the Baker-Akhiezer function $\Omega(u)$ necessary to produce SoV. The construction presented below is taken from Ref. 17). The model is described in terms of the 2×2 matrix $L(u)$ satisfying the relation (2·32) with the r-matrix

$$r(u)=\sum_{\alpha=1}^{3} w_\alpha(u)\overset{1}{\sigma}_\alpha\overset{2}{\sigma}_\alpha\,, \tag{6·1}$$

where σ_α are standard Pauli matrices

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{6·2}$$

and $w_\alpha(u)$ are certain elliptic functions whose exact expression is not important for the moment. It suffices to remark that $r(u)$ is a meromorphic function on $\boldsymbol{C}$ having simple poles on the periodic lattice $\Gamma=\{u\in\boldsymbol{C}|u=m+\tau n;\ m, n\in\boldsymbol{Z};\ \mathrm{Im}\tau>0\}$ and possessing the periodicity properties

$$r(u+1)=\overset{1}{\sigma}_1 r(u)\overset{1}{\sigma}_1=\overset{2}{\sigma}_1 r(u)\overset{2}{\sigma}_1\,,$$
$$r(u+\tau)=\overset{1}{\sigma}_3 r(u)\overset{1}{\sigma}_3=\overset{2}{\sigma}_3 r(u)\overset{2}{\sigma}_3\,. \tag{6·3}$$

The L operator $L(u)$, in turn, is a holomorphic function characterized by the quasiperiodicity properties

$$L(u+1)=(-1)^D\sigma_1 L(u)\sigma_1\,,\qquad L(u+\tau)=(-1)^D e^{-i\pi D(u+\tau)}\sigma_3 L(u)\sigma_3\,, \tag{6·4}$$

where D is a positive integer. The conditions (6·4) determine $L(u)$ up to $4D$ free parameters[46)] which can be considered as the dynamical variables with the Poisson structure defined by (3·11). Since, however, the determinant det $L(u)$ generating the center of the Poisson algebra contains $2D$ parameters (Casimir functions), their values can be fixed which leaves $2D$-dimensional phase space, the constant D being thus the number of degrees of freedom.

In contrast with the XXX magnet, the normalization $\boldsymbol{\alpha}(u)=\text{const}$ of $\Omega(u)$ does not produce SoV for the XYZ magnet for any $\boldsymbol{\alpha}$. The reason is that the corresponding function $B(u)$ has no definite quasiperiodicity for the period lattice Γ and can be characterized only by the periodicity properties in 2Γ

$$B(u+2)=B(u)\,, \quad B(u+2\tau)=e^{-i\pi D(2u+3\tau)}B(u)$$

from which it follows that $B(u)$ has $4D$ zeroes in the fundamental region $\boldsymbol{C}/2\Gamma$ whereas one needs only D separated coordinates x_j.

The correct normalization[17] is given by the holomorphic functions $\boldsymbol{\alpha}(u)$ having the periodicity properties

$$\alpha_1(u+1)=\alpha_1(u)\,, \qquad \alpha_1(u+\tau)=-e^{-i\pi(u+\tau-y)/2}\alpha_1(u)\,,$$
$$\alpha_2(u+1)=-\alpha_2(u)\,, \quad \alpha_2(u+\tau)=e^{-i\pi(u+\tau-y)/2}\alpha_2(u)\,, \qquad (6\cdot5)$$

where y is a parameter. The explicit expressions for $\alpha_n(u)$ can be given in terms of theta-functions for the lattice of periods Γ.[17] The function $B(u)$ corresponding to the normalization vector (6·5) is given by the formula (2·26) and has good periodicity properties on the lattice Γ

$$B(u+1)=(-1)^{D+1}B(u)\,, \quad B(u+\tau)=(-1)^{D+1}e^{-i\pi(D+1)(u+\tau)}e^{-i\pi y}B(u)\,. \qquad (6\cdot6)$$

Consequently, $B(u)$ is a theta-function of order $D+1$ and has $D+1$ zeroes in the fundamental region $\boldsymbol{C}/\Gamma$. It remains to require one superficial zero of $B(u)$ to be a constant (c-number)

$$B(\xi)=0\,, \qquad (6\cdot7)$$

which can be considered as the equation determining the parameter y as a function on the phase space. The remaining D zeroes of $B(u)$ are candidates for the separated coordinates x_j. A direct, though cumbersome, calculation of the Poisson brackets[17] validates successfully the conjecture.

An open question is the quantization of the above construction. The difficulty is the ordering problem in the expression (2·26) since the quantities $\alpha_n(u)$ contain dynamical variables through their dependence on y.

The example of XYZ model raises the question if the correct normalization producing SoV could be found for other integrable models where the simplest choice $\boldsymbol{\alpha}(u)=\text{const}$ is known to fail.[29]

§ 7. 3-particle Calogero-Moser model

In this section we present new results for the classical and quantum 3-particle elliptic Calogero-Moser model. The model is interesting as an example of SoV in case of a dynamical r-matrix.

In the classical case the model has 3 degrees of freedom and is described in terms of the canonical variables (π_n, q_n)

$$\{q_m, q_n\}=\{\pi_m, \pi_n\}=0\,, \qquad \{\pi_m, q_n\}=\delta_{mn}\,, \qquad m, n=1, 2, 3\,. \tag{7·1}$$

The commuting hamiltonians are[47)]

$$H_1=\pi_1+\pi_2+\pi_3\,,$$
$$H_2=\pi_1\pi_2+\pi_1\pi_3+\pi_2\pi_3-g^2(\wp(q_{12})+\wp(q_{13})+\wp(q_{23}))\,,$$
$$H_3=\pi_1\pi_2\pi_3-g^2(\pi_1\wp(q_{23})+\pi_2\wp(q_{13})+\pi_3\wp(q_{12}))\,, \tag{7·2}$$

where $q_{mn}=q_m-q_n$, $\wp$ is Weierstrass elliptic function, and g is the coupling constant. The corresponding L operator is

$$L_{\mathrm{CM}}(u)=\begin{pmatrix} \pi_1 & -igQ_{12}(u) & -igQ_{13}(u) \\ -igQ_{21}(u) & \pi_2 & -igQ_{23}(u) \\ -igQ_{31}(u) & -igQ_{32}(u) & \pi_3 \end{pmatrix}, \tag{7·3}$$

where

$$Q_{mn}=\frac{\sigma(u+q_{mn})}{\sigma(u)\sigma(q_{mn})}\,, \tag{7·4}$$

and $\sigma(u)$ is Weierstrass sigma function.

The hamiltonians (7·2) can be obtained from the spectral invariants of the L operator

$$\det(z-L(u))=z^3-z^2t_1(u)+zt_2(u)-t_3(u)\,, \tag{7·5}$$

$$t_1(u)=H_1\,,$$
$$t_2(u)=H_2+3g^2\wp(u)\,,$$
$$t_3(u)=H_3+g^2\wp(u)H_1-ig^3\wp'(u)\,. \tag{7·6}$$

The L operator (7·3) satisfies the identity (2·29) with rather complicated r-matrix depending on q.[31)] In the absence of general theory of dynamical r-matrices the only available strategy is to try one-by-one possible ansätze for the normalization $\boldsymbol{\alpha}(u)$ of the Baker-Akhiezer function $\Omega(u)$. Fortunately, the very first attempt succeeds: the simplest normalization (3·1) which was applied to the $GL(N)$-magnet does also produce SoV for the Calogero-Moser model.

For our purposes it is convenient to write down the set of equations for the pair (x, z) as (3·9) where the functions $A_{1,2}(u)$ are given by the formulas (3·8) and, in our case, are

$$A_1(u)=\pi_1+ig[\zeta(u)-\zeta(u-q_{23})+\zeta(q_{12})-\zeta(q_{13})]\,,$$
$$A_2(u)=\pi_2+ig[\zeta(u)-\zeta(u-q_{13})-\zeta(q_{12})-\zeta(q_{23})]\,, \tag{7·7}$$

where $\zeta(u)$ is Weierstrass zeta function.

Since the r-matrix for the CM model is different from (2·37) we cannot rely on the results obtained for the $GL(N)$ magnet and have to calculate the Poisson brackets between z and x directly. It turns out that Eqs. (3·9) have only two solutions: (z_1, x_1) and (z_2, x_2). For the third pair of variables one has to take (P, Q)

$$P=\pi_1+\pi_2+\pi_3\,, \qquad Q=q_3\,. \tag{7·8}$$

The calculation shows that the variables $(P, z_1, z_2; Q, x_1, x_2)$ are canonical and satisfy the relations

$$P-H_1=0\,,$$
$$z_j^3-z_j^2H_1+z_j(H_2+3g^2\wp(x_j))-(H_3+g^2\wp(x_j)H_1-ig^3\wp'(x_j))=0\,, \tag{7·9}$$

which fit the form (2·3) and provide thus a SoV.

In the quantum case the momenta π_j are realized as the differentiations $\pi_j=-i\partial_{q_j}$. The hamiltonians (7·2), respectively, are replaced by the differential operators

$$H_1=-i(\partial_{q_1}+\partial_{q_2}+\partial_{q_3})\,,$$
$$H_2=-\partial^2_{q_1q_2}-\partial^2_{q_1q_3}-\partial^2_{q_2q_3}-g(g-1)[\wp(q_{12})+\wp(q_{13})+\wp(q_{23})]\,,$$
$$H_3=i\partial^3_{q_1q_2q_3}+ig(g-1)[\wp(q_{23})\partial_{q_1}+\wp(q_{13})\partial_{q_2}+\wp(q_{12})\partial_{q_3}]\,, \tag{7·10}$$

which do commute,[47] as their classical counterparts.

Since it is still unknown how to quantize the relation (2·29) in case of the dynamical r-matrices, we again have to rely on good luck trying to find a quantum SoV. An additional obstacle is provided by the fact that in our case, even classically, $\{B(u), B(v)\}\neq 0$. Therefore, there is little hope to construct quantum x_j as zeroes of a commuting family of operators $B(u)$ like in case of the $GL(N)$ magnet. Instead, we shall rather look for the kernel $K(x_1, x_2, Q|q_1, q_2, q_3)$ of the integral operator (classically, canonical transformation) intertwining the xQ and q representations.

The first of the classical separated equations (7·9) is easy to quantize. It expresses the conservation of the total momentum and allows to eliminate one pair of variables from K:

$$\begin{cases}P=-i(\partial_{q_1}+\partial_{q_2}+\partial_{q_3})=-i\partial_Q\,,\\ Q=q_3\,,\end{cases} \Longrightarrow \begin{cases}(\partial_{q_1}+\partial_{q_2}+\partial_{q_3}+\partial_Q)K=0\,,\\ \qquad\qquad (Q-q_3)K=0\,,\end{cases}$$

$$(Q-q_3)K=0 \Longrightarrow K=\delta(Q-q_3)\tilde{K}(x_1, x_2|q_1, q_2, q_3)\,,$$

$$(\partial_{q_1}+\partial_{q_2}+\partial_{q_3}+\partial_Q)K=0 \Longrightarrow (\partial_{q_1}+\partial_{q_2}+\partial_{q_3})\tilde{K}=0\,,$$
$$\Longrightarrow \tilde{K}=\tilde{K}(x_1, x_2|q_{13}, q_{23})\,.$$

To determine the kernel $\tilde{K}$, let us try to quantize Eqs. (7·7). Making the substitutions

$$\pi_j\longrightarrow i\partial_{q_{j3}}\,, \qquad z_j\longrightarrow -i\partial_{x_j}\,, \qquad g\longrightarrow g-1$$

(the last one is a quantum correction found experimentally) one obtains from (7·7) the system of 4 first order differential equations for $\tilde{K}$

$$(\partial_{x_1}+\partial_{q_{13}})\tilde{K}+(g-1)[\zeta(x_1)-\zeta(x_1-q_{23})-\zeta(q_{13})+\zeta(q_{13}-q_{23})]\tilde{K}=0\,,$$
$$(\partial_{x_2}+\partial_{q_{13}})\tilde{K}+(g-1)[\zeta(x_2)-\zeta(x_2-q_{23})-\zeta(q_{13})+\zeta(q_{13}-q_{23})]\tilde{K}=0\,,$$
$$(\partial_{x_1}+\partial_{q_{23}})\tilde{K}+(g-1)[\zeta(x_1)-\zeta(x_1-q_{13})-\zeta(q_{23})-\zeta(q_{13}-q_{23})]\tilde{K}=0\,,$$

$$(\partial_{x_2}+\partial_{q_{23}})\tilde{K}+(g-1)[\zeta(x_2)-\zeta(x_2-q_{13})-\zeta(q_{23})-\zeta(q_{13}-q_{23})]\tilde{K}=0\,, \tag{7·11}$$

which are easily solved producing the result which most conveniently can be written using the variables $x_\pm=x_1\pm x_2$ and $\xi_\pm=q_{13}\pm q_{23}$:

$$\tilde{K}=\delta(x_+-\xi_+)\mathcal{K}(x_+,x_-;\xi_-)\,,$$

$$\mathcal{K}=\left[\frac{\sigma\left(\frac{\xi_-+x_-}{2}\right)\sigma\left(\frac{\xi_--x_-}{2}\right)\sigma\left(\frac{x_++\xi_-}{2}\right)\sigma\left(\frac{x_+-\xi_-}{2}\right)}{\sigma(x_1)\sigma(x_2)\sigma(\xi_-)}\right]^{g-1}. \tag{7·12}$$

The above argument, of course, has only heuristic value and provides no guarantee that the kernel K thus constructed would produce SoV. What is necessary to verify is that the integral operator with the kernel K transforms an eigenfunction $\Psi(q_1,q_2,q_3)$ of the hamiltonians H_n (7·2) satisfying (2·13) into the function $\tilde{\Psi}(x_1,x_2,Q)$ satisfying separated equations of the type (2·14).

The observation which is crucial for establishing SoV is that the kernel K satisfies the differential equations

$$[-i\partial_Q-H_1{}^*]K=0\,, \tag{7·13a}$$

$$[i\partial_{x_j}^3+H_1{}^*\partial_{x_j}^2-i(H_2{}^*+3g(g-1)\wp(x_j))\partial_{x_j}$$
$$-(H_3{}^*+g(g-1)H_1{}^*\wp(x_j)-ig(g-1)(g-2)\wp'(x_j))]K=0\,, \tag{7·13b}$$

where H_n^* is the Lagrange adjoint of H_n

$$\int\varphi(q)(H\psi)(q)dq=\int(H^*\varphi)(q)\psi(q)dq\,.$$

Equations (7·13) can be interpreted as the quantum analog of Eqs. (7·9) (note the quantum corrections in g!). Consider now the integral transform

$$\tilde{\Psi}(x_1,x_2,Q)=\iiint dq_1dq_2dq_3K(x_1,x_2,Q|q_1,q_2,q_3)\Psi(q_1,q_2,q_3)\,. \tag{7·14}$$

Acting on $\tilde{\Psi}$ with the differential operators

$$\mathcal{Q}=-i\partial_Q-h_1\,, \tag{7·15a}$$

$$\mathcal{D}_j=i\partial_{x_j}^3+h_1\partial_{x_j}^2-i(h_2+3g(g-1)\wp(x_j))\partial_{x_j}$$
$$-(h_3+g(g-1)h_1\wp(x_j)-ig(g-1)(g-2)\wp'(x_j)) \tag{7·15b}$$

and supposing that $\Psi(\boldsymbol{q})$ is an eigenfunction of H_n, perform the integration by parts using the relations (7·13). The resulting bulk part of the integral is zero. It remains only to find such the limits of integration which would not contribute to the result. Omitting the details of the guesswork we report only the final result.

The function $\tilde{\Psi}(x_+,x_-,Q)$ obtained from an eigenfunction $\Psi(\xi_+,\xi_-,q_3)$ of H_n via the integral transform

$$\tilde{\Psi}(x_+,x_-,Q)=\int_{x_-}^{x_+}d\xi_-\mathcal{K}(x_+,x_-;\xi_-)\Psi(x_+,\xi_-,Q) \tag{7·16}$$

satisfies the differential equations

$$Q\tilde{\Psi}=0, \qquad \mathcal{D}_j\tilde{\Psi}=0$$

which imply the SoV

$$\tilde{\Psi}(x_1, x_2, Q)=e^{ih_1 Q}\psi(x_1)\psi(x_2),$$

where $\psi(x)$ satisfies a third-order analog of the Lamé differential equation

$$i\psi'''+h_1\psi''-i(h_2+3g(g-1)\wp(x))\psi'$$
$$-(h_3+g(g-1)\wp(x)h_1-ig(g-1)(g-2)\wp'(x))\psi=0. \qquad (7\cdot 17)$$

The question of the correct boundary conditions for the separated equation (7·17) is presently under study.

For the degenerate case of trigonometric potential which corresponds to replacing $\wp(q)$ by $1/\sin^2 q$ in (7·2) and $\sigma(u)$ by $\sin u$ in (7·4) the spectrum and eigenfunctions of H_n have been well known quite a while ago.[47] The eigenfunctions are labelled by triplets $\boldsymbol{\nu}=(\nu_1, \nu_2, \nu_3)$ of integers ν_j such that $\nu_1\leq\nu_2\leq\nu_3$. The corresponding eigenvalues of H_n are given by

$$\begin{cases} h_1=2(\mu_1+\mu_2+\mu_3), \\ h_2=4(\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3), \\ h_3=8\mu_1\mu_2\mu_3, \end{cases} \qquad \begin{cases} \mu_1=\nu_1-g, \\ \mu_2=\nu_2, \\ \mu_3=\nu_3+g. \end{cases} \qquad (7\cdot 18)$$

The eigenfunctions Ψ_ν have the structure $\Psi_\nu(\boldsymbol{q})=\Psi_{000}(\boldsymbol{q})\Phi_\nu(\boldsymbol{q})$ where $\Psi_{000}(\boldsymbol{q})=\sin^g q_{12}\sin^g q_{13}\sin^g q_{23}$ is the vacuum eigenfunction corresponding to $\boldsymbol{\nu}=(0,0,0)$ and $\Phi_\nu(\boldsymbol{q})$ are symmetric Laurent polynomials in variables $t_j=e^{2iq_j}$ known as *Jack polynomials*.[48]

The SoV is produced by the same integral kernel K up to replacing σ with sin in (7·12). The separated eigenfunctions $\psi(y)$ satisfy the differential equation

$$i\psi'''+h_1\psi''-i\left(h_2+3\frac{g(g-1)}{\sin^2 x}\right)\psi'$$
$$-\left(h_3+\frac{g(g-1)}{\sin^2 x}h_1+2ig(g-1)(g-2)\frac{\cos x}{\sin^3 x}\right)\psi=0 \qquad (7\cdot 19)$$

and can be factorized $\psi_\nu(x)=\psi_{000}(x)\varphi_\nu(x)$ into the product of the vacuum factor $\psi_{000}(x)=\sin^{2g}x$ and a Laurent polynomial φ_ν in variable $t=e^{2ix}$. Despite the huge body of facts known about the Jack polynomials, the last factorization property seems to be a new result.

A more detailed exposition of the above results will be published elsewhere.

§ 8. Discussion

The above examples show the diversity of models allowing SoV and give a support to the opinion that the domain of SoV method might be very large, even including all the models subject to the classical Inverse Scattering Method and their quantum counterparts.

Let us enlist, in conclusion, some problems whose solution could strengthen the positions of SoV. The most obvious object of study is provided by the class of integrable models described by numeric unitary R-matrices, and in the first turn, $sl(N)$-invariant magnets. The $sl(2)$ case being well enough studied, the $sl(N)$ case seems to present only calculational difficulties. The case of trigonometric R-matrices should not differ considerably from the rational ($sl(N)$-invariant) case. For instance, it would be interesting to generalize the results of § 5 to the relativistic Toda field theories. In the case of elliptic R-matrices discussed in § 6 the problem of nontrivial normalization of the Baker-Akhiezer function arises which leads to the complications with the quantization. However, the success with the classical XYZ magnet allows to hope that the quantization problem would be solved. A similar, but more difficult problem of choosing the correct normalization of B-A function arises in case of R-matrices corresponding to the simple Lie algebras other than $SL(N)$. The problem is not yet solved even in the classical case. The most difficult problems arise in case of dynamical r-matrices where neither general theory exists, no quantization rules are known.

The problem which may be more important than studying all the particular examples is to understand the algebraic structures underlying the SoV and to explain why the "magic recipe" of taking the poles of B-A function does work. The recent paper[42] might be the first step in that direction.

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