ERRATA

Page ligne	Au lieu de	Lire
142^{14}	$sF\|_{rr=r_0}$	$sF_r\|_{r=r_0}$
195^{17}	$\bar{A}\cap\bar{N}\neq 0$	$\bar{A}\cap\bar{B}\neq 0$
196^{2}	R	R^*

Separation principles in the hierarchies of classical and effective descriptive set theory

by

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The so-called first and second separation principles have long played a principal role in the theories of the Borel and projective hierarchies of classical descriptive set theory. More recently they have been considered in the hyperarithmetical and analytical hierarchies (2) studied in recursive function theory. Here the discovery (cf. [7] and [6]) of their fundamental relationship with questions of the essential undecidability of formal systems is an indication of a possibly still more important role for them in the future.

The status of the principles in the Borel hierarchy and at the first two levels of the projective hierarchy was known already by 1935. But despite intensive efforts by Luzin and his school the status of the principles at the third and higher levels of the projective hierarchy has remained largely unknown (3).

Various results concerning the status of the first separation principle in the hyperarithmetical hierarchy can be found, for example, in Kleene's [7], in Mostowski's [15], and in our [1] (cf. also [2]). In [3] we announced its status at the first level of the analytical hierarchy. The status of either principle at the second and higher levels of this hierarchy has remained unknown.

We present here a treatment of the second separation principle in the arithmetical (i. e. finite hyperarithmetical) hierarchy, together with the solution of both separation problems for both the first and second

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(2) Cf. [11] for a good introduction to these hierarchies.

(3) Cf. [17], p. 24. P. S. Novikov stated in 1951 (cf. [16]) that it is consistent with the axioms of set theory to assume that for some natural number k the separation principles at the kth and higher levels of the projective hierarchy behave the way they do at the second level. However, at present writing his promised proof of this assertion has not yet appeared.

levels of the analytical hierarchy. We present further, under the assumption of the axiom of constructibility (which was shown by Gödel [5] to be consistent with the axioms of set theory), the solutions of the separation problems for the third and higher levels of both the projective and analytical hierarchies.

At the same time, relying heavily on the work of Luzin, Sierpiński, Novikov and Kuratowski ([4]), we have summarized here the proofs of the known separation results for the finite Borel, projective, arithmetical, and analytical ([5]) hierarchies. The proofs are presented with the end in mind of (1) emphasizing the simple fundamental unifying principle underlying all the proofs, and (2) illustrating the very essential analogies between the Borel and hyperarithmetical hierarchies and between the projective and analytical hierarchies. This formulation of the analogies, and its preferability to others discussed in the literature, was presented in [1], [2], and [3].

Actually our treatment handles the separation principles for a whole spectrum of hierarchies of which the classical (i. e. the Borel and projective) and effective (i. e. the hyperarithmetical and analytical) hierarchies are the particular examples comprising the opposite extremes. And as a byproduct of this treatment we obtain new strong forms of the negations of the separation principles for the effective and classical hierarchies by using the classical hierarchies in the discussion of the effective ones and vice versa.

1. The spaces $\mathcal{N}^{n,f}$. We denote by N the set of natural numbers $0, 1, 2, \ldots$ and by N^N the set of functions from N into N. The discrete topology is assigned to N and then N^N, considered as the Cartesian product of ω copies of N, is assigned the induced product topology. We use lower case Roman letters other than "s" as variables over N, lower case Greek letters other than "ν" as variables over N^N, and "ν" as a variable over Ω (the set of countable ordinal numbers). Our considerations will be carried out in spaces $\mathcal{N}^{n,f}$ which are the Cartesian products of n copies of N and f copies of N^N, where $n+f>0$ ([6]). As a variable over such a space $\mathcal{N}^{n,f}$ we use "$\mathfrak{a}^{n,f}$". The superscripts will sometimes be omitted when they are arbitrary or it is clear from the context what they should be.

The spaces $\mathcal{N}^{0,f}$ have been traditionally used in classical descriptive set theory, where they are usually visualized as Cartesian products of f copies of the set of irrational numbers with the Baire topology assigned, because the results take on a particularly simple form there. Several discussions (cf. e. g. [14], p. 23 and p. 344) of this simplicity have been given — we mention here only that in these spaces dimension is unimportant because the spaces of all dimensions are homeomorphic.

The hierarchies of recursive function theory were first considered on the spaces $\mathcal{N}^{n,0}$, but were later generalized ([7]) to include all $\mathcal{N}^{n,f}$. Here in considering N^N the functions were envisaged as logical entities with no particular geometric interpretation in mind; this was natural because in recursive function theory one dealt in a logical rather than in a geometric context. It is thus very interesting that although the classical and effective hierarchies were approached from two rather different points of view, they were considered on what is abstractly the same space.

We would note that although copies of N are included in our spaces principally because there is a rich theory here (even when $f=0$) in the case of the effective hierarchies, there is also a notational convenience in including them in the study of the classical hierarchies. Thus, for example, the F_σ-subsets of N^N can be defined as the sets of the form $\hat{\varphi}(Ex)(y)P(\varphi, x, y)$, where P is an open and closed predicate on $N^N\times N\times N$. Because of the discreteness of the topology on N the consideration of P on $N^N\times N\times N$, rather than simply on N^N, adds nothing significant topologically, but simply becomes a convenient way of discussing countability.

As a second illustration of this notational convenience we consider the analytic sets. For any α we denote by $\bar{\alpha}$ the course-of-values function corresponding to α, i. e.

$$\bar{\alpha}(x) = \prod_{i<x} p_i^{\alpha(i)+1},$$

where p_i is the $(i+1)$-st prime number. Then the analytic subsets of N^N can be defined as the sets of the form $\hat{\varphi}(E\alpha)(x)P(\varphi, \bar{\alpha}(x))$, where P is

([4]) For specific historical references the reader is referred to [13] and [17].

([5]) We restrict ourselves here to the finite hierarchies to avoid involving ourselves in definitions and technical problems not connected with the separation principles themselves. However the results do carry over to the extended hierarchies. For the Borel hierarchy this is well-known and for one formulation of the hyperarithmetical hierarchy these results are illustrated in [1]. Furthermore one can define "hyperprojective" (cf. [13], p. 360, Footnote 1) and "hyperanalytical" hierarchies to which the separation results extend.

([6]) To avoid a cumbersome notation we write, for example, $\mathcal{N}^{2,1}$ for both $N\times N^N\times N$ and $N\times N\times N^N$, leaving it always to the reader to decide from the context the order of the factors in our product spaces.

([7]) This generalization, implicitly known since Kleene's fundamental definition of recursive functional in 1950 (cf. [8]), is first given explicitly in [9] in Theorem V* (Part II), as a part of Theorem X (p. 292), when read in light of the parenthetical sentence about uniformity just preceding Theorem X. (Cf. also Theorem XI* for further hierarchy properties.)

an open and closed predicate on $N^N \times N$. For this form simply prescribes that the set is obtained by applying the operation (A) to open and closed sets.

2. The analogies. Now as was intimated in the preceding section the finite Borel (the projective) hierarchy is obtained by applying quantifiers over N (over N^N) to open and closed predicates on $\mathcal{N}$, the arithmetical (the analytical) hierarchy by applying quantifiers over N (over N^N) to recursive predicates on $\mathcal{N}$. Thus without any further analysis we would already expect some analogy between the finite Borel and arithmetical hierarchies and between the projective and analytical hierarchies. But the analogy is even closer because of the following easily proved relationship between the open and closed predicates on $\mathcal{N}$ and the recursive predicates on $\mathcal{N}$:

PRINCIPLE. *A predicate P on $\mathcal{N}$ is open and closed if and only if there exists an a such that P is recursive in a.*

Generalizing from this principle, we consider for every subset C of N^N the class of predicates recursive in some function in C. Each such C yields a class of predicates which, when quantified over N (over N^N), yields hierarchies similar to the finite Borel and arithmetical (to the projective and analytical) hierarchies. (Different C may, of course, lead to the same hierarchy, however.) We call such a hierarchy the *hierarchy arithmetical in functions of C (analytical in functions of C)* or, more briefly, the *C-arithmetical (C-analytical) hierarchy*. The finite Borel, projective, arithmetical, and analytical hierarchies are thus, respectively, the N^N-arithmetical, N^N-analytical, $\emptyset$-arithmetical, and $\emptyset$-analytical hierarchies.

In what follows it will be useful to have the concept of a linked subset of N^N. A subset C of N^N is *linked* if and only if

$$(1) \qquad (\varrho)(\sigma)\big[\varrho, \sigma \in C \rightarrow (E\tau)\ [\tau \in C \ \&\ \varrho, \sigma \text{ are recursive in } \tau]\big].$$

For example, $\emptyset$, N^N, and all singletons are linked.

It seems particularly desirable at this time to introduce simple, uniform, and easy-to-remember notations for the classes of the various hierarchies. Mostowski has used $P_1, Q_1, P_2, \ldots$ for the hyperarithmetical hierarchy; no notation seems to have been introduced for the analytical hierarchy; for the Borel hierarchy, in addition to the inner-quantifier notation $F_0, G_0, F_1, \ldots$ (cf. [13], p. 251-252), a variety of outer-quantifier notations seem to have been used, including Lebesgue's O of class 0, F of class 0, O of class 1, ... and Hausdorff's $P^1, Q^1, P^2, \ldots$; and for the projective hierarchy both $A_1, CA_1, A_2, \ldots$ and $P_1, C_1, P_2, \ldots$ (cf. [17]) seem to be in use. Mostowski has decided to abandon his $P_1, Q_1, P_2, \ldots$, however, because the P_k conflicts with the P_k of the projective hierarchy.

After lengthy discussions here in Warszawa it has been decided to propose $\Sigma_k^{0(C)}\Pi_k^{0(C)}$ ($\Sigma_k^{1[C]}, \Pi_k^{1[C]}$) for the hierarchies built on the class of predicates recursive in functions in C by quantification over N (over N^N). The superscripts "0" and "[$\emptyset$]" are to be omitted, except for emphasis, as an abbreviation and $\Sigma_k^{t[N^N]}$, $\Pi_k^{t[N^N]}$ are abbreviated as $\boldsymbol{\Sigma}_k^t$, $\boldsymbol{\Pi}_k^t$. The "Σ" classes are of course those with an outer Σ or existential quantifier and the "Π" classes those with an outer Π or universal quantifier. Thus the Borel, projective, hyperarithmetical, and analytical hierarchies are respectively denoted by $\boldsymbol{\Sigma}_1, \boldsymbol{\Pi}_1, \boldsymbol{\Sigma}_2, \ldots$, by $\boldsymbol{\Sigma}_1^1, \boldsymbol{\Pi}_1^1, \boldsymbol{\Sigma}_2^1, \ldots$, by $\Sigma_1, \Pi_1, \Sigma_2, \ldots$, and by $\Sigma_1^1, \Pi_1^1, \Sigma_2^1, \ldots$

Among the advantages of this notation over other possible choices we cite: (I) by its uniformity it points up both the classical vs. effective and the number-quantifier vs. function-quantifier analogies; (II) it is easy-to-remember, suggesting directly the definitions through a quantifier notation with deep roots in the history of modern logic; (III) it is easily extended to hierarchies defined by quantification of variables of higher type now under investigation by Kleene (cf. [10], p. 312 and [11], p. 212) and others; (IV) it is an outer-quantifier, rather than an inner-quantifier, notation, and our knowledge of the hierarchies (including preliminary knowledge of hierarchies based on quantification of higher types) indicates that this will be more useful — cf. e. g. the separation principles; (V) it is easily and uniformly pronounced (as opposed, for example, to notation based on $\exists$, $\forall$ or $\vee$, $\wedge$); (VI) it does not permanently "tie up" Roman letters, a large stock of which are convenient for temporary notations; and (VII) it is not subject to confusion with any of the earlier notations which seem to have been used for any of the hierarchies.

3. The separation principles. Our approach to the first and second separation principles will be, like that of Kuratowski [12], through the reduction principle. To say that these principles are true of a class Q of subsets of an arbitrary space we write, respectively, "$Sep_{\mathrm{I}}(Q)$", "$Sep_{\mathrm{II}}(Q)$", and "$Red(Q)$". The predicates Sep_{I}, Sep_{II}, and Red are defined as follows:

$$(2) \qquad Sep_{\mathrm{I}}(Q) \equiv (X)(Y)\big[X, Y \in Q \ \&\ X \cap Y = \emptyset \rightarrow (EX_1)[X_1, \overline{X}_1 \in Q \ \&\ X_1 \supseteq X \ \&\ X_1 \cap Y = \emptyset]\big];$$

$$(3) \qquad Sep_{\mathrm{II}}(Q) \equiv (X)(Y)\big[X, Y \in Q \rightarrow (EX_1)(EY_1)[\overline{X}_1, \overline{Y}_1 \in Q \ \&\ X_1 \supseteq X - Y \ \&\ Y_1 \supseteq Y - X \ \&\ X_1 \cap Y_1 = \emptyset]\big];$$

(4) $Red(Q) \equiv (X)(Y)\big[X, Y \in Q \rightarrow (EX_1)(EY_1)[X_1, Y_1 \in Q \;\&\; X_1 \subseteq X \;\&\; Y_1 \subseteq Y \;\&\; X_1 \cup Y_1 = X \cup Y \;\&\; X_1 \cap Y_1 = \emptyset]\big].$

It is easily shown that:

(5) $Red(Q) \rightarrow Sep_{\mathrm{I}}(cQ) \;\&\; Sep_{\mathrm{II}}(cQ),$

where cQ denotes the class of complements of sets in Q. To separate the required sets one has only to reduce their complements. The reader should observe in checking the proof that $Red(Q)$ implies a very strong form of $Sep_{\mathrm{II}}(cQ)$. And indeed nothing more can be said in general about $Red(Q)$, $Sep_{\mathrm{I}}(cQ)$, and $Sep_{\mathrm{II}}(cQ)$, for one can show that:

(6) $(EQ_1)[Sep_{\mathrm{I}}(cQ_1) \;\&\; \overline{Sep_{\mathrm{II}}}(cQ_1)],$

(7) $(EQ_2)[\overline{Sep_{\mathrm{I}}}(cQ_2) \;\&\; Sep_{\mathrm{II}}(cQ_2)],$

(8) $(EQ_3)[Sep_{\mathrm{I}}(cQ_3) \;\&\; Sep_{\mathrm{II}}(cQ_3) \;\&\; \overline{Red}(Q_3)].$

It suffices to take for cQ_1 the class of Σ_1 subsets of N containing 0, for cQ_2 the class of closed subsets of the real line, and for cQ_3 the class of closed subsets of the real line containing 0 (examples of Sierpiński, Kuratowski).

In view of (5) we can concentrate our attention on the reduction principle.

4. The fundamental reduction technique. Let X and Y be two sets from class Q. To "reduce" these to disjoint sets X_1 and Y_1 of class Q with the same union it is of course clear that the points of $X-Y$ must go into X_1 and those of $Y-X$ into Y_1. All that is needed is a criterion by which to decide for each point of $X \cap Y$ whether it should go into X_1 or into Y_1 — and the criterion must be sufficiently "orderly" that X_1 and Y_1, like X and Y, are in Q.

In case the sets of Q are obtained by unions over an index set Γ of "simpler" sets one criterion promptly suggests itself. It is to well-order Γ and to put a point of $X \cap Y$ into X_1 if and only if the index of the first set of the union forming X in which it appears is less than the index of the first set of the union forming Y in which it appears. The success of this criterion rests of course only on whether it is orderly enough to place X_1 and Y_1 in Q.

It turns out that this criterion can indeed be used to establish the reduction principle, assuming in some cases the axiom of constructibility, on one side of each of the levels of the C-arithmetical and C-analytical hierarchies for any linked C. The constructions in all of our proofs that the reduction principle holds for a class thus rest on a uniform underlying principle.

At all levels of the C-arithmetical hierarchies one can use as the unions those induced by the outer existential quantifier, so that $\Gamma = N$ and we have $Red(\Sigma_k^{[C]})$ for any k $(k \geqslant 1)$ and any linked C. For the C-analytical hierarchies it immediately suggests itself to use as the unions those induced by the outer existential quantifier so that we would have $\Gamma = N^N$. But unlike the situation for N for which there is a simple (viz. $\Sigma_1 \cap \Pi_1$) well-ordering, there is no simple well-ordering of N^N. In fact it is known from classical results of descriptive set theory that there can be no $\boldsymbol{\Sigma}_1^1$ or $\boldsymbol{\Pi}_1^1$ (and a fortiori no Σ_1^1 or Π_1^1) well-ordering of N^N, for otherwise there would be $\boldsymbol{\Sigma}_1^1$ or $\boldsymbol{\Pi}_1^1$ non-measurable sets.

On the other hand, following ideas of Gödel (cf. [5], Note 1, p. 67), we have recently shown (cf. [4]) that, under the assumption of the axiom of constructibility, there is a $\Sigma_2^1 \cap \Pi_2^1$ well-ordering of N^N (which we will denote by "$<$"). This well-ordering turns out to be sufficiently nice to enable one to use as the unions those induced by the outer existential quantifier (so that $\Gamma = N^N$) to prove $Red(\Sigma_k^{1[C]})$ for any k $(k \geqslant 2)$ and any linked C, under the assumption of the axiom of constructibility.

This leaves unsettled only the first level of the C-analytical hierarchies, and here an independent approach is needed. Such an approach, which still falls under the general scheme outlined above, is provided by an ingenious device (discovered by Luzin and Sierpiński in 1918 for the case of the N^N-analytical hierarchy, and rediscovered independently 35 years later by Kleene, in what was seemingly an entirely different field, for the $\emptyset$-analytical hierarchy) which expresses in a simple way the $\Pi_1^{1[C]}$ sets as unions with $\Gamma = \Omega$ and enables one to prove $Red(\Pi_1^{1[C]})$ for any linked C.

This same device can be extended, as Novikov first observed (for the case of the N^N-analytical hierarchy), to the second level of the C-analytical hierarchies. This enables one to conclude $Red(\Sigma_2^{1[C]})$ for any linked C without using the axiom of constructibility. Whether similarly ingenious devices will someday enable man to prove $Red(\Sigma_k^{1[C]})$ in case $k \geqslant 3$ and C is linked without using the axiom of constructibility or whether these principles will turn out to be independent of the accepted axioms of set theory remains a rather fascinating open question.

5. The detailed proofs. We proceed now to a detailed consideration of the proofs, which we consider in four cases. In each case we let X and Y be the two sets to be reduced, and construct reduced sets X_1 and Y_1. Since we have outlined above how the constructions are to proceed our major task remaining is to show that X_1 and Y_1 are in the same class as X and Y. Y_1 is always defined by interchanging "R^e" and "S^a" and replacing "$<$" by "$\leqslant$" — hence the proof that it is of the proper class is in each case so similar to that for X_1 that we omit it. Throughout this section we assume C is an arbitrary linked subset of N^N.

CASE I. $X, Y \in \Sigma_k^{0[C]}$ $(k \geqslant 1)$. To illustrate we fix $k = 2$. Then for some ϱ, σ in C and R^ϱ, S^σ recursive in ϱ, σ, respectively,

$$X = \hat{\mathfrak{a}}(Ex)(y)R^\varrho(\mathfrak{a}, x, y), \qquad Y = \hat{\mathfrak{a}}(Ex)(y)S^\sigma(\mathfrak{a}, x, y).$$

So

$$X_1 = \hat{\mathfrak{a}}(Ex)[(y)R^\varrho(\mathfrak{a}, x, y) \,\&\, \overline{(Ex_1)}_{x_1<x}(y)S^\sigma(\mathfrak{a}, x_1, y)]$$
$$= \hat{\mathfrak{a}}(Ex)\big[(y)R^\varrho(\mathfrak{a}, x, y) \,\&\, (Ey)(x_1)_{x_1<x}\overline{S}^\sigma(\mathfrak{a}, x_1, (y)_{x_1})\big].$$

Now bringing quantifiers to the front and contracting like quantifiers [8] we have the desired prefix. And since $\lambda yx(y)_x$ (which maps (y, x) into the greatest z such that $y/p_x^z \in N$) and $\lambda x_1 x\; x_1 < x$ are in $\Sigma_1^0 \cap \Pi_1^0$, the resulting matrix will be recursive in ϱ, σ, and hence by (1), in some τ in C. So $X_1 \in \Sigma_2^{0[C]}$.

The device of interchanging quantifiers of opposite kind used in the above proof breaks down in case $k = 1$. But there it may be replaced by a simple application of E of p. 228 of [9] and of Post's theorem (cf. p. 293 of [9]). This latter argument works equally well for all k $(k \geqslant 1)$, but we have presented the first argument here to set the stage for the analogous device to be used in Case IV.

CASE II. $X, Y \in \Pi_1^{1[C]}$. Then for some ϱ, σ in C and R^ϱ, S^σ recursive in ϱ, σ, respectively,

$$X = \hat{\mathfrak{a}}(a)(Ex)R^\varrho\big(\mathfrak{a}, \overline{a}(x)\big), \qquad Y = \hat{\mathfrak{a}}(a)(Ex)S^\sigma\big(\mathfrak{a}, \overline{a}(x)\big).$$

Now by the device of Luzin and Sierpiński and Kleene mentioned above R^ϱ, S^σ are easily chosen so that

$$X = \hat{\mathfrak{a}}(E\nu)|\hat{s}\overline{R}^\varrho(\mathfrak{a}, s)| = \nu, \qquad Y = \hat{\mathfrak{a}}(E\nu)|\hat{s}\overline{S}^\sigma(\mathfrak{a}, s)| = \nu,$$

where "s" is a variable over the set of sequence numbers (numbers of the form $\overline{a}(x)$ for some a and x) and $|F|$, for a set F of sequence numbers, is the order type of F under the ordering $\prec$ (which orders the sequence numbers according to the backwards lexicographical ordering using the alphabet $\ldots, 2, 1, 0$ of the sequences they represent). So

$$X_1 = \hat{\mathfrak{a}}(E\nu)[|\hat{s}\overline{R}^\varrho(\mathfrak{a}, s)| = \nu \,\&\, \overline{(E\nu_1)}_{\nu_1<\nu}|\hat{s}\overline{S}^\sigma(\mathfrak{a}, s)| = \nu_1]$$
$$= [\text{by the rules of the predicate calculus with equality}],$$
$$\hat{\mathfrak{a}}\big[(E\nu)|\hat{s}\overline{R}^\varrho(\mathfrak{a}, s)| = \nu \,\&\, \overline{(E\nu_1)}[|\hat{s}\overline{S}^\sigma(\mathfrak{a}, s)| = \nu_1 \,\&\, |\hat{s}\overline{S}^\sigma(\mathfrak{a}, s)| < |\hat{s}\overline{R}^\varrho(\mathfrak{a}, s)|]\big]$$
$$= \hat{\mathfrak{a}}\big[(a)(Ex)R^\varrho\big(\mathfrak{a}, \overline{a}(x)\big) \,\&\, \overline{(E\psi)}[\psi \text{ is a } \prec\text{-isomorphism of } \hat{s}\overline{S}^\sigma(\mathfrak{a}, s) \text{ into a proper segment of } \hat{s}\overline{R}^\varrho(\mathfrak{a}, s)]\big].$$

[8] Cf. e. g. (1)-(6) of [10], p. 315.

Now the scope of $(E\psi)$ is easily shown to be arithmetical in ϱ, σ, so bringing quantifiers to the front, absorbing quantifiers over N into those over N^N [8], and contracting like quantifiers [8], we have the desired prefix. The resulting matrix will be recursive in ϱ, σ, and hence by (1), in some τ in C. So $X_1 \in \Pi_1^{1[C]}$.

CASE III. $X, Y \in \Sigma_2^{1[C]}$. Then for some ϱ, σ in C and R^ϱ, S^σ recursive in ϱ, σ, respectively,

$$X = \hat{\mathfrak{a}}(E\beta)(E\nu)|\hat{s}\overline{R}^\varrho(\mathfrak{a}, \beta, s)| = \nu,$$
$$Y = \hat{\mathfrak{a}}(E\beta)(E\nu)|\hat{s}\overline{S}^\sigma(\mathfrak{a}, \beta, s)| = \nu.$$

The possibility of representation in this form follows from the discussion in Case II, of course. Now like quantifiers over any ranges commute so

$$X = \hat{\mathfrak{a}}(E\nu)(E\beta)|\hat{s}\overline{R}^\varrho(\mathfrak{a}, \beta, s)| = \nu,$$
$$Y = \hat{\mathfrak{a}}(E\nu)(E\beta)|\hat{s}\overline{S}^\sigma(\mathfrak{a}, \beta, s)| = \nu.$$

Hence

$$X_1 = \hat{\mathfrak{a}}(E\nu)[(E\beta)|\hat{s}\overline{R}^\varrho(\mathfrak{a}, \beta, s)| = \nu \,\&\, \overline{(E\nu_1)}_{\nu_1<\nu}(E\beta)|\hat{s}\overline{S}^\sigma(\mathfrak{a}, \beta, s)| = \nu_1]$$
$$= [\text{by the rules of the predicate calculus with equality}],$$
$$\hat{\mathfrak{a}}(E\beta)\big[(E\nu)|\hat{s}\overline{R}^\varrho(\mathfrak{a}, \beta, s)| = \nu \,\&\, (\beta_1)\overline{(E\nu_1)}[|\hat{s}\overline{S}^\sigma(\mathfrak{a}, \beta_1, s)| = \nu_1 \,\&\, |\hat{s}\overline{S}^\sigma(\mathfrak{a}, \beta_1, s)| < |\hat{s}\overline{R}^\varrho(\mathfrak{a}, \beta, s)|]\big].$$

Now the argument concluding Case II is repeated exactly to give $X_1 \in \Sigma_2^{1[C]}$.

CASE IV. $X, Y \in \Sigma_k^{1[C]}$ $(k \geqslant 3)$. To illustrate we fix $k = 3$. Then for some ϱ, σ in C and R^ϱ, S^σ recursive in ϱ, σ, respectively,

$$X = \hat{\mathfrak{a}}(E\gamma)(\beta)(Ea)(x)R^\varrho(\mathfrak{a}, \gamma, \beta, a, x),$$
$$Y = \hat{\mathfrak{a}}(E\gamma)(\beta)(Ea)(x)S^\sigma(\mathfrak{a}, \gamma, \beta, a, x).$$

So

$$X_1 = \hat{\mathfrak{a}}(E\gamma)[(\beta)(Ea)(x)R^\varrho(\mathfrak{a}, \gamma, \beta, a, x) \,\&\, \overline{(E\gamma_1)}_{\gamma_1<\gamma}(\beta)(Ea)(x)S^\sigma(\mathfrak{a}, \gamma_1, \beta, a, x)]$$
$$= \hat{\mathfrak{a}}(E\gamma)[(\beta)(Ea)(x)R^\varrho(\mathfrak{a}, \gamma, \beta, a, x) \,\&\, (E\beta)(\gamma_1)_{\gamma_1<\gamma}(Ei)(a)(Ex)\overline{S}^\sigma(\mathfrak{a}, \gamma_1, \lambda t\beta(p_i^t), a, x)].$$

Now writing $\gamma_1 < \gamma$ as $(E\chi_1)(\chi_2)(Ey)T(\gamma_1, \gamma, \chi_1, \chi_2, y)$ for some recursive T (possible since by [4] $<$ is in $\Sigma_2^1 \cap \Pi_2^1$), bringing quantifiers to the front, absorbing quantifiers over N into those over N^N [8], and contracting like quantifiers [8], we have the desired prefix. The resulting matrix will be recursive in ϱ, σ, and hence by (1) in some τ in C. So $X_1 \in \Sigma_3^{1[C]}$.

Note that the device of interchanging quantifiers of opposite kind used in Case IV, which is a generalization of the device employed in Case I, succeeds because the Gödel well-ordering $<$ is an Ω_1-well-ordering (and would have failed if, for example, $<$ had been an Ω_2-well-ordering).

Just as the device for interchanging quantifiers of opposite kind used in Case I fails to handle the case of $k = 1$, so here the generalization of this device fails to handle the case of $k = 2$. And just as in Case I a more complicated argument can be used to handle the case of $k = 1$, so here a more complicated argument can be used to push through the case of $k = 2$ for X_1, Y_1 defined in exact analogy with those in Case IV. (Of course we have seen in Case III that the case of $k = 2$ can be handled using different X_1, Y_1 even without the use of the axiom of constructibility; yet it is still of considerable interest to see that the same kind of X_1, Y_1 which work for $k \geqslant 3$ also works for $k = 2$). We will present this argument, together with other applications of the same argument, elsewhere (⁹).

6. The negations of the separation principles. A subset $\hat{\mathfrak{a}}^{n,f}\hat{\mathfrak{a}}^{n',f'}U(\mathfrak{a}^{n,f},\mathfrak{a}^{n',f'})$ of $\mathcal{N}^{n+n',f+f'}$ is called *universal* for a class Q of subsets of $\mathcal{N}^{n,f}$ if and only if

$$X \in Q \equiv (E\mathfrak{a}^{n',f'})[X = \hat{\mathfrak{a}}^{n,f}U(\mathfrak{a}^{n,f},\mathfrak{a}^{n',f'})].$$

A pair $\big(\hat{\mathfrak{a}}^{n,f}\hat{\mathfrak{a}}^{n',f'}U(\mathfrak{a}^{n,f},\mathfrak{a}^{n',f'}), \hat{\mathfrak{a}}^{n,f}\hat{\mathfrak{a}}^{n',f'}V(\mathfrak{a}^{n,f},\mathfrak{a}^{n',f'})\big)$ of subsets of $\mathcal{N}^{n+n',f+f'}$ is called *doubly universal* for a class Q of subsets of $\mathcal{N}^{n,f}$ if and only if

$$X, Y \in Q \equiv (E\mathfrak{a}^{n',f'})[X = \hat{\mathfrak{a}}^{n,f}U(\mathfrak{a}^{n,f},\mathfrak{a}^{n',f'})\ \&\ Y = \hat{\mathfrak{a}}^{n,f}V(\mathfrak{a}^{n,f},\mathfrak{a}^{n',f'})].$$

The variable $\mathfrak{a}^{n',f'}$ is referred to as the *parameter of universality* (or *of double universality*). In our considerations it will be sufficient always to assume that $n' = 0$, $f' = 1$ or that $n' = 1$, $f' = 0$. It is important to note that the definitions imply that Q must have the cardinality of the range of the parameter of universality (or of double universality).

It is easy to see that if there is a pair (U, V) of sets from Q doubly universal for Q, whereas there is no set in $Q \cap cQ$ universal for $Q \cap cQ$, then $Red(Q) \to \overline{Sep_I}(Q)$. For if U, V were reduced to U_1, V_1 (according to $Red(Q)$), and these were separated by U_2 (according to $Sep_I(Q)$), U_2 would be a $Q \cap cQ$ set universal for $Q \cap cQ$.

(⁹) There does not appear to be any difficulty in generalizing the results of this section from the case of two sets to the case of a countable infinity of sets, if the indices (defined in a natural manner) of the infinity of sets are enumerable by a function recursive in some function in C, or alternatively, if the infinity of sets is represented by one predicate with a parameter ranging over N which is of the class in question as a predicate of all its variables, including the parameter.

It is also easily seen that for an arbitrary class Q of sets

$$(9) \qquad Red(Q) \to [Sep_I(Q) \equiv Sep_{II}(Q)].$$

We now consider the C-arithmetical and C-analytical hierarchies in the light of these remarks, breaking our discussion up into two cases according to the type of our space $\mathcal{N}^{n,f}$.

CASE I. $f > 0$. It is well known that for any t, k ($t = 0, 1$; $k \geqslant 1$) there is a $\boldsymbol{\Sigma}_k^t$ set universal for $\boldsymbol{\Sigma}_k^t$ and a $\boldsymbol{\Pi}_k^t$ set universal for $\boldsymbol{\Pi}_k^t$. Using simple pairing functions one can then use these to obtain pairs of $\boldsymbol{\Sigma}_k^t$ sets doubly universal for $\boldsymbol{\Sigma}_k^t$ and pairs of $\boldsymbol{\Pi}_k^t$ sets doubly universal for $\boldsymbol{\Pi}_k^t$. On the other hand for no t, k ($t = 0, 1$; $k \geqslant 1$) is there a $\boldsymbol{\Sigma}_k^t \cap \boldsymbol{\Pi}_k^t$ set universal for $\boldsymbol{\Sigma}_k^t \cap \boldsymbol{\Pi}_k^t$, as the standard diagonal argument shows. Thus the argument outlined above applies, and we can conclude form the results of section 4 that $\overline{Sep_I}(Q)$, and by (9) and (5) also $\overline{Sep_{II}}(Q)$ and $\overline{Red}(cQ)$, for $Q = \boldsymbol{\Sigma}_k^0, \boldsymbol{\Pi}_1^1, \boldsymbol{\Sigma}_{k+1}^1$ ($k \geqslant 1$).

But actually we can strengthen our argument a little and conclude much more. Consider, for example, the $\boldsymbol{\Sigma}_1$ or open sets. It is easy to show that there is even a Σ_1 subset of $\mathcal{N}^{n,f+1}$ universal for $\boldsymbol{\Sigma}_1$ subsets of $\mathcal{N}^{n,f}$. (For example, $\hat{\varphi}_1 \ldots \hat{\varphi}_f \hat{x}_1 \ldots \hat{x}_n \hat{\psi}(Ea_1) \ldots (Ea_f)(Eb)[p_1^{\varphi_1(a_1)} \cdot \ldots \cdot p_f^{\varphi_f(a_f)} \cdot p_{f+1}^{x_1} \cdot \ldots \cdot p_{f+n}^{x_n} = \psi(b)]$ is such a set.) Similarly there is a pair of Σ_1 sets doubly universal for $\boldsymbol{\Sigma}_1$ sets. Now this pair can be reduced, as argued above, to yield two disjoint Σ_1 sets which are not separable even by a $\boldsymbol{\Sigma}_1 \cap \boldsymbol{\Pi}_1$ set. The same argument can be applied to actually name two disjoint Σ_{k+1} sets (Π_1^1 sets, Σ_{k+1}^1 sets) not separable by $\boldsymbol{\Sigma}_{k+1} \cap \boldsymbol{\Pi}_{k+1}$ sets (by $\boldsymbol{\Sigma}_1^1 \cap \boldsymbol{\Pi}_1^1$ sets, by $\boldsymbol{\Sigma}_{k+1}^1 \cap \boldsymbol{\Pi}_{k+1}^1$ sets), for $k \geqslant 1$. Similar strong forms of the negation of the second separation principle can likewise be obtained at all levels.

Now for any subsets C_1, C_2 of N^N, if $C_1 \subseteq C_2$, then $\Sigma_k^{t[C_1]} \subseteq \Sigma_k^{t[C_2]}$ and $\Pi_k^{t[C_1]} \subseteq \Pi_k^{t[C_2]}$. Thus the above results yield, by weakening them in two directions, that for any C $\overline{Sep_I}(\Sigma_k^{t[C]})$, $\overline{Sep_I}(\Pi_1^{1[C]})$, in case $t = 0$, $k \geqslant 1$ or $t = 1$, $k \geqslant 2$. And these results yield a simple proof, incidentally, that for any C the C-arithmetical and C-analytical hierarchies are true hierarchies.

CASE II. $f = 0$. Here there is much degeneracy and many C do not yield true hierarchies. This is true, for example, when $C = N^N$, and for this case the argument given above does not work. For although there exists a pair of $\boldsymbol{\Sigma}_1$ sets doubly universal for $\boldsymbol{\Sigma}_1$, there also exists a $\boldsymbol{\Sigma}_1 \cap \boldsymbol{\Pi}_1$ set universal for $\boldsymbol{\Sigma}_1 \cap \boldsymbol{\Pi}_1$. The diagonal argument which would usually preclude the latter conclusion breaks down here since the parameter of universality ranges over N^N, which has greater cardinality than the space $\mathcal{N}^{n,0}$.

For some C an argument similar to the above can be applied, however. We illustrate with the case where C is finite. Here we can choose a $\Sigma_1^{[C]}$ subset of $\mathcal{N}^{n+1,0}$ universal for the $\Sigma_1^{[C]}$ subsets of $\mathcal{N}^{n,0}$, whereas the diagonal argument does prevent the existence of a $\Sigma_1^{[C]} \cap \Pi_1^{[C]}$ subset of $\mathcal{N}^{n+1,0}$ universal for $\Sigma_1^{[C]} \cap \Pi_1^{[C]}$ subsets of $\mathcal{N}^{n,0}$. So we can conclude $\overline{Sep}_{\mathrm{I}}(\Sigma_1^{[C]})$ as before. Similar arguments work, of course, at all levels of the C-arithmetical and C-analytical hierarchies with finite C.

7. Summary. The first and second separation principles hold on the universal side and fail on the existential side of the arithmetical, finite Borel, analytical, and projective hierarchies, with the exception of the first level of the analytical and projective hierarchies, where the situation is exactly reversed. (At the third and higher levels of the analytical and projective hierarchies our proofs require the axiom of constructibility.)

For all four hierarchies the reduction principle holds wherever the separation principles fail and fails wherever they hold. Whenever the reduction or separation principles fail for a class in these hierarchies they fail, in a certain sense, by very much. The results can be extended to wide classes of hierarchies of which the four mentioned are special cases.

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