

Sequence Space $m(M, \phi)^F$ of Fuzzy Real Numbers Defined by Orlicz Functions with Fuzzy Metric

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ABSTRACT. The sequence space $m(M, \phi)^F$ of fuzzy real numbers is introduced. Some properties of this sequence space like solidness, symmetricity, convergence-free etc. are studied. We obtain some inclusion relations involving this sequence space.

1. Introduction

The concept of *fuzzy set theory* was introduced by L.A. Zadeh in the year 1965. Later on different classes of sequences of fuzzy numbers have been investigated by Esi [2], Nuray and Savas [6], Syau [9], Tripathy and Baruah ([13], [14], [15]), Tripathy and Borgohain [16], Tripathy and Dutta ([17], [18]), Tripathy and Sarma [20] and many others.

An *Orlicz function* is a function $M : [0, \infty) \rightarrow [0, \infty)$, which is *continuous*, *non-decreasing* and *convex* with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \rightarrow \infty$, as $x \rightarrow \infty$.

If the *convexity* of the Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called as *modulus function*.

Remark. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Sargent [8] introduced the crisp set sequence space $m(\phi)$ and studied some properties of this space. Later on it was studied from the sequence space point of view and some matrix classes were characterized with one member as $m(\phi)$ by Rath and Tripathy [7], Tripathy [10] and others. In this article we introduce the space $m(M, \phi)^F$ of fuzzy real numbers defined by Orlicz function.

Throughout the article $w^F, \ell^F, \ell_\infty^F$ represent the classes of *all*, *absolutely*

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summable and *bounded* sequences of fuzzy real numbers respectively.

2. Definitions and Background

Definition 2.1. A *fuzzy real number* X is a fuzzy set on R i.e. a mapping $X : R \rightarrow I (= [0, 1])$ associating each real number t with its grade of membership $X(t)$.

Definition 2.2. A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

Definition 2.3. If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

Definition 2.4. A fuzzy real number X is said to be *upper semi-continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in I$ is open in the usual topology of R .

The class of all *upper semi-continuous, normal, convex* fuzzy real numbers is denoted by $R(I)$.

Definition 2.5. For $X \in R(I)$, the α -level set X^α , for $0 < \alpha \leq 1$ is defined by, $X^\alpha = \{t \in R : X(t) \geq \alpha\}$. The 0-level set of X i.e. X^0 is the closure of strong 0-cut, i.e. $cl\{t \in R : X(t) > 0\}$.

Definition 2.6. The *absolute value* of $X \in R(I)$ is defined by,

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{for } t \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.7. For $r \in R$ and $\bar{r} \in R(I)$ is defined as,

$$\bar{r}(t) = \begin{cases} 1 & \text{if } t = r; \\ 0 & \text{if } t \neq r. \end{cases}$$

Definition 2.8. The *additive* and *multiplicative identities* of $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$.

Definition 2.9. Let D be the set of all closed bounded intervals $X = [X^L, X^R]$.

Define $d : D \times D \rightarrow R$ by $d(X, Y) = \max\{|X^L - Y^L|, |X^R - Y^R|\}$. Then clearly (D, d) is a *complete metric space*.

Define $\bar{d} : R(I) \times R(I) \rightarrow R$ by $\bar{d}(X, Y) = \sup_{0 < \alpha \leq 1} d(X^\alpha, Y^\alpha)$, for $X, Y \in R(I)$.

Then it is well known that $(R(I), \bar{d})$ is a *complete metric space*.

Definition 2.10. A sequence $X = (X_k)$ of fuzzy real numbers is said to *converge* to the fuzzy number X_0 , if for every $\varepsilon > 0$, there exists $k_0 \in N$ such that $\bar{d}(X_k, X_0) < \varepsilon$, for all $k \geq k_0$.

Definition 2.11. A sequence space E is said to be *solid* if $(Y_n) \in E$, whenever $(X_n) \in E$ and $|Y_n| \leq |X_n|$, for all $n \in N$.

Definition 2.12. Let $X = (X_n)$ be a sequence, then $S(X)$ denotes the set of all permutations of the elements of (X_n) i.e. $S(X) = \{(X_{\pi(n)}) : \pi \text{ is a permutation of } N\}$. A sequence space E is said to be *symmetric* if $S(X) \subset E$ for all $X \in E$.

Definition 2.13. A sequence space E is said to be *convergence-free* if $(Y_n) \in E$ whenever $(X_n) \in E$ and $X_n = \bar{0}$ implies $Y_n = \bar{0}$.

Definition 2.14. A sequence space E is said to be *monotone* if E contains the canonical pre-images of all its step spaces.

Lemma 2.1. A sequence space E is solid implies that E is monotone.

Definition 2.15. Let \wp_s be the class of all subsets of N those do not contain more than s number of elements. Throughout (ϕ_n) is a non-decreasing sequence of positive real numbers such that $n\phi_{n+1} \leq (n + 1)\phi_n$ for all $n \in N$.

The space $m(\phi)$ introduced by Sargent [8] is defined by,

$$m(\phi) = \left\{ (x_k) \in w : \|x\|_{m(\phi)} = \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| < \infty \right\}.$$

Afterwards different types of generalizations of the classes of sequences $m(\phi)$ was introduced and investigated by Rath and Tripathy [7], Tripathy ([10], [11]) and many others.

Definition 2.16. Lindenstrauss and Tzafriri [5] used the notion of Orlicz function and introduced the sequence space:

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right), \text{ for some } \rho > 0 \right\}$$

The space ℓ_M with the norm,

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a *Banach space*, which is called an *Orlicz sequence space*. The space ℓ_M is closely related to the space ℓ_p , which is an Orlicz sequence space with $M(x) = x^p$, for $1 \leq p \leq \infty$.

In the later stage different classes of Orlicz sequence spaces were introduced and studied by Altin, Et and Tripathy [1], Esi [2], Tripathy, Altin and Et [12], Tripathy and Mahanta [19], Tripathy and Sarma ([20], [21]) and many others.

Definition 2.17. Let $d_F : R(I) \times R(I) \rightarrow R(I)$ be the *fuzzy metric*. Let the mappings $L, M : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be symmetric, non-decreasing in both arguments and satisfy, $L[0, 0] = 0$ and $M[1, 1] = 1$. i.e. $L = \min\{p, q\}$ and $M = \max\{p, q\}$, where $p, q \in [0, 1]$.

Let $\lambda : R(I) \times R(I) \rightarrow R$ such that $\lambda(X, Y) = \sup_{0 < \alpha \leq 1} \lambda_\alpha(X^\alpha, Y^\alpha)$, where $\lambda_\alpha : R \times R \rightarrow R$ and $\lambda_\alpha(X^\alpha, Y^\alpha) = \min\{|X_1^\alpha - Y_1^\alpha|, |X_2^\alpha - Y_2^\alpha|\}$.

Similarly, let $\rho : R(I) \times R(I) \rightarrow R$ be such that $\rho(X, Y) = \sup_{0 < \alpha \leq 1} \rho_\alpha(X^\alpha, Y^\alpha)$, where $\rho_\alpha : R \times R \rightarrow R$ and $\rho_\alpha(X^\alpha, Y^\alpha) = \max\{|X_1^\alpha - Y_1^\alpha|, |X_2^\alpha - Y_2^\alpha|\}$.

Since the distance between two fuzzy numbers is again a fuzzy number, so the α -level set of this distance d_F between the fuzzy real numbers X and Y is denoted by,

$$[d(X, Y)]_\alpha = [\lambda_\alpha(X^\alpha, Y^\alpha), \rho_\alpha(X^\alpha, Y^\alpha)], 0 < \alpha \leq 1.$$

The quadruple $(R(I), d_F, M, N)$ is called a *fuzzy metric space* and d_F is a fuzzy metric, if,

1. $d_F(X, Y) = \bar{0}$ if and only if $X = Y$.
2. $d_F(X, Y) = d_F(Y, X)$, for all $X, Y \in R(I)$.
3. For all $X, Y, Z \in R(I)$,
 - (i) $d_F(X, Y)(s + t) \geq L(d_F(X, Z)(s), d_F(Z, Y)(t))$, whenever $s \leq \lambda_1(X, Z)$, $t \leq \lambda_1(Z, Y)$ and $s + t \leq \lambda_1(X, Y)$.
 - (ii) $d_F(X, Y)(s + t) \leq R(d_F(X, Z)(s), d_F(Z, Y)(t))$, whenever $s \geq \lambda_1(X, Z)$, $t \geq \lambda_1(Z, Y)$ and $s + t \geq \lambda_1(X, Y)$.

Using the concept of Orlicz function and fuzzy metric, we introduce the following sequence spaces,

$$m(M, \phi)^F = \left\{ (X_k) \in w^F : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right); \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k, \bar{0})}{r} \right) \right\},$$

for all $r > 0$

3. Main Results

Theorem 3.1. *The sequence space $m(M, \phi)^F$ is a metric space with the metric defined by,*

$$\bar{d}(X, Y)_M = \inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) \leq 1; \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k, \bar{0})}{r} \right) \leq 1 \right\},$$

for $X, Y \in m(M, \phi)^F$

Proof. Let $X, Y, Z \in m(M, \phi)^F$.

(i) $\bar{d}(X, Y)_M = 0$.

This implies,

$$\lambda(X_k, Y_k) = 0 \text{ and } \rho(X_k, Y_k) = 0, \text{ for all } k \in N. (\text{ Since } M(0) = 0).$$

Which implies,

$$\sup_{0 < \alpha \leq 1} \lambda_\alpha(X_k^\alpha, Y_k^\alpha) = 0 \Rightarrow \lambda_\alpha(X_k^\alpha, Y_k^\alpha) = 0, \text{ for all } \alpha \in (0, 1].$$

$$(3.1) \quad \Rightarrow \min\{|X_{k1}^\alpha - Y_{k1}^\alpha|, |X_{k2}^\alpha - Y_{k2}^\alpha|\} = 0, \text{ for all } \alpha \in (0, 1].$$

$$\sup_{0 < \alpha \leq 1} \rho_\alpha(X_k^\alpha, Y_k^\alpha) = 0 \Rightarrow \rho_\alpha(X_k^\alpha, Y_k^\alpha) = 0, \text{ for all } \alpha \in (0, 1].$$

$$(3.2) \quad \Rightarrow \max\{|X_{k1}^\alpha - Y_{k1}^\alpha|, |X_{k2}^\alpha - Y_{k2}^\alpha|\} = 0, \text{ for all } \alpha \in (0, 1].$$

From (3.1) and (3.2), it follows that, $X_k = Y_k \Rightarrow X = Y$.

Conversely, assume that, $X = Y$. Then, using the definition of λ and ρ , we get,

$$\lambda_\alpha(X_k^\alpha, Y_k^\alpha) = 0 \text{ and } \rho_\alpha(X_k^\alpha, Y_k^\alpha) = 0, \text{ for all } k \in N, \alpha \in (0, 1].$$

Which implies,

$$\sup_{0 < \alpha \leq 1} \lambda_\alpha(X_k^\alpha, Y_k^\alpha) = 0 \text{ and } \sup_{0 < \alpha \leq 1} \rho_\alpha(X_k^\alpha, Y_k^\alpha) = 0, \text{ for all } k \in N.$$

It follows that, $\lambda(X_k, Y_k) = 0$ and $\rho(X_k, Y_k) = 0$.

Using the continuity of M , we get, $\bar{d}(X, Y)_M = 0$. Which shows that, $\bar{d}(X, Y)_M = 0$ if and only if $X = Y$.

(ii) $\bar{d}(X, Y)_M$

$$= \inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, Y_k)}{r} \right) \leq 1; \sup_{s \geq 1, \sigma \in \varphi_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k, Y_k)}{r} \right) \leq 1 \right\}.$$

From the definition of λ , it follows,

$$\begin{aligned} \lambda(X_k, Y_k) &= \sup_{0 < \alpha \leq 1} \lambda_\alpha(X_k^\alpha, Y_k^\alpha) \\ &= \sup_{0 < \alpha \leq 1} [\min\{|X_{k1}^\alpha - Y_{k1}^\alpha|, |X_{k2}^\alpha - Y_{k2}^\alpha|\}] \\ &= \sup_{0 < \alpha \leq 1} [\min\{|Y_{k1}^\alpha - X_{k1}^\alpha|, |Y_{k2}^\alpha - X_{k2}^\alpha|\}] \\ &= \sup_{0 < \alpha \leq 1} \lambda_\alpha(Y_k^\alpha, X_k^\alpha) \\ &= \lambda(Y_k, X_k). \end{aligned}$$

Proceeding in the same way, we get, $\rho(X_k, Y_k) = \rho(Y_k, X_k)$. Thus we get,

$$\begin{aligned} & \inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, Y_k)}{r} \right) \leq 1; \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k, Y_k)}{r} \right) \leq 1 \right\} \\ &= \inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(Y_k, X_k)}{r} \right) \leq 1; \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(Y_k, X_k)}{r} \right) \leq 1 \right\} \\ &= \bar{d}(Y, X)_M. \text{ Hence, } \bar{d}(X, Y)_M = \bar{d}(Y, X)_M. \end{aligned}$$

(iii) Let $r_1, r_2 > 0$ such that,

$$\begin{aligned} \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, Z_k)}{r_1} \right) &\leq 1. \\ \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(Z_k, Y_k)}{r_2} \right) &\leq 1. \end{aligned}$$

Let $r = r_1 + r_2$, then we have,

$$\begin{aligned} & \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, Y_k)}{r} \right) \\ &\leq \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, Z_k)}{r_1 + r_2} + \frac{\lambda(Z_k, Y_k)}{r_1 + r_2} \right) \\ &\leq \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{r_1}{r_1 + r_2} \left(\frac{\lambda(X_k, Z_k)}{r_1} \right) + \frac{r_2}{r_1 + r_2} \left(\frac{\lambda(Z_k, Y_k)}{r_2} \right) \right) \\ &\leq \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \frac{r_1}{r_1 + r_2} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, Z_k)}{r_1} \right) \\ &\quad + \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \frac{r_2}{r_1 + r_2} \sum_{k \in \sigma} M \left(\frac{\lambda(Z_k, Y_k)}{r_2} \right) \\ &\leq 1. \end{aligned}$$

Since r 's are non-negative, so taking the infimum of such r 's, we get,

$$\begin{aligned} & \inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, Y_k)}{r} \right) \leq 1 \right\} \\ &\leq \inf \left\{ r_1 > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, Z_k)}{r_1} \right) \leq 1 \right\} \\ &\quad + \inf \left\{ r_2 > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(Z_k, Y_k)}{r_2} \right) \leq 1 \right\} \end{aligned}$$

Proceeding in the same way, we get,

$$\inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k, Y_k)}{r} \right) \leq 1 \right\}$$

$$\leq \inf \left\{ r_1 > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k, Z_k)}{r_1} \right) \leq 1 \right\} \\ + \inf \left\{ r_2 > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(Z_k, Y_k)}{r_2} \right) \leq 1 \right\}$$

Thus we have,

$$\inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, Y_k)}{r} \right) \leq 1; \right. \\ \left. \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k, Y_k)}{r} \right) \leq 1 \right\} \\ \leq \inf \left\{ r_1 > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, Z_k)}{r_1} \right) \leq 1; \right. \\ \left. \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k, Z_k)}{r_1} \right) \leq 1 \right\} \\ + \inf \left\{ r_2 > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(Z_k, Y_k)}{r_2} \right) \leq 1; \right. \\ \left. \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(Z_k, Y_k)}{r_2} \right) \leq 1 \right\}$$

$$\Rightarrow \bar{d}(X, Y)_M \leq \bar{d}(X, Z)_M + \bar{d}(Z, Y)_M.$$

This proves that $m(M, \phi)^F$ is a metric space. □

Theorem 3.2. *The sequence space $m(M, \phi)^F$ is a complete metric space with the metric defined by,*

$$\bar{d}(X, Y)_M = \inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, Y_k)}{r} \right) \leq 1; \right. \\ \left. \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k, Y_k)}{r} \right) \leq 1 \right\},$$

for $X, Y \in m(M, \phi)^F$

Proof. Let $(X^{(i)})$ be a Cauchy sequence in $m(M, \phi)^F$ such that, $X^{(i)} = (X_n^{(i)})_{n=1}^\infty$.

Let $\varepsilon > 0$ be given. For a fixed $x_0 > 0$, choose $p > 0$ such that $M \left(\frac{px_0}{2} \right) \geq 1$.

Then there exists a positive integer $n_0 = n_0(\varepsilon)$ such that,

$$\bar{d}(X^{(i)}, X^{(j)})_M < \frac{\varepsilon}{px_0}, \text{ for all } i, j \geq n_0.$$

By the definition of \bar{d}_M , we get;

$$(3.3) \quad \inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k^{(i)}, X_k^{(j)})}{r} \right) \leq 1; \right. \\ \left. \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k^{(i)}, X_k^{(j)})}{r} \right) \leq 1 \right\} < \varepsilon$$

for all $i, j \geq n_0$. Which implies,

$$(3.4) \quad \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k^{(i)}, X_k^{(j)})}{r} \right) \leq 1$$

$$(3.5) \quad \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k^{(i)}, X_k^{(j)})}{r} \right) \leq 1$$

From (3.4) we get,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k^{(i)}, X_k^{(j)})}{r} \right) \leq 1$$

On taking $s = 1$ and varying σ over \wp_s , we get,

$$\sum_{k \in \sigma} M \left(\frac{\lambda(X_k^{(i)}, X_k^{(j)})}{r} \right) \leq \phi_1, \text{ for all } i, j \geq n_0. \\ \Rightarrow M \left(\frac{\lambda(X_k^{(i)}, X_k^{(j)})}{\bar{d}(X^{(i)}, X^{(j)})} \right) \leq \phi_1 \leq M \left(\frac{px_0}{2} \right).$$

Using the continuity of M , we get,

$$\lambda_\alpha(X_k^{(i)}, X_k^{(j)}) \leq \frac{px_0}{2} \cdot \frac{\varepsilon}{px_0} = \frac{\varepsilon}{2},$$

i.e $(X_k^{(i)})$ is a Cauchy sequence of $R(I)$. Since $R(I)$ is complete, so it follows that, $(X_k^{(i)})$ is also convergent.

Let, $\lim_i X_k^{(i)} = X_k$, for each $k \in N$. We have to prove that,

$$\lim_i X^{(i)} = X \text{ and } X \in m(M, \phi)^F.$$

Since M is continuous, so on taking $j \rightarrow \infty$ and fixing i , we get from (3.4);

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k^{(i)}, X_k)}{r} \right) \leq 1, \text{ for some } r > 0 \text{ and } i \geq n_0.$$

Proceeding in the same way, we get from (3.5):

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k^{(i)}, X_k)}{r} \right) \leq 1, \text{ for some } r > 0 \text{ and } i \geq n_0.$$

Now on taking the infimum of such r 's together, we get from (3.3):

$$\inf \left\{ r > 0 : \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k^{(i)}, X_k)}{r} \right); \right. \\ \left. \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k^{(i)}, X_k)}{r} \right) \leq 1 \right\} < \varepsilon,$$

for some $r > 0$ and $i \geq n_0$. Which shows, $\bar{d}(X^{(i)}, X)_M < \varepsilon$, for all $i \geq n_0$. i.e. $\lim_i X^{(i)} = X$.

Now, to show that $X \in m(M, \phi)^F$. We have,

$$\begin{aligned} \bar{d}(X, \bar{\theta})_M &\leq \bar{d}(X, X^{(i)})_M + \bar{d}(X^{(i)}, \bar{\theta})_M \\ &< \varepsilon + M, \text{ for all } i \geq n_0(\varepsilon). \end{aligned}$$

i.e. $\bar{d}(X, \bar{\theta})_M$ is finite. Which implies $x \in m(M, \phi)^F$. Hence $m(M, \phi)^F$ is a complete metric space. This completes the proof of the theorem. Proofs are similar for other spaces also. \square

Theorem 3.3. *The sequence space $m(M, \phi)^F$ is solid.*

Proof. Let $(X_k) \in m(M, \phi)^F$. Then we have, for some $r > 0$,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda_\alpha(X_k, \bar{0})}{r} \right) < \infty; \quad \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho_\alpha(X_k, \bar{0})}{r} \right) < \infty.$$

Let (Y_k) be a sequence of fuzzy numbers with,

$$[d(Y_k, \bar{0})]_\alpha = [\lambda_\alpha(Y_k^\alpha, 0), \rho_\alpha(Y_k^\alpha, o)], \text{ for } 0 < \alpha \leq 1,$$

Such that, $\lambda(Y_k, \bar{0}) \leq \lambda(X_k, \bar{0})$ and $\rho(Y_k, \bar{0}) \leq \rho(X_k, \bar{0})$.

Since M is non-decreasing continuous function, so we get, for some $r > 0$,

$$M \left(\frac{\lambda(Y_k, \bar{0})}{r} \right) \leq M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) \text{ and } M \left(\frac{\rho(Y_k, \bar{0})}{r} \right) \leq M \left(\frac{\rho(X_k, \bar{0})}{r} \right).$$

Which implies,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left(\frac{\lambda(Y_k, \bar{0})}{r} \right) \leq \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) < \infty, \text{ for some } r > 0.$$

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left(\frac{\rho(Y_k, \bar{0})}{r} \right) \leq \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left(\frac{\rho(X_k, \bar{0})}{r} \right) < \infty, \text{ for some } r > 0.$$

Which implies,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left(\frac{\lambda(Y_k, \bar{0})}{r} \right) < \infty, \text{ for some } r > 0.$$

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left(\frac{\rho(Y_k, \bar{0})}{r} \right) < \infty, \text{ for some } r > 0.$$

Which shows, $(Y_k) \in m(M, \phi)^F$. Hence, $m(M, \phi)^F$ is solid. This completes the proof. \square

Theorem 3.4. *The sequence space $m(M, \phi)^F$ is symmetric.*

Proof. Let $(X_k) \in m(M, \phi)^F$ and (Y_k) be a rearrangement of (X_k) , such that,

$$X_k = Y_{m_k}, \text{ for each } k \in N.$$

Then, we have, $\lambda(X_k, \bar{0}) = \lambda(Y_{m_k}, \bar{0})$ and $\rho(X_k, \bar{0}) = \rho(Y_{m_k}, \bar{0})$.

Using the continuity of M , we get,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) = \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left(\frac{\lambda(Y_{m_k}, \bar{0})}{r} \right), \text{ for some } r > 0.$$

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left(\frac{\rho(X_k, \bar{0})}{r} \right) = \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left(\frac{\rho(Y_{m_k}, \bar{0})}{r} \right), \text{ for some } r > 0.$$

Which implies,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left(\frac{\lambda(Y_{m_k}, \bar{0})}{r} \right) < \infty \text{ and } \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} M \left(\frac{\rho(Y_{m_k}, \bar{0})}{r} \right) < \infty,$$

for some $r > 0$. Which shows, $(Y_k) \in m(M, \phi)^F$. Hence $m(M, \phi)^F$ is symmetric. This completes the proof. \square

Proposition 3.1. *The sequence space $m(M, \phi)^F$ is not convergence-free.*

Proof. The result follows from the following example.

Example 3.1. Consider the sequence (X_k) defined as follows:

$$X_k(t) = \begin{cases} 1 + kt, & \text{for } t \in [-\frac{1}{k}, 0] \\ 1 - kt, & \text{for } t \in [0, \frac{1}{k}] \\ 0 & \text{otherwise} \end{cases}$$

Then we have, for some $r > 0$,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) < \infty$$

and

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k, \bar{0})}{r} \right) < \infty$$

Which shows, $(X_k) \in m(M, \phi)^F$.

Now, let us take another sequence (Y_k) such that,

$$Y_k(t) = \begin{cases} 1 + \frac{t}{k^2}, & \text{for } t \in [-k^2, 0] \\ 1 - \frac{t}{k^2}, & \text{for } t \in [0, k^2] \end{cases}$$

But,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(Y_k, \bar{0})}{r} \right) = \infty \text{ and } \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\rho(Y_k, \bar{0})}{r} \right) = \infty$$

Thus, $(Y_k) \notin m(M, \phi)^F$. Thus $m(M, \phi)^F$ is not convergence-free. This completes the proof. \square

Proposition 3.2. $m(M, \phi)^F \subseteq m(M, \phi, p)^F$, for $1 \leq p < \infty$.

Proof. Let $X \in m(M, \phi)^F$, then we have, for some $r > 0$,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) = K (< \infty)$$

Hence, for each fixed s , we have,

$$\begin{aligned} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) &\leq K \phi_s, \text{ for } \sigma \in \wp_s. \\ \Rightarrow \left[\sum_{k \in \sigma} \left(M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) \right)^p \right]^{\frac{1}{p}} &\leq K \phi_s, \text{ for } \sigma \in \wp_s. \\ \Rightarrow \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \left[\sum_{k \in \sigma} \left(M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) \right)^p \right]^{\frac{1}{p}} &\leq K. \\ \Rightarrow \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \left[\sum_{k \in \sigma} \left(M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) \right)^p \right]^{\frac{1}{p}} &< \infty. \end{aligned}$$

Proceeding in the same way, we get,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \left[\sum_{k \in \sigma} \left(M \left(\frac{\rho(X_k, \bar{0})}{r} \right) \right)^p \right]^{\frac{1}{p}} < \infty.$$

Which implies $X \in m(M, \phi, p)^F$, for $1 \leq p < \infty$.

This completes the proof. \square

Proposition 3.3. $m(M, \phi)^F \subseteq m(M, \psi)^F$ if and only if $\sup_{s \geq 1} \left(\frac{\phi_s}{\psi_s} \right) < \infty$, for $0 < p < \infty$.

Proof. Suppose, $\sup_{s \geq 1} \left(\frac{\phi_s}{\psi_s} \right) = K (< \infty)$, then we have, $\phi_s \leq K\psi_s$.

Now, if $(X_k) \in m(M, \phi)^F$, then,

$$\begin{aligned} & \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) < \infty. \\ \Rightarrow & \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{K\psi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) \leq \infty. \\ & \Rightarrow (X_k) \in m(M, \psi)^F. \end{aligned}$$

Hence, $m(M, \phi)^F \subseteq m(M, \psi)^F$.

Conversely, suppose that $m(M, \phi)^F \subseteq m(M, \psi)^F$. To show that, $\sup_{s \geq 1} \left(\frac{\phi_s}{\psi_s} \right) = \sup_{s \geq 1} (\eta_s) < \infty$.

Suppose, $\sup_{s \geq 1} (\eta_s) = \infty$. Then there exists a subsequence (η_{s_i}) of (η_s) such that,

$$\lim_{i \rightarrow \infty} (\eta_{s_i}) = \infty.$$

Then for $(X_k) \in m(M, \phi)^F$, we have,

$$\begin{aligned} \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\psi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) & \geq \sup_{s_i \geq 1, \sigma \in \wp_{s_i}} \frac{\eta_{s_i}}{\phi_{s_i}} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) = \infty. \\ \sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\psi_s} \sum_{k \in \sigma} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) & = \infty. \end{aligned}$$

Proceeding in the same way, we get,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\psi_s} \sum_{k \in \sigma} M \left(\frac{\rho(X_k, \bar{0})}{r} \right) = \infty.$$

Which implies that $(X_k) \notin m(M, \psi)^F$, a contradiction. This completes the proof. \square

Corollary 3.1. $m(M, \phi)^F = m(M, \psi)^F$ if and only if $\sup_{s \geq 1} (\eta_s) < \infty$ and $\sup_{s \geq 1} (\eta_s^{-1}) < \infty$, where $\eta_s = \frac{\phi_s}{\psi_s}$.

Theorem 3.5. $\ell_p(M)^F \subseteq m(M, \phi, p)^F \subseteq \ell_\infty(M)^F$, for $1 \leq p < \infty$.

Proof. On taking $M(x) = x^p$, for $1 \leq p < \infty$ and $\phi_n = 1$, for all $n \in N$. We get, $m(M, \phi, p)^F = \ell_p(M)^F$. So, the first inclusion is clear.

Next, suppose that, $(X_k) \in m(M, \phi, p)^F$ that implies that,

$$\sup_{s \geq 1, \sigma \in \wp_s} \frac{1}{\phi_s} \left(\sum_{k \in \sigma} \left(M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) \right)^p \right)^{\frac{1}{p}} = K (< \infty).$$

For, $s = 1, M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) \leq K \phi_1, k \in \sigma$. Which implies that, $\sup_{k \geq 1} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) < \infty$.

Following the same way, we get,

$$\sup_{k \geq 1} M \left(\frac{\lambda(X_k, \bar{0})}{r} \right) < \infty.$$

Which implies, $(X_k) \in \ell_\infty(M)^F$. This completes the proof. □

Putting $\psi_n = 1$, for all $n \in N$, in Corollary 3.1, we get,

Proposition 3.4. $m(M, \phi, p)^F = \ell_p(M)^F$ if and only if $\sup_{s \geq 1} (\phi_s) < \infty$ and $\sup_{s \geq 1} (\phi_s^{-1}) < \infty$.

Putting $\psi_n = n$, for all $n \in N$, in Corollary 3.1, we get,

Corollary 3.2. $m(M, \phi, p)^F = \ell_\infty(M)^F$ if $\lim_{s \rightarrow \infty} \left(\frac{\phi_s}{s} \right) > 0$, for $0 < p < \infty$.

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