Sequences – Basic Elements for Discrete Mathematics

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Abstract: Sequences are fundamental mathematical objects with a long history in mathematics. Sequences are also tools for the development of other concepts (e. g. the limit concept), as well as tools for the mathematization of real-life situations (e. g. growth processes). But, sequences are also interesting objects in themselves, with lots of surprising properties (e. g. Fibonacci sequence, sequence of prime numbers, sequences of polygonal numbers). Nowadays, new technologies provide the possibility to generate sequences, to create symbolic, numerical and graphical representations, to change between these different representations. Examples of some empirical investigation are given, how students worked with sequences in a computer-supported environment.

Kurzreferat: Folgen sind grundlegende mathematische Objekte mit einer langen Entwicklungsgeschichte in der Mathematik. Folgen sind zum einen Grundlage und Hilfsmittel für Begriffsentwicklungen (etwa des Grenzwertbegriffs) oder zur Modellierung von Umweltsituationen. Zum anderen sind Folgen aber auch als eigenständige Objekte interessant, die eine Vielzahl an Eigenschaften aufweisen (z. B. Fibonacci-Folgen oder die Folgen der Polygonalzahlen). Heute ergibt sich mit Hilfe neuer Technologien die Möglichkeit, Folgen auf Knopfdruck zu erzeugen und sie symbolisch, numerisch oder graphisch darzustellen. Verschiedene empirische Untersuchungen zeigen, wie Studierende mit Folgen in einer computerunterstützten Lernumgebung arbeiten.

1 Understanding the Sequence Concept

Sequences are prototypes of discrete objects in mathematics. On the one hand, sequences are easily defined as functions having the natural numbers as their domain of definition; but, on the other hand, there exist a wide variety of representations, related concepts and perceptions connected with the sequence concept. Sequences can be represented by explicit or recursive formulas, graphs, arrow diagrams, or tables. Sequences appear in all areas of mathematics, e.g. sequences of numbers, mappings or geometric figures. Algorithms may be thought of as sequences of well-defined single steps. The sequence concept also has an intuitive basis in everyday life situations—think about sequences of playing cards, stamps, days, years, or even proverbs like "The punishment follows close on the heels of an evil deed". According to ideas of FREUDENTHAL's "Didactical Phenomenology of Mathematical Structure," (1983) the understanding of a concept presupposes the development of a wide variety of perceptions. We see the development of understanding as a long-term process and refer to the models of SKEMP (1979) and VOLLRATH (1984). SKEMP distinguishes between "instrumental" and "relational understanding"; VOLLRATH's model of understanding develops on different levels:

- *Intuitive understanding*: E.g. knowing examples and representations of sequences;
- *Content-oriented understanding*: E.g. knowing properties of sequences, like monotony, convergence;
- *Integrated understanding*: E. g. seeing relationships between properties of sequences and between sequences and concepts like functions or mappings;
- *Formal understanding*: E. g. knowing definitions of sequences and being able to prove properties of sequences;
- *Structural understanding*: E.g. working with compositions of sequences or seeing sequences as elements of a vector space.

2 The Sequence Concept in German Schools

The way sequences are taught in school mathematics in Germany has changed over the last two decades. Up to 1980 sequences were taught in a wide range of precalculus lessons in order to build a basis for the limit concept in calculus. The definitions used were quite formal, like:

$$\forall \varepsilon > 0 \ \exists n_o \in \mathbb{N} \ \forall n > n_o : \left| a_n - A \right| < \varepsilon .$$

The consequences of this formal approach included the following: it took students a long time become familiar with the notation; (many) students worked mainly at a formal level (sometimes without understanding of the concepts); and real world problems were integrated (if at all) only at the end of the course. In the last few years, school calculus has started immediately with continuous functions, based on an "intuitive limit concept" (which goes back to EMIL ARTIN 1957 and SERGE LANG 1964). Students work on an intuitive level of understanding, while developing ideas related to the limit concept like "… getting closer" or "… as close as you want".

The advantages of this new approach of that important concepts like the derivative and applications are integrated into the courses right from the beginning. The negative result is that the majority of students leave school without having any idea of the sequence concept and it seems to be hard to have conceptions of the basic ideas of calculus without knowing sequences.

3 Revitalization of the Sequence Concept

Over the last few years, as a result of the increasing role of computers in mathematics and mathematics education, discrete mathematics, and hence sequences, have gained in importance. This is also emphasized by the *NCTM-STANDARDS* (1989), which has discrete mathematics as a separate standard for grade 9 to 12: "Sequences and series ... should receive more attention, with a greater emphasis on their descriptions in terms of recurrence relations."[1]. In the *NCTM-PRINCIPLES AND STANDARDS FOR SCHOOL MATHEMATICS* of the year 2000 discrete mathematics is no longer a separate standard, but is now distributed across

[1] http://standards.nctm.org/Previous/CurrEvStds/9-12s12.htm

the *Standards* and spans the years from pre-kindergarten through grade 12. *Iteration and Recursion* are explicitly emphasized as one of the three important areas of discrete mathematics.

I see three reasons for a need to revitalize the sequence concept in school mathematics (in Germany), and these reasons should be viewed as closely related to the ideas of discrete mathematics. First, many real life problems allow mathematical representations with sequences, e.g. growth processes or goods-cost-problems. Second, many mathematical problems can be solved with special sequences, e.g. triangular numbers, polygonal numbers. Third, sequences are tools for the development of continuous concepts; e.g., the difference quotient can be taken as a basis for the understanding of the differential quotient.

New technologies can serve as a catalyst for revitalizing sequences in school mathematics. Nowadays, computers make it possible to generate sequences, to create symbolic, numerical and graphical representations, and to change between different representations—just by the press of a button. Moreover, we think that spreadsheets are the right tools in the context of discrete mathematics, especially for working with sequences.

In the following, we describe two empirical investigations, in which we evaluated students' working with sequences in a computer-supported environment.

4 Empirical Investigations

4.1 Questions of the Investigations

The aims of the investigations were to document and to analyze the working styles of students solving problems in a computer-based environment—in contrast to traditional paper and pencil activities, and to evaluate their understanding of the used concepts. We were especially interested in the following questions:

- How do students use a computer while they are solving problems involving the sequence concept? How does this differ compared to paper-and-pencil work?
- *What* do students understand about (recursively given) sequences and their properties?
- *What* do students understand about the concept of difference sequences?

4.2. Working Styles and Representations

We see *a "working style"* as *sequences of user's actions* affecting and altering mathematical objects, e.g., changing variables, multiplying an equation by a number, or differentiating a function. We speak of a *local working style* if particular terms of the sequence are changed (e.g., the initial values of a recursively defined sequence). If all the terms of a sequence are varied, we speak of a *global working style,* e.g., finding approximations to sequences and curves (see WEIGAND a. WELLER 2001).

The concept of mathematical working styles is closely related to the working with representations, like symbols, graphs, tables, diagrams or pictures, as well as working with working with menus, buttons and rollbars of (computer-)tools. The computer is a tool with special mathematical notations for the objects and special menus, or mouse-driven actions and commands. It allows a person to work in new ways with objects – or rather, the representations of these objects – on the screen. Representation is one of the five "Process Standards" in the already mentioned *NCTM-PRINCIPLES AND STANDARDS FOR SCHOOL MATHEMATICS* (2000). This stresses again the importance of representations in mathematical learning and problem-solving, which become even more important when you work with computers (see also GOLDIN 2002).

In these investigations, we were interested in operations that are aimed at solving a problem. In order to be able to distinguish such purposeful activities from non-reflective guess-and-check strategies, it is necessary to analyze the working processes, and students' actions must be seen in relation to the understanding that correspond to those actions.

4.3 Computer Protocols

During this study, the working styles of the subjects were recorded with the help of "computer protocols". While the students work on the computer, a recording program runs in the background and saves all user inputs via keyboard or mouse.[2] Compared to videotapes and interviews, computer protocols make it possible to observe a large group of students simultaneously. Computer protocols are produced simultaneously with the students' problem-solving processes. The computer protocols can be replayed later as a 'film', which shows all screen activities of the pupil in real time. These protocols can then be analyzed, to determine whether and how a subject has solved a problem, how much time s/he spent with a given problem, how many and what kind of representations s/he has seen on the screen, and when and how often s/he switched over to a different representation. Moreover, it is also possible to evaluate the students' less successful strategies.

4.4 Investigation Programs

With the help of the spreadsheet EXCEL, two teaching and learning programs were developed. The programs introduce the contents, the level of difficulty is increased in a step by-step manner, and they allow selecting different solution strategies.

The first program is about "Sequences and Growth processes" (see WEIGAND 1999 and THIES a. WEIGAND 2003) and uses the following types of sequences $(a_n)_{\mathbb{N}_0}$:

- *Linear* growth: $a_{n+1} = a_n + B$;
- *Exponential* growth.: $a_{n+1} = A \cdot a_n$;
- *Limited* growth: $a_{n+1} = a_n + P \cdot (B - a_n)$.

A, B, P $\in \mathbb{R}$, initial value a_0. We used graphs, tables and formulas as representations. The students are asked to solve geometric and real-life problems related to these sequences.

The second program is about "Difference Sequences" (THIES 2002). This concept was introduced in connection

[2] There are different types of programs available. We used the program *Camtasia*: http://www.techsmith.com/products/studio/default.asp

with average air-temperature per year. The real-life-situation was represented in tables and graphs. Fig. 1a shows the graph of the average air-temperature per year and Fig 1b shows the difference sequence. The students were asked to supply the missing values (see arrows).

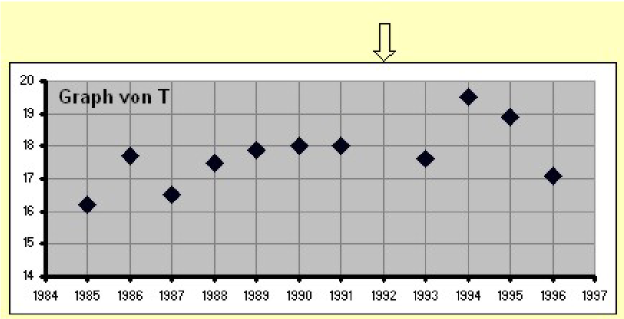

Fig. 1a

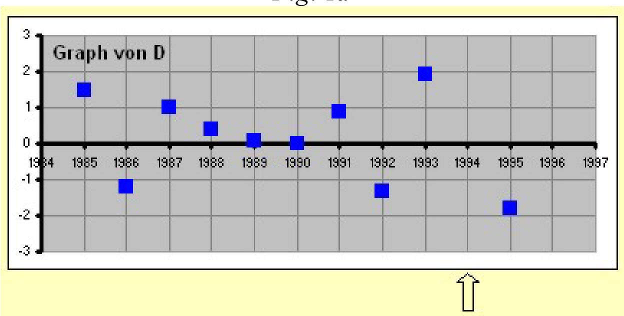

Fig. 1b:

We chose this example, because the given relation is not based on an algebraic formula, and we did not want the students to work immediately on a formal level. The aim of the investigation was to analyze the students' understanding of the concept of difference sequences. We extended the concept of sequence to functions defined on $\mathbb{Z}$, f: $\mathbb{Z} \rightarrow \mathbb{R}$, and spoke of "Z-functions".

4.5 Participants of the Investigation

The first program was given to 28 students in two different 12^{th} grade classes. The students worked individually with the program for two two-hour sessions. All subjects had a good, or very good, knowledge of Excel, and all the subjects had already worked with (arithmetic and geometric) sequences in the 11^{th} grade, when the concept of limit was introduced.

The second program, "Difference Sequences," was given to 53 high school students, who had not been taught the concept of derivative, and to 21 elementary and secondary pre-service teachers, who had taken a major in mathematics at the university. They too worked individually with the program in two two-hours.

4.6 Results

In the following, we only give a few highlights of these two investigations. The complete studies are reported in WEIGAND (1999), THIES a. WEIGAND (2003), THIES (2002).

4.6.1 The Sequence $(a_n)_{\mathbb{N}_0}$ *with* $a_{n+1} = a_n + B$

The sequence was introduced using three representations: formula, table and graph. The students were asked about how the terms of the sequence a_n (with $a_{n+1} = a_n + B$) would change, if the value of the initial term a_0 was increased by five. The students were also asked to give some reasons for their answer. They had the choice of working with the graph or table (Fig.2-4).

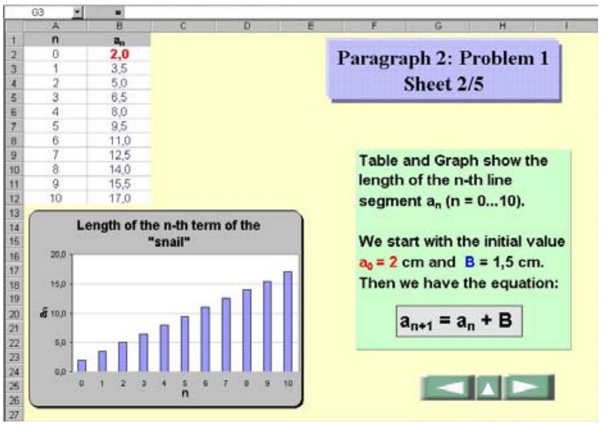

Fig. 2

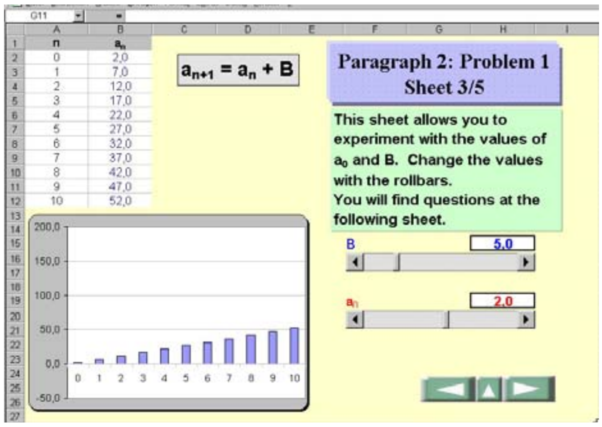

Fig. 3

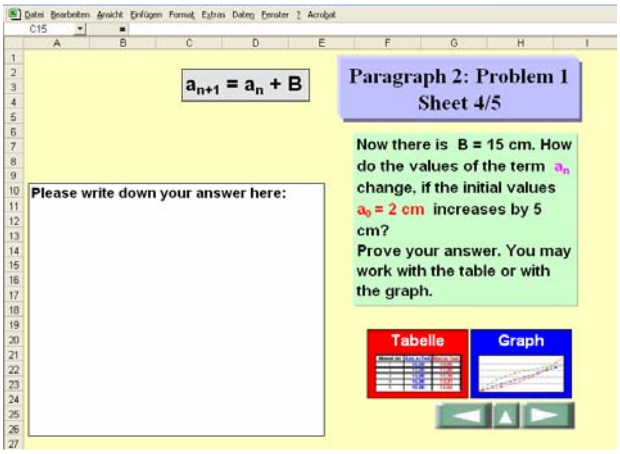

Fig. 4

82% of the subjects answered correctly. However, the verbal reasons given by the subjects very often only described what would happen to the later terms of the sequence, but did not give arguments or proofs for this behavior. The students worked significantly more often with the table than with the graph. The chosen representation influenced the students' arguments justifying the observed behavior of the sequence. The graphic representation goes along with a global view of the sequence.

4.6.2 The Sequence $(a_n)_{IN}$ with $a_{n+1} = B \cdot a_n$

The starting point is a figure with circles with increasing radius. Each successive radius is increased by the constant factor B. The students were asked, how the value of a_5 changed, when the initial value a_o is multiplied by three (Fig. 5-6).

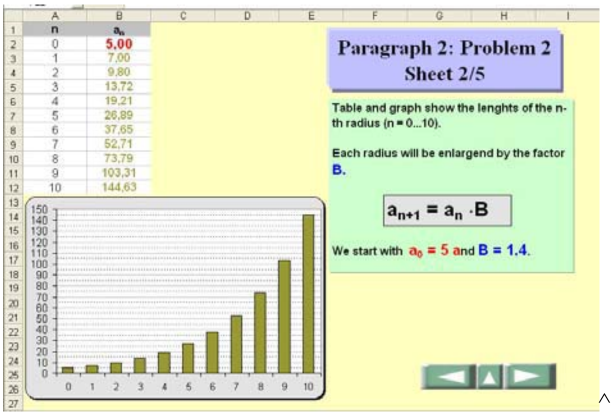

Fig. 5

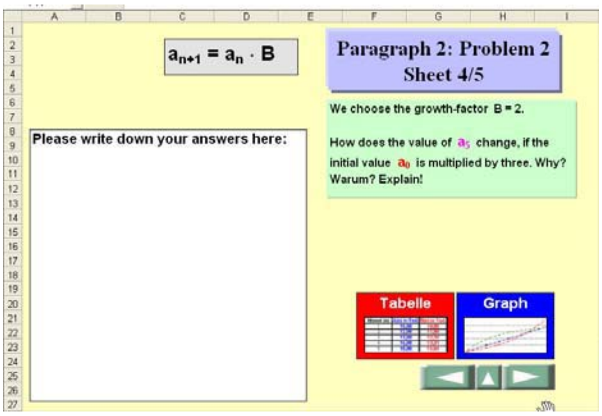

Fig. 6

78% of the subjects correctly recognized the proportional relationship between the terms a_n of the sequence and the initial value a_o. In contrast, students—as expected—did not do nearly as well in answering the question about what would happen to a_5 if the value of B was doubled (correct percentage 35%). The students worked mainly with the table; some students did not consider the graph at all, while some other subjects frequently switched back and forth between graph and table. Some subjects' arguments show step-by-step iterative thinking (i.e., the students successively proceeded from a_o to a_5). Other subjects argued about the exponential behavior of the sequence which is especially evident in the graphic and the symbolic representations ($a_{n+1} = a_o \cdot B^n$).

4.6.3 The Sequence $(a_n)_{\mathbb{N}_0}$ with

$$a_{n+1} = a_n + P \cdot (G - a_n)$$

The first two problems were related to linear and exponential functions, a topic the students were familiar with. In contrast, the sequences describing limited growth were completely new to the students. The dependence of the terms a_n of the sequence on the variables a_o, P, G cannot be described by simple relations. Therefore, we posed the questions in a way intended to provoke intuitive descriptions. This is shown in the following two examples.

Geometric example: Limited Growth

The dynamic view of the students' descriptions are expressed by phrases like, "getting closer", "quickly approaching" or "successively approaching". Students transferred knowledge they had developed in the context of continuous functions to the discrete case. Compared to the previous problems the students worked more intensively with the graph. Some subjects hardly considered the table or simply did not use it at all (Fig. 7-9).

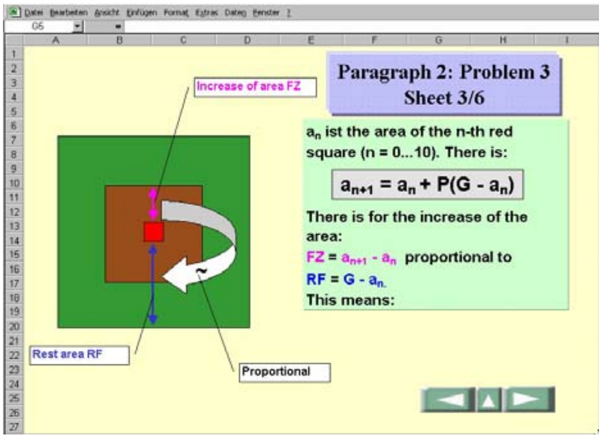

Fig. 7

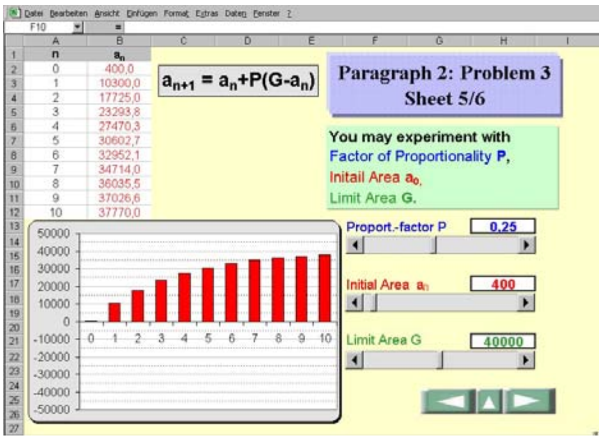

Fig. 8

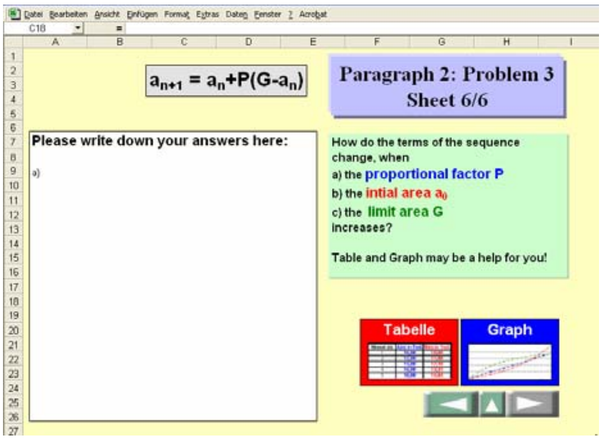

Fig. 9

The Traffic Jam Problem: An application to $a_{n+1} = a_n + P\cdot(G - a_n)$

Interpreting the representations of the traffic jam problem caused bigger difficulties than the previous problem. For the initial term a_o, student answers contained expressions like "number of cars per meter and per second" or "number of the cars during usual traffic conditions." In some cases, students made connections to the geometric example (for a_o: "initial field - number of cars before the jam started"; for G: "Limited area that can be filled with cars"). Some students described only the effect of the change of the parameters ("the less a_o, the lower the curve"). The explanation of the parameter P and the continuation of the sentence "The larger P the ..." was made in a qualitative style by all subjects. Almost no one saw a relationship to the difference $a_{n+1} - a_n$. However, the responses reflect the dynamics that were suggested by the learning program: "More cars are entering the jam" or "The number of cars is increasing" (Fig. 10-13).

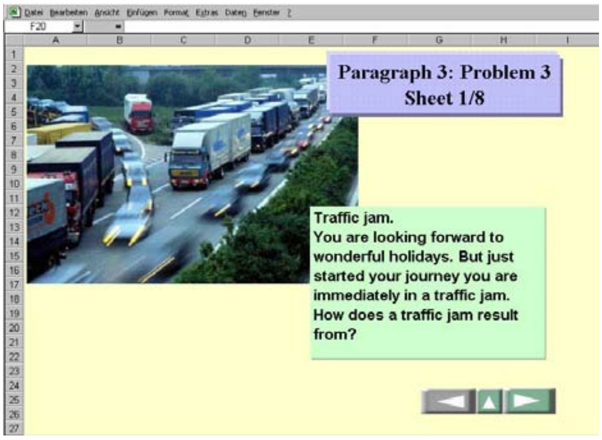

Fig. 10

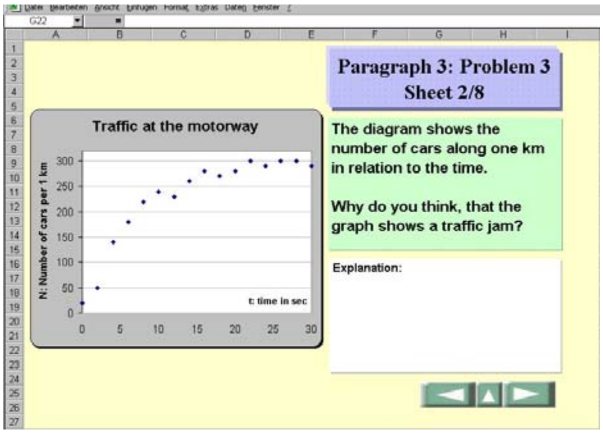

Fig. 11

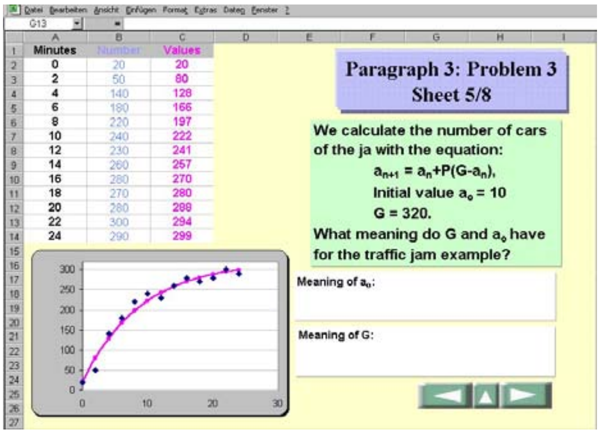

Fig. 12

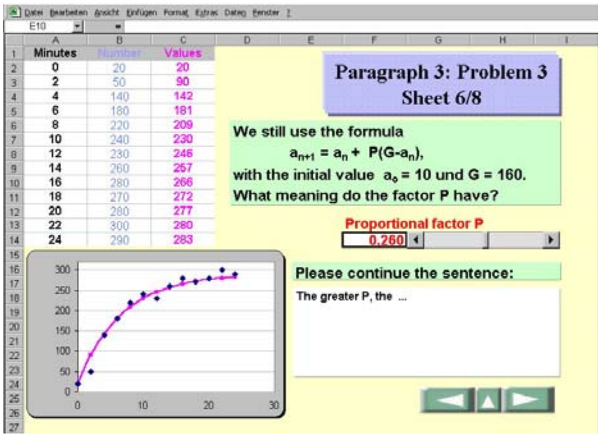

Fig. 13

4.6.4 Difference sequences

This topic was a part of the second investigation. The concept of a difference sequence of a Z-function was developed using a time–air-temperature relation represented in tables and graphs. Then, this concept was explained in connection with linear and quadratic Z-functions (Fig. 14). There were some interactive exercises. The following is one example (Fig. 15).

Changing c in $f(n) = 2n^2 – 8n + c$

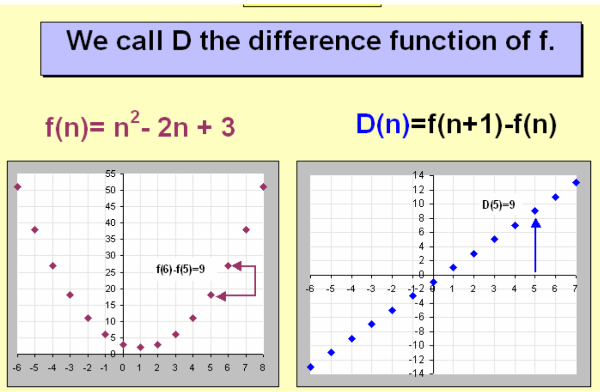

Fig. 14

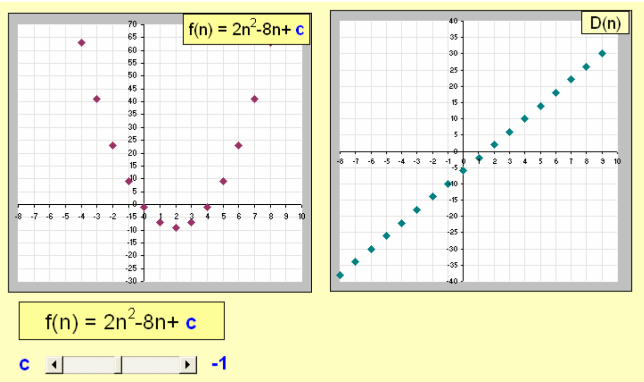

Fig. 15

The function with $f(n) = 2n^2 – 8n + c$ was given and the students were asked to explain verbally the “strange” behavior of D when changing c.[3]

About one third of the high school students gave acceptable reasons for this behavior, like:

- “There is only a translation of the parabola, which doesn’t change the difference of the values.”
- “It is a translation of the parabola, which doesn’t affect the ratio of the values to each other.”
- “The difference function gives you the slope of f. If c changes, this doesn’t change the slope.”

Most of the other students only described the change in the graph, without giving any reasons for it. Two thirds of the university students gave acceptable answers for the behavior of the sequence.

Changing b in $f(n) = 3n^2 + bn + 40$
The function with $f(n) = 3n^2 + bn + 40$ was given and the students were asked to explain the “strange” behavior of D when changing the parameter b (Fig. 16-17).[4]

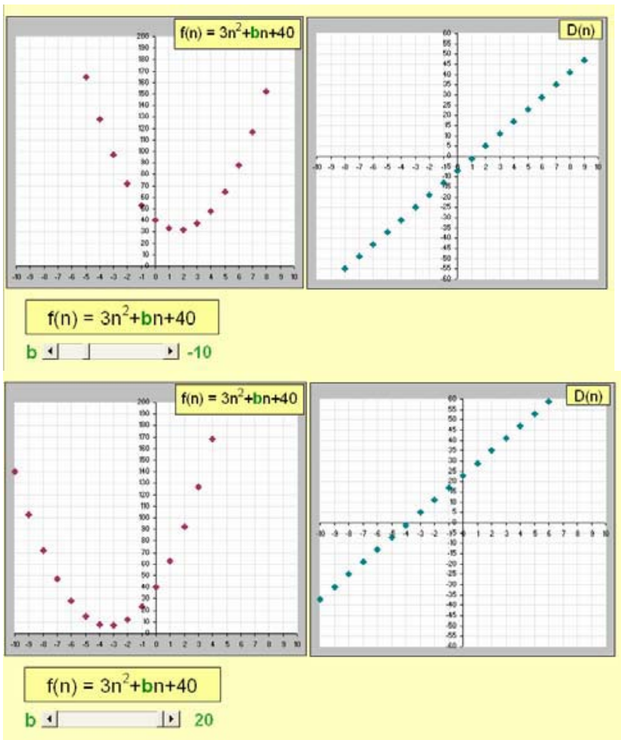

Fig. 16 and 17: The graphs of f and D for b = -10 and b = 20

In the context of the learning program, the calculation of the terms of the difference sequence of a quadratic Z-function was explained on the symbolic level. Starting with $f(n) = a{\cdot}n^2 + b{\cdot}n + c$ the students were asked to calculate the term of D(n) using paper and pencil. This calculation gives $D(n) = f(n+1) – f(n) = 2an + a + b$.

However, to interpret the behavior of the difference function resulting from changes of the parameters was a challenging problem for the students. We emphasize again that these students had not yet studied derivatives. Thus, the majority of the students were out of their depth in this problem. Only one tenth of the students gave acceptable reasons, e.g.,

- “f doesn’t change the slope, only the locus. The differences lie only at a different height.”
- “D(n) = 2an + a + b. If you change b, the intersection of the line with the y-axis changes.”

Many students were not able to see the difference function as a whole—they only had a local view of this function as a collection of discrete points, without grasping the global aspect of the sequence.

As one might expect, the university students did better with this problem, but some of them also got in trouble with the concept of difference sequence and Z-functions. Surprisingly, one third of them were unable to transfer their knowledge of derivatives, developed in connection with continuous functions, to the discrete situation.

5 Summary of the Results

Spreadsheets support the process of learning of the relationship between *local* and *global aspects of sequences*. They make it possible to evaluate the global changes in the values of the terms of a sequence as a consequence of local variations of parameters, like the initial value or the parameters in a recursively given sequence. This knowl-

[3] The difference sequence and hence the graph do not change..
[4] There is a translation of the graph of D parallel to the y-axis.

edge can be used to solve interpolation problems and finding regression functions (or sequences). Thus, spreadsheets can be seen as step-by-step, expandable, powerful, didactic tools for mathematics instruction (see also HEALY a. SUTHERLAND 1990, NEUWIRTH 1991).

A well-structured learning process is necessary for learning to see and understand the relationship between a *sequence* and its *difference sequence*. To develop this relationship exclusively in the context of a computer-based learning program requires concentrated and careful work on the topic from the students. (see also Borneleit 2001). For most of the students, this seemed to be asking too much.

Working experimentally is also an important method in mathematics for getting ideas about, and a "feeling" for, the – possible – solutions of a problem. However, to overcome thoughtless button-pressing, and, to get into thoughtful problem solving strategies, theoretical considerations about the problem are necessary. The students had to be forced – e.g. by asking related questions – to get into theoretical considerations, while working experimentally.

The results from the university students show – again – the danger of tying concepts too closely, or *basing them entirely on a formal level*. Many of these students were unable to transfer their knowledge – derived in the context of (continuous) calculus – to discrete problems. Our results support the importance of students also having an intuitive access to concepts while working with computers. The formation of concepts (like the concept of difference sequence or function) should start on a non-formal level, e.g. with non-rule-based sequences, like the empirically observed relation between air-temperature and calendar year. Otherwise, there is the danger that students only work on a symbolic level—manipulating symbols without any understanding.

The *high density of information* presented by the computer program requires a concentrated working style. Especially some students are out of their depth when working with dynamic representations. These students quit trying to solve the problem, or went ahead with mindless button pressing activities. For this reason, convenient work tools (like scroll bars in the spreadsheet) should be used extensively. In constructing learning programs, it is important to try to slow down the students' work pace.

References:

ARTIN, E., A Freshman Honors Course in Calculus and Analytic Geometry, Taught at Princeton University, Charlottesville, Virginia 1957

BORNELEIT, P., DANCKWERTS, R., HENN, H.-W., WEIGAND, H.-G. (2001): Mathematikunterricht in der gymnasialen Oberstufe. In: Tenorth, E. (Hrsg.): Kerncurriculum Oberstufe, Beltz Verlag, Weinheim u. Basel, 26-53

FREUDENTHAL, H., Didactical Phenomenology of Mathematical Structure, Kluwer, Dordrecht, 1983

GOLDIN, G., Representation in Mathematical Learning and Problem Solving, in: English, L. D., Handbook of International Research in Mathematics Education, Lawrence Erlbaum, Mahwah, New Jersey a. London, 2002, 197-218

HEALY, L, SUTHERLAND, R., The use of spreadsheets within the mathematics classroom, Int. J. Math. Educ. Sci. Technol. 21 (Nov-Dec 1990), No 6, 847 – 862 http://standards.nctm.org/Previous/CurrEvStds/index.htm

LANG, S., A first Course in Calculus, Addison-Wesley, Massachusetts 1964

NCTM, 1989 Curriculum and Evaluation Standards,

NCTM, 2000, Principles and Standards for School Mathematics, http://standards.nctm.org/

NEUWIRTH, E., Visualizing Formal and Structural Relationships with Spreadsheets, in: DiSessa, A. a. o (ed.), Computers and Exploratory Learning, NATO ASI Series, Series F: Computer and Systems Sciences Vol. 146, Springer 1991, 155 – 174

SKEMP, R. R., Goals of Learning and Qualities of Understanding, Mathematics Teaching 88 (1979), 44 – 49

SKEMP, R. R., Relational Understanding and Instrumental Understanding, Mathematics Teaching 77 (1976), 20 – 26

THIES, S., Zur Bedeutung dieskreter Arbeitsweisen im Mathematikunterricht, Giessen 2002, full text available under http://bibd.uni-giessen.de/ghtm/2002/uni/d020154.htm

THIES, S., WEIGAND, H.-G. Working Styles of Students in a computer-based Environment - Results of a DFG Project, in: Törner, G. a. o. (Ed.), Selected Papers form the Annual Conferences on Didactics of Mathematics, Ludwigsburg 2001, Verlag Franzbecker, Hildesheim 2003, also: http://webdoc.sub.gwdg.de/ebook/e/gdm/2001/index.html,

VOLLRATH, H.-J., Methodik des Begriffslehrens, Stuttgart 1984

WEIGAND, H.-G., Evaluating Working Styles of Students in a Computer-Based Environment - A Comparative Study Between the USA and Germany - Proceedings of the 4th International Conference on Technology in Mathematics Teaching, Aug. 9 - 13 1999 in Plymouth, 10 pages

WEIGAND, H.-G., Tabellenkalkulation - ein schrittweise erweiterbares didaktisches Werkzeug, Der Mathematikunterricht 47 (2001), H. 3, 16 - 27

WEIGAND, H.-G., Weller, H., Changes of Working Styles in a Computer Algebra Environment - The Case of Functions, International Journal of Computers for Mathematical Learning, Kluwer, 6 (March 2001), 87-111

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