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Sequences of Topological Near Open and Near Closed Sets with Rough Applications

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Abstract. In this paper, we initiated new types of near open and closed sets called higher order sets. These sets generated by much iteration of topological interior and closure operations for a given set. These new sets are the terms of many sequences of topological near open and near closed sets. We studied many generalizations of the classical near open sets to these sequences. This paper is the starting point of a new way for researchers to study the high order topology.

1. Introduction

Topology in this time is essential and rich field of applications for many other branches of the real world. For instance, in the computer science field and especially in information retrieval category, topological neighborhood systems are used in granular information [11]. In medicine, many generalizations of rough sets using topological spaces are done and give good new results [12]. In addition, in feature selection approach documents classification based on topology are achieved in [13]. For estimating missing values, topological approaches to retrieve missing values in incomplete information systems are done [2,14, 15].

Many researchers in the field of general topology strive to obtain new sets and research into the interrelationship between them. Everyone knows that working on it is very difficult because the topological structure is so complex. When the number of topologies that can be formed into a set with three elements are 29 then, how many topologies can formed if that set contains 15 elements?

In this paper, we want to study the possibility of finding new sets of higher orders using intended methods regardless of the topological structure. Therefore, what if we wanted to find a closure and interior to a set of n (any finite positive integer number) times alternating. The new sets will be generated anyway, but their knowledge and association with their predecessors is the main significance of this paper. To know this, we define our goal of comparing the new generated sets with the original one using three comparisons: equal, containment or containing. When the new set is equal to the original set, we call there is an equal correlation. However, when the set we obtained content within the original one, we say that this is a relationship of the type of containment. Finally, when the original set contented within the new generated one, it is a correlation of the reverse containment type.

The topological concept semi-open set [5] for example is of type three (the reverse containment type) above such that the original set is contained in the closure of the interior of the set ($A \subseteq cl(int(A))$) (the set

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obtained by iterate closure and interior one time only). In addition, the concept $\beta-$closed set [1] is of type 2 (containment) such that the interior of closure of interior of the original set ($int(cl(int(A))) \subseteq A$) (the new calculated set) contained in the original set. The final example is the regular open set [4] is of type 1 such that the original set is equal to the interior of the closure of the given set ($A = int(cl(A))$).

Many researchers in the field of topology are still searching for new sets by many methods but in classically way. In this paper, we employ a new computational technique to obtain new classes of sets, called *high ordered sets* by using an iteration method. We proved that the class of *high ordered* sets at a certain level contains the class of semi-closed sets. In addition, it is properly contained in the class of semi-pre-closed sets at another certain level. Further, it is observed that the class of *high ordered sets* is independent from other classes of near open or near closed sets.

In this paper as listed sections, the first section is for the introduction to this work. In Section 2, we defined a high order interior and closure operators. In Section 3, we give definitions and some properties of some hybrid topological sets, that generated by repeated the processes of interior and closure operators. Section 4 addressed rough set properties in terms of topological concepts. We give two important algorithms that help researchers to achievement many calculations of this paper in Section 5. Illustrated examples and hint for application are given in Section 6. The paper conclusion and future work are given in Section 7.

2. High order interior and closure operators

Topology is a rich field of exciting new results that can apply in various fields. The topological identifiers interior and closure are the starting point for many useful applications. Because there is no connection between topologists and non-specialists in topology, there is a deficiency in linking topology applications with these disciplines.

From this point, we established the link between topology and other sciences through computer science. The bridge in this paper is the reuse of topological concepts based on interior and closure from a different viewpoint.

In the residue of this section, we will give other styles of interior and closure operators. We will connect the new styles with some classical ones as an illustration example. Then in the next sections, we will give more details about the generalizations and applications of these styles.

The definition of the topological space and their basics such as open sets, closet sets, clopen sets, base and subbase can be found in [7, 8].

If (U, τ) is a topological space, then $cl(X) = \bigcap\{Y \subseteq U : X \subseteq Y, U - Y \in \tau\}$ is the closure of a subset $X \subseteq U$. Also, $int(X) = \bigcup\{Y \subseteq U : Y \subseteq X, Y \in \tau\}$is the interior of $X \subseteq U$. We will rewrite the above notions in a new form that help us in this work to generate it for high-level sets. To end this, we write $\frac{dX}{d^-} = cl(X)$ and $\frac{dX}{d^o} = int(X)$. Make use of these forms we can generate them to n^{th} level by the forms: $\frac{d^nX}{d^{-n}} = cl^n(X)$(the closure of X, n times), $\frac{d^nX}{d^{o^n}} = int^n(X)$(the interior of X, n times). We write interior closure as $\frac{d^2X}{d^{o-}} = int(cl(X))$, the closure, interior written as $\frac{d^2X}{d^{-o}} = cl(int(X))$. The general forms in this case are $\frac{d^{2n}X}{d^{o^n-^n}} = int^n(cl^n(X))$and $\frac{d^{2n+1}X}{d^{-^no^{n+1}}} = cl^n(int^{n+1}(X))$(closure n times and interior $n+1$ times or vice versa).

According to the above notations a subset $A \subseteq U$in $(U, \tau)$is said to be α-open [1] (resp. semi-open, pre-open [3], β-open [4], semi-pre-open) if $A \subseteq \frac{d^3A}{d^{o-o}}$(resp. $A \subseteq \frac{d^2A}{d^{-o}}$,$A \subseteq \frac{d^2A}{d^{o-}}$, $A \subseteq \frac{d^3A}{d^{-o-}}$,$A \subseteq \frac{d^2A}{d^{-o}} \cup \frac{d^2A}{d^{o-}}$).

The following example shows that according to the chosen subset under the same topology on a unique universe, the iteration process of the closure and interior can ended for even or odd integers.

Example 2.1. *Consider the universe $U = \{a, b, c, d, e\}$with the topology$\tau = \{U, \varphi, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}\}$. If we choose$n = 5$, then for the subset$A = \{a, b, c\}$ we have $\frac{dA}{d^-} = \frac{d^2A}{d^{-o}} = U = \frac{d^{2n}A}{d^{-^no^n}}$for any integer$n$. For the subset, $B = \{a, d, e\}$we have $\frac{d^3B}{d^{o^2-}} = \frac{d^5B}{d^{o^3-^2}} = \{a, c, d\} \neq \frac{dB}{d^o} = \{a\}and\frac{d^2B}{d^{-o}} = \frac{d^4B}{d^{-^2o^2}} = \{a, c, d, e\}$. Also, $\frac{dB}{d^-} = \frac{d^3B}{d^{o-^2}} = \frac{d^5B}{d^{o^2-^3}} = \{a, c, d, e\}$.*

We define the operator $\frac{d}{d^-} : P(U) \to P(U)$to be a closure function from the power set of the universe $U$to itself. According to this, the closure of any subset $A \subseteq U$is $\frac{dA}{d^-}$. By the same manner the interior function can defined as $\frac{d}{d^o} : P(U) \to P(U)$, such that $\frac{dA}{d^o}$is the interior of $A \subseteq U$. The interior function can see as the dual closure function if we put $\frac{dA}{d^o} = -\frac{d(-A)}{d^-}$, where $-A = U - A$ is the complement subset. Moreover, the closure function can calculate by interior function as $\frac{dA}{d^-} = -\frac{d(-A)}{d^o}$.

Using interior function and closure function, we can define the neighborhood function as $N : U \to P(P(U))$, where $N(x) = \{A \in P(U) : x \in \frac{dA}{d^o}\}$and $N^*(x) = \{A \in P(U) : x \in \frac{dA}{d^-}\}$are the first order neighborhood systems of $x \subseteq U$using interior and closure respectively. The high order neighborhood system of $x \subseteq U$can defined as $\frac{d^{2n}N(x)}{d^{o^n-^n}} = \{A \in P(U) : x \in \frac{d^{2n}A}{d^{o^n-^n}}\}$. These families of neighboring systems can be moral knowledge bases in applications combining with rough approximations.

The closure and interior functions are satisfying the following lemmas.

Lemma 2.1. *The following conditions are equivalent for any closure function* $\frac{d}{d^-} : P(U) \to P(U)$.
(C1) $A \subseteq B \Rightarrow \frac{dA}{d^-} \subseteq \frac{dB}{d^-}, \forall A, B \in P(U)$.
(C2) $\frac{dA}{d^-} \bigcup \frac{dB}{d^-} \subseteq \frac{d(A\cup B)}{d^-}, \forall A, B \in P(U)$.
(C3) $\frac{d(A\cap B)}{d^-} \subseteq \frac{dA}{d^-} \bigcap \frac{dB}{d^-}, \forall A, B \in P(U)$.

Proof. Suppose $A \subseteq B \Rightarrow \frac{dA}{d^-} \subseteq \frac{dB}{d^-}, \forall A, B \in P(U)$. Then $A \subseteq A \cup B, B \subseteq A \cup B$ implies $\frac{dA}{d^-} \subseteq \frac{d(A\cup B)}{d^-}, \frac{dB}{d^-} \subseteq \frac{d(A\cup B)}{d^-}$, therefore $\frac{dA}{d^-} \bigcup \frac{dB}{d^-} \subseteq \frac{d(A\cup B)}{d^-}$. Analogously, $A \cap B \subseteq A, A \cap B \subseteq B$ this yields that $\frac{d(A\cap B)}{d^-} \subseteq \frac{dA}{d^-}, \frac{d(A\cap B)}{d^-} \subseteq \frac{dB}{d^-}$, therefore $\frac{d(A\cap B)}{d^-} \subseteq \frac{dA}{d^-} \bigcap \frac{dB}{d^-}$. Next, assume $\frac{dA}{d^-} \bigcup \frac{dB}{d^-} \subseteq \frac{d(A\cup B)}{d^-}$and assume $A \subseteq B$. Then $\frac{dA}{d^-} \subseteq \frac{dA}{d^-} \bigcup \frac{dB}{d^-} \subseteq \frac{d(A\cup B)}{d^-} = \frac{dB}{d^-}$. Finally, if $\frac{d(A\cap B)}{d^-} \subseteq \frac{dA}{d^-} \bigcap \frac{dB}{d^-}$and $A \subseteq B$we have $\frac{dA}{d^-} = \frac{d(A\cap B)}{d^-} \subseteq \frac{dA}{d^-} \cap \frac{dB}{d^-} \subseteq \frac{dB}{d^-}$. □

Lemma 2.2. *The following conditions are equivalent for any interior function* $\frac{d}{d^o} : P(U) \to P(U)$.
(I1) $A \subseteq B \Rightarrow \frac{dA}{d^o} \subseteq B, \forall A, B \in P(U)$.
(I2) $\frac{dA}{d^o} \bigcup \frac{dB}{d^o} \subseteq \frac{d(A\cup B)}{d^o}, \forall A, B \in P(U)$.
(I3) $\frac{d(A\cap B)}{d^o} \subseteq \frac{dA}{d^o} \bigcap \frac{dB}{d^o}, \forall A, B \in P(U)$.

Proof. Similar to proof of Lemma 1 by repeated applications of $\frac{dA}{d^-} = -\frac{d(-A)}{d^o}$and $\frac{dA}{d^o} = -\frac{d(-A)}{d^-}$. □

The next theorem shows that the closure function and neighborhood function are equivalent.

Theorem 2.1. 1. $x \in \frac{dA}{d^-} \Leftrightarrow (-A) \notin N(x)$.
2. $x \in \frac{dA}{d^o} \Leftrightarrow (-A) \notin N^*(x)$.

Proof. (1)

$$\begin{aligned} & x \in \frac{dA}{d^-} = \frac{d(-(-A))}{d^-} \Leftrightarrow (-A) \in \{N : x \in \frac{d(-N)}{d^-}\} \\ & \Leftrightarrow (-A) \notin \{N : x \notin \frac{d(-N)}{d^-}\} = \{N : x \in (-\frac{d(-N)}{d^-})\} \\ & = \{N : x \in \frac{dN}{d^o}\} = N(x) \end{aligned}$$

(2)

$$\begin{aligned} & x \in \frac{dA}{d^o} = \frac{d(-(-A))}{d^o} \Leftrightarrow (-A) \in \{M : x \in \frac{d(-M)}{d^o}\} \\ & \Leftrightarrow (-A) \notin \{M : x \notin \frac{d(-M)}{d^O}\} = \{M : x \in (-\frac{d(-M)}{d^O})\} \\ & = \{M : x \in \frac{dN}{d^-}\} = N^*(x) \end{aligned}$$

□

A function $f : (X, \frac{d}{d^-}) \to (Y, \frac{d}{d^-})$is closure preserving if for all $A \subseteq X$yields $f(\frac{dA}{d^-}) \subseteq \frac{d}{d^-}(f(A))$and it is continuous if for all $B \subseteq Y$holds $\frac{d}{d^-}(f^{-1}(B)) \subseteq f^{-1}(\frac{dB}{d^-})$.

Theorem 2.2. *Let* $f : (X, \frac{d}{d^-}) \to (Y, \frac{d}{d^-})$*be a closure function, then the following conditions are equivalent:*

1. $\frac{d}{d^-}(f^{-1}(B)) \subseteq f^{-1}(\frac{dB}{d^-})$*, for all* $B \subseteq Y$.

2. $f^{-1}(\frac{dB}{d^o}) \subseteq \frac{d}{d^o}(f^{-1}(B))$, *for all* $B \subseteq Y$.

Proof. We show that (1) implies to (2):

$$f^{-1}(\frac{dB}{d^o}) = -f^{-1}(-\frac{dB}{d^o}) = -f^{-1}(\frac{d(-B)}{d^-}) \subseteq -\frac{d}{d^-}(f^{-1}(-B))$$
$$= -\frac{d}{d^-}(-f^{-1}(B)) = \frac{d}{d^o}(f^{-1}(B))$$

Now we show that (2) implies to (1):

$$\frac{d}{d^-}(f^{-1}(B)) = -\frac{d}{d^-}(-f^{-1}(B)) = -\frac{d}{d^o}(-f^{-1}(B)) = -\frac{d}{d^o}(f^{-1}(-B))$$
$$\subseteq -f^{-1}(\frac{d}{d^o}(-B)) = -f^{-1}(-\frac{d}{d^-}B)) = f^{-1}(\frac{d}{d^-}B))$$

□

3. Sequences of topological near open and near closed sets

In this section, we give definitions and some properties of some hybrid topological sets, that generated by repeated the processes of interior and closure operators.

We define for any subset A of a topological space (U, τ)sequences of near open sets and near closed sets corresponding to the natural numbers as follows:

1. Sequence of the semi-open sets: $A \subseteq \frac{d^2A}{d^{-o}}, A \subseteq \frac{d^4A}{d^{-^2o^2}}, ..., A \subseteq \frac{d^{2n}A}{d^{-^no^n}}, n = 1, 2, 3, ...$
2. Sequence of the semi-closed sets: $\frac{d^2A}{d^{-o}} \subseteq A, \frac{d^4A}{d^{-^2o^2}} \subseteq A, ..., \frac{d^{2n}A}{d^{-^no^n}} \subseteq A,\ n = 1, 2, 3, ...$
3. Sequence of the pre-open sets: $A \subseteq \frac{d^2A}{d^{o-}}, A \subseteq \frac{d^4A}{d^{o^2-^2}}, ..., A \subseteq \frac{d^{2n}A}{d^{o^n-^n}},\ n = 1, 2, 3, ...$
4. Sequence of the pre-closed sets: $\frac{d^2A}{d^{o-}} \subseteq A, \frac{d^4A}{d^{o^2-^2}} \subseteq A, ..., \frac{d^{2n}A}{d^{o^n-^n}} \subseteq A,\ n = 1, 2, 3, ...$
5. Sequence of the $\alpha-$open sets: $A \subseteq \frac{d^3A}{d^{o-o}}, A \subseteq \frac{d^6A}{d^{o-o^2-o}}, A \subseteq \frac{d^9A}{d^{o-o^2-o^2-o}}, ...$
6. Sequence of the $\alpha-$closed sets: $\frac{d^3A}{d^{-o-}} \subseteq A, \frac{d^6A}{d^{-o-^2o-}} \subseteq A, \frac{d^9A}{d^{-o-^2o-^2o-}} \subseteq A, ...$
7. Sequence of the $\beta-$open sets: $A \subseteq \frac{d^3A}{d^{-o-}}, A \subseteq \frac{d^6A}{d^{-o-^2o-}}, A \subseteq \frac{d^9A}{d^{-o-^2o-^2o-}}, ...$
8. Sequence of the semi-pre-open sets: $A \subseteq \frac{d^3A}{d^{-o-}}, A \subseteq \frac{d^6A}{d^{-o-^2o-}}, A \subseteq \frac{d^9A}{d^{-o-^2o-^2o-}}, ...$
9. Sequence of the semi-pre-closed sets: $A \subseteq \frac{d^3A}{d^{o-o}}, A \subseteq \frac{d^6A}{d^{o-o^2-o}}, A \subseteq \frac{d^9A}{d^{o-o^2-o^2-o}}, ...$
10. Sequence of the regular-open sets: $\frac{d^2A}{d^{o-}} = A, \frac{d^4A}{d^{o^2-^2}} = A, ..., \frac{d^{2n}A}{d^{o^n-^n}} = A,\ n = 1, 2, 3, ...$
11. Sequence of the regular-closed sets: $\frac{d^2A}{d^{-o}} = A, \frac{d^4A}{d^{-^2o^2}} = A, ..., \frac{d^{2n}A}{d^{-^no^n}} = A,\ n = 1, 2, 3, ...$
12. Sequence of $\delta-$closed sets: $A = \frac{dA}{\delta^-}, A = \frac{d^2A}{\delta^{-2}}, ..., A = \frac{d^nA}{\delta^{-n}}$, $n = 1, 2, 3, ...$, where $\frac{dA}{\delta^-} = \{x \in U : \frac{d^2G}{d^{o-}} \cap A \neq \varphi, x \in G, G \in \tau\}$.
13. Sequence of generalized-closed sets: $\frac{dA}{d^-} \subseteq G, \frac{d^2A}{d^{-2}} \subseteq G, ..., \frac{d^nA}{d^{-n}} \subseteq G, A \subseteq G, G \in \tau$
14. Sequence of Q-sets: $\frac{d^2A}{d^{o-}} = \frac{d^2A}{d^{-o}}, \frac{d^4A}{d^{o^2-^2}} = \frac{d^4A}{d^{-^2o^2}}, ..., \frac{d^{2n}A}{d^{o^n-^n}} = \frac{d^{2n}A}{d^{-^no^n}}, n = 1, 2, 3, ...$

All the above sequences depends on the selected topology (this means that each topology can generate different sequences).

4. Rough sets of closure and interior operators

In this section, we addressed rough set properties in terms of topological concepts. The complete revision about the basic concepts of rough sets can be found in [8, 9, 10].

From rough sets view a subset $X \subseteq U$ has two possibilities either rough or exact. However, in the view of topology the subset $X \subseteq U$ has the following types of higher orders:

1. X is totally definable of order 1, if $\frac{dX}{d^-} = \frac{dX}{d^o} = X$and of order $2n$ if$\frac{d^{2n}X}{d^{-^no^n}} = \frac{d^{2n}X}{d^{o^n-^n}} = X$.

2. X is internally definable order 1, if$\frac{dX}{d^-} \neq X = \frac{dX}{d^o}$, and of order $2n$ if $\frac{d^{2n}X}{d^{-^n o^n}} \neq X = \frac{d^{2n}X}{d^{o^n -^n}}$.
3. $X$is externally definable order 1, if $\frac{dX}{d^-} = X \neq \frac{dX}{d^o}$, and of order $2n$ if $\frac{d^{2n}X}{d^{-^n o^n}} = X \neq \frac{d^{2n}X}{d^{o^n -^n}}$.
4. X is undefinable order 1, if $\frac{dX}{d^-} \neq X \neq \frac{dX}{d^o}$, and of order $2n$ if $\frac{d^{2n}X}{d^{-^n o^n}} \neq X \neq \frac{d^{2n}X}{d^{o^n -^n}}$.

From the viewpoint of topology the rough membership function of order 1 is defined as follows: $\mu_X(x) = \frac{\left|X \cap \{\bigcap_{x \in G} \frac{dG}{d^o}\}\right|}{\left|\bigcap_{x \in G} \frac{dG}{d^o}\right|}, G \subset U, x \in G$, where $|Y|$is the cardinality of$Y$.
The $2n$ order rough membership function is defined as follows:

$$\mu_X^{2n}(x) = \frac{\left|X \cap \{\bigcap_{x \in G} \frac{d^{2n}G}{d^{-^n o^n}}\}\right|}{\left|\bigcap_{x \in G} \frac{d^{2n}G}{d^{-^n o^n}}\right|}, G \subset U, x \in G.$$

If we have a binary relation$R$defined on the universe U, then using the topology generated by this relation, we can define the lower and upper approximations of $X \subset U$as follows:
$\underline{R}^{2n}(X) = \{G \subseteq U : \frac{d^{2n}G}{d^{-^n o^n}} \subseteq X\}$ and $\overline{R}^{2n}(X) = \{G \subseteq U : \frac{d^{2n}G}{d^{-^n o^n}} \cap X \neq \varphi\}$.
The following properties hold for any $X, Y \subseteq U$ and R be an equivalence relation:

(L1) $\underline{R}^{2n}(X) = -\overline{R}^{2n}(-X)$.
(L2) $\underline{R}^{2n}(U) = U$.
(L3) $\underline{R}^{2n}(X \cap Y) = \underline{R}^{2n}(X) \cap \underline{R}^{2n}(Y)$.
(L4) $\underline{R}^{2n}(X \cup Y) \supseteq \underline{R}^{2n}(X) \cup \underline{R}^{2n}(Y)$.
(L5) $X \subseteq Y \Rightarrow \underline{R}^{2n}(X) \subseteq \underline{R}^{2n}(Y)$.
(L6) $\underline{R}^{2n}(\varphi) = \varphi$.
(L7) $\underline{R}^{2n}(X) \subseteq X$.
(L8) $X \subseteq \underline{R}^{2n}(\overline{R}^{2n}(X))$.
(L9) $\underline{R}^{2n}(X) = \underline{R}^{2n}(\underline{R}^{2n}(X))$.
(L10) $\overline{R}^{2n}(X) = \underline{R}^{2n}(\overline{R}^{2n}(X))$.
(U1) $\overline{R}^{2n}(X) = -\underline{R}^{2n}(-X)$.
(U2) $\overline{R}^{2n}(\varphi) = \varphi$.
(U3) $\overline{R}^{2n}(X \cup Y) = \overline{R}^{2n}(X) \cup \overline{R}^{2n}(Y)$.
(U4) $\overline{R}^{2n}(X \cap Y) \subseteq \overline{R}^{2n}(X) \cap \overline{R}^{2n}(Y)$.
(U5) $X \subseteq Y \Rightarrow \overline{R}^{2n}(X) \subseteq \overline{R}^{2n}(Y)$.
(U6) $\overline{R}^{2n}(U) = U$.
(U7) $X \subseteq \overline{R}^{2n}(X)$.
(U8) $X \supseteq \overline{R}^{2n}(\underline{R}^{2n}(X))$.
(U9) $\overline{R}^{2n}(X) = \overline{R}^{2n}(\overline{R}^{2n}(X))$.
(U10) $\underline{R}^{2n}(X) = \overline{R}^{2n}(\underline{R}^{2n}(X))$.

$(CO)\underline{R}^{2n}(-X \cup Y) \subset -\underline{R}^{2n}(X) \cup \underline{R}^{2n}(Y)$.

$(LU)\underline{R}^{2n}(X) \subset \overline{R}^{2n}(X)$.

5. Algorithms

In this section, we give two important algorithms that help researchers to achievement many calculations of this paper.
In Algorithm 5.1, we give the basic concepts of rough approximations of knowledge bases.

Algorithm 5.1. *Rough knowledge bases and approximations*
Define the universe of discourse $U = \{u_1, u_2, ..., a_n\}$

$$\forall u_i, u_j \in U$$

Define a relation$R = \{(u_i, u_j) : \forall u_i, u_j \in U\}$ *on* U
Initial right base $N_r(U) = \{\}$
Initial left base $N_l(U) = \{\}$
Initial maximum base $N_m(U) = \{\}$
for $(i = 1; i \leq n; i++)$
{
for (j=1;j≤ n;j++)
{
If $((u_i, u_j) \in R)$ *then*
$N_r(u_i) = \{u_j\}, j++$
$N_l(u_i) = \{u_i\}, i++$
Collected right base $N_r(U) = N_r(U) \cup N_r(ui)$
Collected left base $N_l(U) = N_l(U) \cup N_l(ui)$
Collected maximum base $N_m(U) = \{N_m(u_i)\}, N_m(ui) = \{N_r(u_i), N_l(u_i)\}$
End if
Take any subset $X \subseteq U$,
For $(i \in \{r, l, m\})$
{
The initial lower $\underline{R}_i(X) = \{\}$
The initial upper $\overline{R}_i(X) = \{\}$
for (i=1;i≤ n;i++)
{
If $(N_i(X) \subset X)$ *then*

$$\underline{R}_i(X) = \{u_i\}$$

Else if $(N_i(X) \cap X \neq \varphi)$ *then*

$$\overline{R}_i(X) = \{u_i\}$$

Collected lower $\underline{R}_i(X) = \underline{R}_i(X) \cup \{u_i\}$
Collected upper $\overline{R}_i(X) = \overline{R}_i(X) \cup \{u_i\}$
End if
}
}

Generating the sequences of near open sets are one of the most important achievements of this paper. Algorithm 5.2 gives the way to define the families of these sequences in special cases when a binary general relation is defined on the universe. This algorithm discussed three different ways for generating those sequences, namely, the right, left and maximum neighborhood approaches.

Algorithm 5.2. *Families of sequences of near open sets*
Define the universe $U = \{u_1, u_2, ..., a_n\}$
Derive the Power set of this universe $P(U) = \{X_i : X_i \subseteq U\}$
The initial right sequence $Seq_r(U) = \{\}$
The initial left sequence $Seq_l(U) = \{\}$
The initial maximum sequence $Seq_m(U) = \{\}$
For (i=1;i≤ n;i++)

{ *For (j=1;j≤ n;j++)*
Call Algorithm 1
If $(X_j \subseteq \underline{R}_i(\overline{R}_i(X_j))$*then*

$Seq_r(X_j) = \{X_j\}$

else if $(X_j \subseteq \overline{R}_i(\underline{R}_i(X_j))$ *then*

$Seq_l(X_j) = \{X_j\}$

else if $(X_j \subseteq \overline{R}_i(\underline{R}_i(X_j)) \cup \underline{R}_i(\overline{R}_i(X_j))$ *then*

$Seq_m(X_j) = \{X_j\}$

end if

$Seq_r(U) = Seq_r(U) \cup Seq_r(X_j)$

$Seq_l(U) = Seq_l(U) \cup Seq_l(X_j)$

$Seq_m(U) = Seq_m(U) \cup Seq_m(X_j)$

}

6. Examples and Applications

Researchers that work in computer science need some new mathematical results to apply in artificial intelligence and data mining. Some of them make use of the generated topological operators and their generalizations in rough set theory for applications in their fields. The results addressed in this paper are the starting point to solve many of the problems in different areas.

Example 6.1. *If our universe contains 10 objects say,* $U = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}\}$. *Then the power set* $P(U)$ *(all subsets of the universe U) contains* 2^{10} *subsets. In addition, to select some of these subsets among the 1024 subsets that satisfied some relations are very difficult and cumbersome. Imagine that you are looking for specific of recipes that have something to do or to check it, and may these categories require some calculations dozens of times. With respect to the topology* $\tau = \{U, \varphi, \{a_1\}, \{a_1, a_2\}, \{a_1, a_9\}, \{a_1, a_2, a_9\}\}$, *the sequences of the regular-open sets and of the regular-closed sets are reduced to the first terms only such that:* $\frac{d^2A}{d^{o-}} = \frac{d^4A}{d^{o^2-2}} = \ldots = \frac{d^{10}A}{d^{o^5-5}} = \ldots = A$,*and* $\frac{d^2A}{d^{-o}} = \frac{d^4A}{d^{-2}o^2} = \ldots = \frac{d^{10}A}{d^{-5}o^5} = \ldots = A$, $\forall A \in P(U)$*and for any natural number these sequences are constant. When we choose another topology the results are totally different.*

In this example, we try to discover the nature of the sequences generating by topological near open sets with selecting different topologies on the universe of discourse.

Example 6.2. *Consider the universe* $U = \{a, b, c, d, e\}$*and define on it the topology* $\tau = \{U, \varphi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. *We need to generate the first three terms of the sequence of semi-pre-open sets* $A \subseteq \frac{d^3A}{d^{-o-}}, A \subseteq \frac{d^6A}{d^{-o-2}o-}, A \subseteq \frac{d^9A}{d^{-o-2}o-2}o-}$, *...and the sequence of semi-pre-closed sets* $A \subseteq \frac{d^3A}{d^{o-o}}, A \subseteq \frac{d^6A}{d^{o-o^2-o}}, A \subseteq \frac{d^9A}{d^{o-o^2-o^2-o}}$, *.... The results are given in Table 1 below. From the results of Table 1 the sequences are constant with the selected topology.*

Example 6.3. *If our universe of discourse is* $U = \{a, b, c, d\}$*and we define on it the topology* $\tau = \{U, \varphi, \{d\}, \{a, b\}, \{a, b, d\}\}$. *Then the first term* A_1*of the sequence of the regular-open sets is* $Seq_{regular}(A_1) = \{U, \varphi, \{d\}, \{a, b\}\}$. *In addition, the first term* A_1*of the sequence of the semi-pre-open sets are given as follows:*

$Seq_{\beta-open}(A_1) = \{U, \varphi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, c\},$
$\{b, c\}, \{b, d\}, \{c, d\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$

The second term A_2*of the sequence of the* $\alpha-$*open sets is given by:*

$Seq_{\alpha-open}(A_2) = \{U, \varphi, \{d\}, \{a, b\}, \{a, b, d\}\}$

Terms of sequence of semi-pre-open sets			Terms of sequence of semi-pre-closed sets		
First term	Second term	Third term	First term	Second term	Third term
$A \subseteq \frac{d^3A}{d^{-o-}}$	$A \subseteq \frac{d^6A}{d^{-o-^2o-}}$	$A \subseteq \frac{d^9A}{d^{-o-^2o-^2o-}}$	$A \subseteq \frac{d^3A}{d^{o-o}}$	$A \subseteq \frac{d^6A}{d^{o-o^2-o}}$	$A \subseteq \frac{d^9A}{d^{o-o^2-o^2-o}}$
{{a},{b},{a,b}, {a,c},{b,c}, {a,d},{b,d}, {a,e},{b,e}, {a,b,c},{a,b,d}, {a,c,d},{b,c,d}, {a,b,e},{a,c,e}, {b,c,e},{a,d,e}, {b,d,e},{a,b,c,d}, {a,b,c,e}, {a,b,d,e}, {a,c,d,e}, {b,c,d,e},U}	{{a},{b},{a,b}, {a,c},{b,c}, {a,d},{b,d}, {a,e},{b,e}, {a,b,c},{a,b,d}, {a,c,d},{b,c,d}, {a,b,e},{a,c,e}, {b,c,e},{a,d,e}, {b,d,e},{a,b,c,d}, {a,b,c,e}, {a,b,d,e}, {a,c,d,e}, {b,c,d,e},U}	{{a},{b},{a,b}, {a,c},{b,c}, {a,d},{b,d}, {a,e},{b,e}, {a,b,c},{a,b,d}, {a,c,d},{b,c,d}, {a,b,e},{a,c,e}, {b,c,e},{a,d,e}, {b,d,e},{a,b,c,d}, {a,b,c,e}, {a,b,d,e}, {a,c,d,e}, {b,c,d,e},U}	{{a},{b}, {a,b}, {a,b,c}, {a,b,d}, {a,b,e}, {a,b,c,d}, {a,b,c,e}, {a,b,d,e}}	{{a},{b}, {a,b}, {a,b,c}, {a,b,d}, {a,b,e}, {a,b,c,d}, {a,b,c,e}, {a,b,d,e}}	{{a},{b}, {a,b}, {a,b,c}, {a,b,d}, {a,b,e}, {a,b,c,d}, {a,b,c,e}, {a,b,d,e}}

Table 1: First three terms of sequences of semi-pre-open and closed sets

7. Conclusions and Future Work

In this paper, we tried to introduce new sequences of near open and near closed sets using an iteration process of topological interior and closure operators. These sequences changed according to the selection of topology used. In application each population has its individual topology, then each situation have its suitable sequence. The introduced sequences can be a knowledge base about this population that help in making decisions.

When we define one topology for each attribute in the information system, then we can use the concepts related the predefined sequence in the manuscript to define reductions of the system. Accordingly, the real life applications can do for many fields such as in medical data of diagnosis attributes.

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