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SEQUENT-SYSTEMS FOR MODAL LOGIC

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Abstract. The purpose of this work is to present Gentzen-style formulations of $S5$ and $S4$ based on sequents of higher levels. Sequents of level 1 are like ordinary sequents, sequents of level 2 have collections of sequents of level 1 on the left and right of the turnstile, etc. Rules for modal constants involve sequents of level 2, whereas rules for customary logical constants of first-order logic with identity involve only sequents of level 1. A restriction on *Thinning* on the right of level 2, which when applied to *Thinning* on the right of level 1 produces intuitionistic out of classical logic (without changing anything else), produces $S4$ out of $S5$ (without changing anything else).

This characterization of modal constants with sequents of level 2 is *unique* in the following sense. If constants which differ only graphically are given a formally identical characterization, they can be shown inter-replaceable (not only uniformly) with the original constants *salva* provability. Customary characterizations of modal constants with sequents of level 1, as well as characterizations in Hilbert-style axiomatizations, are not unique in this sense. This parallels the case with implication, which is not uniquely characterized in Hilbert-style axiomatizations, but can be uniquely characterized with sequents of level 1.

These results bear upon theories of philosophical logic which attempt to characterize logical constants syntactically. They also provide an illustration of how alternative logics differ only in their *structural* rules, whereas their rules for *logical constants* are identical.

§0. Introduction. The aim of this work is to present sequent formulations of the modal logics $S5$ and $S4$ based on sequents of higher levels. Sequents of level 1 have collections of formulae of a given formal language on the left and right of the turnstile, sequents of level 2 have collections of sequents of level 1 on the left and right of the turnstile, etc. Rules for modal constants will involve sequents of level 2, whereas rules for other customary logical constants of first-order logic (with identity) will involve only sequents of level 1.

We shall show how a restriction on *Thinning* of level 2, which when applied to *Thinning* of level 1 produces intuitionistic out of classical logic, produces in this case $S4$ out of $S5$. Both in passing from classical to intuitionistic logic and in passing from $S5$ to $S4$, only *Thinning* is changed—all the other assumptions are unchanged. In particular, this means that $S5$ and $S4$ will be formulated with identical assumptions for the necessity operator.

We shall also show in what sense our characterization of the necessity operator is

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unique—a sense in which Hilbert-style or customary Gentzen-style characterizations of this constant are not unique.

After presenting in the next section the sequent language we shall work with, we shall consider first structural systems in this language ("structural" in the sense in which some rules in sequent-systems are called "structural"). Next we shall introduce a way to formulate sequent-rules which we shall call "double-line rules". All rules for logical constants will be given by double-line rules.

Then we shall present sequent-systems for classical, Heyting and Kolmogorov-Johansson ("minimal") propositional logic, and also a sequent-system for an intuitionistic relevant propositional logic, which is essentially logic without *Thinning*. This is followed by a discussion of sequent-systems for propositional $S5$ and $S4$. Next we shall consider the corresponding nonmodal and modal first-order logics.

After a section on uniqueness of characterization, we conclude this work with a brief discussion of some views on the notion of a logical constant and on alternative logics to which our results might be leading.

Most of our demonstrations will be given in a rather sketchy form, or will be omitted altogether, but we suppose that none of them is so difficult that it could not be easily reconstructed. We presuppose for this work a certain acquaintance with the proof theory of classical and intuitionistic first-order logic and modal logic, as expounded, for example, in Kleene [1952, Chapter XV], Curry [1963] and Zeman [1973], and for a minor part of this work a certain acquaintance with Anderson and Belnap [1975].

§1. The language D. Let O be a given formal language. For the moment we assume only that O has at least some formulae. Later on we shall specify particular languages O. Starting with O we shall build a language D as follows ("O" stands for "object", but is also associated with "level 0"; we use "D" to name the sequent language because of the association between sequents and *deductions*):

Vocabulary of D: (1) the vocabulary of O; (2) $\{$, $\}$, the comma, $\varnothing$, $\vdash^1, \vdash^2, \vdash^3, \ldots$.

Formulae of D: (1) The formulae of O, which are called "formulae of level 0".

(2) The constant $\varnothing$ is a *set term* of any level $n \geq 1$. Let $A_1^n, \ldots, A_k^n$, where $k \geq 1$, be formulae of level n; then $\{A_1^n, \ldots, A_k^n\}$ is a set term of level $n + 1$, provided no formula occurs more than once among $A_1^n, \ldots, A_k^n$, and if $B_1^n, \ldots, B_k^n$ are the formulae $A_1^n, \ldots, A_k^n$ taken in any order, the set term $\{B_1^n, \ldots, B_k^n\}$ shall be indistinguishable from the set term $\{A_1^n, \ldots, A_k^n\}$.

(3) Let Γ and Δ be set terms of level $n \geq 1$; then $\Gamma \vdash^n \Delta$ is a formula of level n, called a "sequent".

(4) Nothing is a formula of D save if it can be obtained by (1)–(3).

It should be clear from this specification that a set term of level $n \geq 1$ is in a one-one correspondence with a *finite (possibly empty) set* of formulae of D of level $n - 1$, the constant $\varnothing$ corresponding to the empty set.

The schemata $A^n, B^n, C^n, A_1^n, B_1^n, \ldots$ will be used for formulae of an unspecified level $n \geq 0$. If for n we substitute $0, 1, 2, \ldots$, we obtain schemata for formulae of levels $0, 1, 2, \ldots$, but we shall omit the superscript "O". For example, $\{A^1, B^1\}$ is a schema for a set term of level 2. Substitution for "A^1" and "B^1" is subject to the proviso that the resulting expression is an expression of D. This means that, for example, in $\{A^1, B^1\} \vdash^2 \varnothing$ we can substitute only different expressions for "A^1" and "B^1", whereas in $\{A^1\} \vdash^2 \{B^1\}$ we can substitute also identical expressions.

The schemata $\Gamma, \Delta, \Theta, \Xi, \Pi, \Sigma, \Gamma_1, \Delta_1, \ldots$ will be used for set terms of any level ≥ 1. Substitution for these schemata is subject to the same proviso as above. The allowable substitutions for a schema like "Γ" can be inferred from the schema of the formula in which it occurs. In particular, they will be restricted to some levels in this context. Though in principle we could also note that level by a superscript, no ambiguities can arise in schemata of formulae if we do not.

A schema of the form $\Gamma_1 \cup \cdots \cup \Gamma_k$, where $k \geq 2$, will be used for the set term which corresponds to the union of the sets of formulae $\Gamma_1, \ldots, \Gamma_k$. These sets need not be disjoint, but all the formulae in them must be of the same level.

A schema of the form $\Gamma + \{A_1^n, \ldots, A_k^n\}$, where $n \geq 0$ and $k \geq 1$, will be used for the set term of level $n + 1$ which corresponds to the union of the sets of formulae Γ and $\{A_1^n, \ldots, A_k^n\}$, provided Γ and $\{A_1^n, \ldots, A_k^n\}$ are disjoint, and the formulae in Γ are of level n.

We shall omit quotation marks in this work wherever confusion is not likely.

Next we give the following

DEFINITION *of levels of rules*. A rule is of level n iff the highest level of formulae occurring in it is n.

DEFINITION *of level-preserving rules*. The rule

$$\frac{A_1^{m_1} \cdots A_k^{m_k}}{B^{m_0}}$$

is level-preserving iff for every i, $0 \leq i \leq k$, $m_i = n$ for a given $n \geq 0$.

DEFINITION *of horizontalizations of rules*. The sequent $\{A_1^n, \ldots, A_k^n\} \vdash^{n+1} \{B^n\}$, where $k \geq 1$ and $n \geq 0$, is a horizontalization of the level-preserving rule

$$\frac{\Psi}{B^n}$$

(where Ψ is a set of *occurrences* of formulae) iff, for every i, $1 \leq i \leq k$, an occurrence of the formula A_i^n belongs to Ψ and if an occurrence of a formula C^n belongs to Ψ, either C^n is identical with A_j^n for some j, $1 \leq j \leq k$, or A_j^n is a substitution instance of C^n.

NOTE. A formula can occur only once in $\{A_1^n, \ldots, A_k^n\}$, but more than once in Ψ. So a certain contraction can be involved in producing a horizontalization. Accordingly, different rules like

$$\frac{A \quad A}{B} \quad \text{and} \quad \frac{A}{B}$$

can have the same horizontalization. Also a rule can have more than one horizontalization: for example, to

$$\frac{A \quad B}{A}$$

correspond both $\{A, B\} \vdash^1 \{A\}$ and $\{A\} \vdash^1 \{A\}$ if B can be either different or equal to A in the rule. Every level-preserving rule has at least one and at most a finite number of horizontalizations.

§2. Structural systems. First we give the following

DEFINITION *of expressions essential for a system.* An expression of a language L is essential for a system S in L iff it occurs at least once in a rule, axiom or axiom-schema by which S is presented, or it occurs, or is referred to, at least once in a proviso of a rule, axiom or axiom-schema by which S is presented.

DEFINITION *of structural systems.* A system in D is structural iff no constant of O is essential for it.

We shall also say that rules, deductions, etc., are structural, whenever they do not involve any constant of the object language.

For the structural systems we shall consider we shall give at least some of the following structural rules:

Ascending (***A***). $$\frac{A^n}{\varnothing \vdash^{n+1} \{A^n\}}, \qquad n \geq 0.$$

Descending (***D***). $$\frac{\varnothing \vdash^{n+1} \{A^n\}}{A^n}, \qquad n \geq 0.$$

Iteration (***I***). $$\frac{A^n}{A^n}, \qquad n \geq 0.$$

Cut (***C***). $$\frac{\Gamma \vdash^{n+1} \Delta + \{A^n\} \qquad \Theta + \{A^n\} \vdash^{n+1} \Xi}{\Gamma \cup \Theta \vdash^{n+1} \Delta \cup \Xi}, \qquad n \geq 0,$$

provided either $\Gamma \neq \Theta + \{A^n\}$ or $\Delta + \{A^n\} \neq \Xi$.

Thinning (***T***). $$\frac{\Gamma \vdash^{n+1} \Delta}{\Gamma \cup \Theta \vdash^{n+1} \Delta \cup \Xi}, \qquad n \geq 0,$$

provided either $\Gamma \neq \Gamma \cup \Theta$ or $\Delta \neq \Delta \cup \Xi$.

The provisos for ***C*** and ***T*** are given to make the rules strictly independent from each other and to forestall some trivial considerations.

Note that a certain form of *Contraction* is involved in ***C***.

A rule which is an instance of level n of a rule ***R*** will be called "$\boldsymbol{R}^n$". In proofs we shall write the name of the relevant instance of a rule on the left of the horizontal line. When a rule ***R*** is given only for all levels $\leq n$, we shall call it "$\boldsymbol{R}^{\leq n}$".

Adding to ***T*** the proviso on the right gives

$$\boldsymbol{T}_{\mathrm{H}}: \begin{cases} \text{if } \Delta = \varnothing, \Xi \text{ must be a singleton or empty;} \\ \text{if } \Delta \neq \varnothing, \Xi \text{ must be empty,} \end{cases}$$

$$\boldsymbol{T}_K: \quad \Xi \text{ must be empty.}$$

("H" stands for "Heyting" and "K" for "Kolmogorov-Johansson". Why these indices are chosen will become apparent in §5.)

Axioms for the structural and other systems in D we shall consider will be generated from the rules according to the following principle.

Horizontalizing of rules (***h***). All the horizontalizations of the level-preserving rule ***R*** are axioms or axiom-schemata.

If a rule $\boldsymbol{R}$ is mentioned in $\boldsymbol{h}$ we shall say that it is in the *scope of* $\boldsymbol{h}$. To designate the horizontalizations of a rule $\boldsymbol{R}$ we shall use "$\boldsymbol{h}(\boldsymbol{R})$", also superscribing to "$\boldsymbol{h}$" and "$\boldsymbol{R}$" the level of the instance of $\boldsymbol{R}$ in question. In proofs such a designation will be written on the right of the axiom. When $\boldsymbol{h}$ is restricted only to rules of level n, or only to rules of levels $\leq n$, we shall call it "$\boldsymbol{h}^n$", or "$\boldsymbol{h}^{\leq n}$", respectively.

Using $\boldsymbol{h}$ is mainly a matter of economy. In principle all the needed axioms and axiom-schemata obtained by applying $\boldsymbol{h}$ could be listed, their number being finite. But $\boldsymbol{h}$ also helps to make the articulation of our systems more transparent.

Canonically we shall name a system by listing the names of all of its rules and indicating which of them are in the scope of $\boldsymbol{h}$. This will be done by writing the names of the rules in the scope of $\boldsymbol{h}$ to the right of "$\boldsymbol{h}$". For example, we shall consider the structural system named by $\boldsymbol{ADhICT}$.

A rule such that all of its premises are provable in a system S only if the conclusion is, is called "admissible in S". Analogously to that we have the following

DEFINITION *of admissible sequents.* The sequent $\Gamma \vdash^{n+1} \Delta$ is admissible in the system S iff [if all the formulae in Γ are provable in S, a formula in Δ is provable in S].

A rule, axiom or axiom-schema is *eliminable* from a system S iff the subsystem of S without this rule, axiom or axiom-schema has the same theorems as S. It can easily be shown that a rule $\boldsymbol{R}$ is admissible in a system S iff it is eliminable from the extension of S with $\boldsymbol{R}$.

Our aim now is to show that $\boldsymbol{D}$ is eliminable from the systems $\boldsymbol{ADhICT}$, $\boldsymbol{ADhICT}_{\mathrm{H}}$, $\boldsymbol{ADhICT}_{\mathrm{K}}$ and $\boldsymbol{ADhIC}$. But before showing that we shall make some remarks on the eliminability of other rules from these systems, which are of an independent interest. To simplify our exposition we shall concentrate on the first system only.

It can easily be shown that $\boldsymbol{A}$ is *not* eliminable from $\boldsymbol{ADhICT}$, and that the rule $\boldsymbol{I}$ (not $\boldsymbol{h}(\boldsymbol{I})$) is eliminable from any system. The rule $\boldsymbol{T}$ will be eliminable from $\boldsymbol{ADhICT}$ (where we retain $\boldsymbol{h}(\boldsymbol{T})$ as an axiom-schema), for we have

$$
\boldsymbol{D}^{n+2}\,\dfrac{\boldsymbol{C}^{n+2}\,\dfrac{\boldsymbol{A}^{n+2}\,\dfrac{\Gamma \vdash^{n+1} \Delta}{\varnothing \vdash^{n+2} \{\Gamma \vdash^{n+1} \Delta\}} \qquad \{\Gamma \vdash^{n+1} \Delta\} \vdash^{n+2} \{\Gamma \cup \Theta \vdash^{n+1} \Delta \cup \Xi\} \;\; \boldsymbol{h}^{n+1}(\boldsymbol{T}^{n+1})}{\varnothing \vdash^{n+2} \{\Gamma \cup \Theta \vdash^{n+1} \Delta \cup \Xi\}}}{\Gamma \cup \Theta \vdash^{n+1} \Delta \cup \Xi.}
$$

The rule $\boldsymbol{C}$ of any particular level $n+1$, $n \geq 0$, could be shown to be eliminable in the same way, using $\boldsymbol{h}^{n+1}(\boldsymbol{C}^{n+1})$, $\boldsymbol{A}^{n+2}$, $\boldsymbol{C}^{n+2}$ and $\boldsymbol{D}^{n+2}$. But in general, with $\boldsymbol{C}$ of all levels, we have

LEMMA 1. *The rule* $\boldsymbol{C}$ *is not eliminable from* $\boldsymbol{ADhICT}$.

DEMONSTRATION. We have a proof of the following form in $\boldsymbol{ADhICT}$:

$$
①: \{A^n\} \vdash^{n+1} \{A^n\}\; \boldsymbol{h}^n(\boldsymbol{I}^n),
$$

$$
\boldsymbol{C}^{n+3}\,\dfrac{\begin{array}{l} \boldsymbol{A}^{n+3}\,\dfrac{\boldsymbol{A}^{n+2}\,\dfrac{①}{\varnothing \vdash^{n+2} \{①\}}}{\varnothing \vdash^{n+3} \{\varnothing \vdash^{n+2} \{①\}\}} \\ \{\varnothing \vdash^{n+2} \{①\}, \{①\} \vdash^{n+2} \varnothing\} \vdash^{n+3} \{\varnothing \vdash^{n+2} \varnothing\} \;\; \boldsymbol{h}^{n+2}(\boldsymbol{C}^{n+2}) \end{array}}{\{\{①\} \vdash^{n+2} \varnothing\} \vdash^{n+3} \{\varnothing \vdash^{n+2} \varnothing\}.}
$$

But we can show that a formula proved in such a way is not provable in ***ADhIT*** extended with $\boldsymbol{h}(\boldsymbol{C})$. First we show that $\boldsymbol{D}$ is eliminable from this system by showing that it is admissible in ***AhIT*** extended with $\boldsymbol{h}(\boldsymbol{C})$. Then we proceed by cases inspecting the axioms and rules of this last system. Q.E.D.

More precisely, this demonstration shows that it is impossible to eliminate $\boldsymbol{C}$ of all levels from any system $\boldsymbol{A}^{\leq n+1}\boldsymbol{D}^{\leq n+1}\boldsymbol{h}^{\leq n}\boldsymbol{I}^{\leq n}\boldsymbol{C}^{\leq n+1}\boldsymbol{T}^{\leq n+1}$, where $n \geq 2$. But we can also show that in an extension of $\boldsymbol{A}^{\leq 2}\boldsymbol{D}^{\leq 2}\boldsymbol{h}^{\leq 1}\boldsymbol{I}^{\leq 1}\boldsymbol{C}^{\leq 2}\boldsymbol{T}^{\leq 2}$ with a formula of level 0, $\boldsymbol{C}^2$ is not eliminable: we simply start the proof of the demonstration above with this formula as ①. We have also that $\boldsymbol{C}^2$ is not eliminable from $\boldsymbol{A}^{\leq 2}\boldsymbol{D}^{\leq 2}\boldsymbol{h}^{\leq 1}\boldsymbol{I}^{\leq 1}\boldsymbol{C}^{\leq 2}\boldsymbol{T}^{\leq 2}$ if there are at least two different formulae of level 0. In this system the following is provable only with $\boldsymbol{C}^2$ and not without

$$\{\varnothing \vdash^1 \{A\}, \{A\} \vdash^1 \{B\}, \{B\} \vdash^1 \varnothing\} \vdash^2 \{\varnothing \vdash^1 \varnothing\}.$$

"Cut-elimination" usually consists in showing that a rule corresponding to $\boldsymbol{C}^1$ is eliminable from a system corresponding to $\boldsymbol{A}^1\boldsymbol{D}^1\boldsymbol{h}^0\boldsymbol{I}^0\boldsymbol{C}^1\boldsymbol{T}^1$ or an extension of it.

That a rule is eliminable does not mean that its horizontalizations are eliminable. For example, $\boldsymbol{h}(\boldsymbol{I})$, $\boldsymbol{h}(\boldsymbol{C})$ and $\boldsymbol{h}(\boldsymbol{T})$ are, of course, not eliminable from ***ADhICT***.

To show that $\boldsymbol{D}$ is eliminable from ***ADhICT***, we first state

LEMMA 2. *If $\Gamma \vdash^{n+1} \Delta$ is provable in **AhICT**, $\Gamma \vdash^{n+1} \Delta$ is admissible in every* extension of ***AhICT*** *(including **AhICT**)*.

This lemma can be demonstrated by an induction on the length of proof of $\Gamma \vdash^{n+1} \Delta$. (It is also possible to demonstrate the converse.) As a corollary we get

LEMMA 3. *The rule **D** is eliminable from **ADhICT**.*

For suppose $\varnothing \vdash^{n+1} \{A^n\}$ is provable in ***AhICT***. It follows by Lemma 2 that A^n is provable. Hence, $\boldsymbol{D}$ is admissible in this system.

We could show analogously that $\boldsymbol{D}$ is eliminable from $\boldsymbol{ADhICT}_{\mathrm{H}}$, $\boldsymbol{ADhICT}_{\mathrm{K}}$ and ***ADhIC***.

The elimination of $\boldsymbol{D}$ corresponds somehow to the elimination of a certain form of *Cut*, for it shows that there need not be detours in proofs which consist in ascending to a higher level and then descending to a lower one, and it also makes practicable inductions on the length of proofs. Eliminating $\boldsymbol{D}$ of all levels $\geq n$ shows that the system at all levels $\geq n$ is irrelevant for proving theorems of levels $\leq n-1$.

§3. Double-line rules. Let $B_1^{m_1}, \ldots, B_k^{m_k}$, A^n, where $k \geq 1$, $m_i \geq 0$, $1 \leq i \leq k$, and $n \geq 0$, be formulae of a language D; then all the rules

$$\frac{B_1^{m_1} \cdots B_k^{m_k}}{A^n}, \frac{A^n}{B_1^{m_1}}, \ldots, \frac{A^n}{B_k^{m_k}}$$

will be given by the following expression:

$$\frac{B_1^{m_1} \cdots B_k^{m_k}}{\overline{\overline{A^n}}},$$

called a "double-line rule".

If "$\boldsymbol{R}$" is the name of this double-line rule, "$\boldsymbol{R}\downarrow$" will be the name of the first rule given by it in the list above, and "$\boldsymbol{R}\uparrow$" will be a designation for any of the other rules in this list.

Double-line rules are only an abbreviatory device. For example, a system with exactly the same theorems as ***ADhICT*** could have been given only with double-line rules—viz. the following rules and the horizontalizations of the level-preserving rules given by them:

$$\frac{A^n}{\overline{\varnothing \vdash^{n+1} \{A^n\}}}, \qquad \frac{A^n}{\overline{A^n}}, \qquad \frac{\Gamma \vdash^{n+1} \Delta + \{A^n\} \qquad \Gamma + \{A^n\} \vdash^{n+1} \Delta}{\overline{\Gamma \vdash^{n+1} \Delta}}.$$

The first of these double-line rules is a conflation of $\boldsymbol{A}$ and $\boldsymbol{D}$, and the last a conflation of $\boldsymbol{C}$ and $\boldsymbol{T}$.

We shall now consider the eliminability of $\boldsymbol{D}$ of at least some levels from extensions of structural systems with double-line rules.

LEMMA 4. *Let S be* ***ADhICT***, ***ADhICT***$_{\mathrm{H}}$, ***ADhICT***$_{\mathrm{K}}$ *or* ***ADhIC***, *and let* S_1 *be an extension of S with double-line rules such that all the rules given by these double-line rules are level-preserving, eventually in the scope of* $\boldsymbol{h}$, *and of at most level k, for some* $k \geq 0$. *Then* $\boldsymbol{D}$ *of all levels* $\geq k + 1$ *is eliminable from* S_1.

In the demonstration of this lemma the essential step is to show a corresponding form of Lemma 2 for the subsystem of S_1 without $\boldsymbol{D}$ of all levels $\geq k + 1$, where n in Lemma 2 is $\geq k$. This is done by an induction on the length of proof of $\Gamma \vdash^{n+1} \Delta$.

Lemma 4 will be used for demonstrations of some of the results in the sequel.

§4. Classical propositional logic. Let O be the language of the propositional calculus based on $\rightarrow$, $\wedge$, $\vee$, $\bot$ and $\top$. We define "$\neg A$" as "$A \rightarrow \bot$" and "$A \leftrightarrow B$" as "$(A \rightarrow B) \wedge (B \rightarrow A)$". We give the following double-line rules for propositional constants:

$$(\rightarrow) \quad \frac{\Gamma + \{A\} \vdash^1 \Delta + \{B\}}{\overline{\Gamma \vdash^1 \Delta + \{A \rightarrow B\}}},$$

$$(\wedge) \quad \frac{\Gamma \vdash^1 \Delta + \{A\} \qquad \Gamma \vdash^1 \Delta + \{B\}}{\overline{\Gamma \vdash^1 \Delta + \{A \wedge B\}}},$$

$$(\vee) \quad \frac{\Gamma + \{A\} \vdash^1 \Delta \qquad \Gamma + \{B\} \vdash^1 \Delta}{\overline{\Gamma + \{A \vee B\} \vdash^1 \Delta}},$$

$$(\bot) \quad \frac{\Gamma \vdash^1 \varnothing}{\overline{\Gamma \vdash^1 \{\bot\}}},$$

$$(\top) \quad \frac{\varnothing \vdash^1 \Delta}{\overline{\{\top\} \vdash^1 \Delta}}.$$

We note that substitution for "A" and "B" in these double-line rules is free, so that A and B can be the same formula. Hence, there will be two horizontalizations of $(\wedge)\!\downarrow$ and of $(\vee)\!\downarrow$, viz.

$$\{\Gamma \vdash^1 \Delta + \{A\}, \Gamma \vdash^1 \Delta + \{B\}\} \vdash^2 \{\Gamma \vdash^1 \Delta + \{A \wedge B\}\}$$

and

$$\{\Gamma \vdash^1 \Delta + \{A\}\} \vdash^2 \{\Gamma \vdash^1 \Delta + \{A \wedge A\}\},$$

and analogously with $(\vee)\!\downarrow$. We note also that in spite of our restrictions on disjointness the rules

$$\frac{\Gamma + \{A\} \vdash^1 \Delta + \{A \to B, B\}}{\Gamma \vdash^1 \Delta + \{A \to B\}} \quad \text{and} \quad \frac{\Gamma + \{A\} \vdash^1 \Delta + \{B, A \to B\}}{\Gamma + \{A\} \vdash^1 \Delta + \{B\}}$$

are derivable and their horizontalizations are provable in $\boldsymbol{AhIC}(\to)$ (when the name of a double-line rule is on the right of "$\boldsymbol{h}$", this means that all the rules given by it are in the scope of $\boldsymbol{h}$). We can show something analogous with $(\wedge)$ and $(\vee)$, and with double-line rules for constants we shall consider later. We shall not comment on this topic anymore.

Let Cp/D be the system

$$\boldsymbol{ADhICT}(\to)(\wedge)(\vee)(\bot)(\top)$$

("C" stands for "classical", "p" for "propositional" and "D" for "the language D").

The system Cp/O will be an axiomatization of the classical propositional calculus in O, with *modus ponens* as primitive rule.

We can show the following:

LEMMA 5. *A formula of level* 0 *is provable in* Cp/D *iff it is provable in* Cp/O.

DEMONSTRATION. From right to left the demonstration is a matter of routine. (However, we must pay attention to the restrictions on substitution we have stated: proofs based on our sequents with sets of formulae are not always the same as proofs based on sequents with *sequences of occurrences* of formulae, in which the same formula can occur more than once.)

From left to right we proceed as follows. Consider the following translation:

$$\begin{array}{lll} o_1(A) & \text{is} & A, \\ o_1(\Gamma) & \text{is} & \begin{cases} o_1(A_1^n) \wedge \cdots \wedge o_1(A_k^n), & \text{if } \Gamma = \{A_1^n, \dots, A_k^n\}, k \geq 1, \\ \top, \quad \text{if } \Gamma = \varnothing, \end{cases} \\ \breve{o}_1(\Gamma) & \text{is} & \begin{cases} o_1(A_1^n) \vee \cdots \vee o_1(A_k^n), & \text{if } \Gamma = \{A_1^n, \dots, A_k^n\}, k \geq 1, \\ \bot, \quad \text{if } \Gamma = \varnothing, \end{cases} \\ o_1(\Gamma \vdash^{n+1} \Delta) & \text{is} & \hat{o}_1(\Gamma) \to \breve{o}_1(\Delta), \quad \text{where } n \geq 0. \end{array}$$

We can show by an induction on the length of proof of A^n in Cp/D that $o_1(A^n)$ is provable in Cp/O if A^n, where $n \geq 0$, is provable in Cp/D. If $n = 0$, then $o_1(A^n)$ is A^n. Q.E.D.

It is also possible to show that Cp/D axiomatizes the classical propositional calculus *separatively*, in the sense that all theorems involving only some particular constants are provable only with structural axioms and rules and axioms and rules involving only these constants. This separativeness will also characterize all the systems in D we shall consider subsequently.

§5. Intuitionistic propositional logic. Let O be as in §4, and let Hp/D be the system

$$\boldsymbol{ADhICT}_{\mathrm{H}}(\to)(\wedge)(\vee)(\bot)(\top),$$

Kp/D the system

$$\boldsymbol{ADhICT}_{\mathrm{K}}(\to)(\wedge)(\vee)(\bot)(\top),$$

and RAp/D the system

$$\boldsymbol{ADhIC}(\rightarrow)(\wedge)(\vee)(\bot)(\top)$$

("RA" stands for "R Absolute", a name derived from the name of the relevant logic R—see below).

Each of these systems is a proper subsystem of the preceding one, and they are all proper subsystems of Cp/D. It can easily be shown that Hp/D, in contradistinction to Cp/D, has the property that if $\Gamma \vdash^{n+1} \Delta$, where $n \geq 0$, is provable in it, Δ is either a singleton or empty. Let us call this property "the single-conclusion property". Hence, Kp/D and RAp/D also have the single-conclusion property. It is also easily shown that Hp/D without ($\bot$) and Kp/D without ($\bot$) coincide.

The system Hp/O will be an axiomatization of the Heyting propositional calculus in O, the system Kp/O an axiomatization of the Kolmogorov-Johansson propositional calculus in O (usually called "minimal propositional calculus"), and the system RAp/O will be obtained from the positive fragment of the relevant propositional calculus R (see Anderson and Belnap [1975, p. 341]) by rejecting

$$(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee C)$$

and adding the axiom and the axiom-schema $\rightarrow (A \rightarrow A)$. As for Kp/$O$, no axioms or rules are given for $\bot$. (The logic RAp/O, as well as the corresponding predicate logic which we shall mention in §7, is due to Smirnov [1972, Chapter 6]; the name "RA" is from a later work of Smirnov.)

We can show the following:

Lemma 6. 6.1. *A formula of level* 0 *is provable in* Hp/D *iff it is provable in* Hp/O.

6.2. *A formula of level* 0 *is provable in* Kp/D *iff it is provable in* Kp/O.

6.3. *A formula of level* 0 *is provable in* RAp/D *iff it is provable in* RAp/O.

Demonstration. For 6.1 and 6.2 we proceed as for Lemma 5.

6.3. From right to left the demonstration is a matter of routine. From left to right we use the following translation of formulae of levels 0 and 1:

$$o_2(A) \quad \text{is} \quad A,$$

$$o_2(\{A_1, \ldots, A_k\} \vdash^1 \{B\}) \quad \text{is} \quad \begin{cases} A_1 \rightarrow (A_2 \rightarrow \cdots \rightarrow (A_k \rightarrow B) \cdots), & \text{if } k \geq 1, \\ B, \ \text{if } k = 0, \end{cases}$$

$$o_2(\{A_1, \ldots, A_k\} \vdash^1 \varnothing) \quad \text{is} \quad \begin{cases} A_1 \rightarrow (A_2 \rightarrow \cdots \rightarrow (A_k \rightarrow \bot) \cdots), & \text{if } k \geq 1, \\ \bot, \ \text{if } k = 0. \end{cases}$$

We can show by an induction on the length of proof of A^n in RAp/D that $o_2(A^n)$ is provable in RAp/O if A^n, $0 \leq n \leq 1$, is provable in RAp/D. Here we use Lemma 4 to show that $\boldsymbol{D}^2$ is eliminable. Q.E.D.

§6. Propositional modal logic. Assume for the moment that O is the language of the propositional calculus as in §4 enlarged with the constants $\Box$, $\Diamond$ and $\dashv$ of modal logic. We give the following double-line rules for these constants:

$$(\Box) \quad \frac{\Pi + \{\varnothing \vdash^1 \{A\}\} \vdash^2 \Sigma + \{\Theta \vdash^1 \Xi\}}{\Pi \vdash^2 \Sigma + \{\Theta + \{\Box A\} \vdash^1 \Xi\}},$$

$$(\Diamond)\quad \frac{\Pi + \{\{A\} \vdash^1 \varnothing\} \vdash^2 \Sigma + \{\Theta \vdash^1 \Xi\}}{\Pi \vdash^2 \Sigma + \{\Theta \vdash^1 \Xi + \{\Diamond A\}\}},$$

$$(\dashv)\quad \frac{\Pi + \{\{A\} \vdash^1 \{B\}\} \vdash^2 \Sigma + \{\Theta \vdash^1 \Xi\}}{\Pi \vdash^2 \Sigma + \{\Theta + \{A \dashv B\} \vdash^1 \Xi\}}.$$

We can show that if "$A \dashv B$" is defined as "$\Box(A \to B)$", the rules given by ($\dashv$) are derivable and their horizontalizations are provable in ***ADhIC*** $(\to)(\Box)$. (In a certain sense conversely, we have that if "$\Box A$" is defined as "$\top \dashv A$", the rules given by ($\Box$) are derivable and their horizontalizations are provable in ***ADhIC*** $(\top)(\dashv)$.) We can also show that if "$\Diamond A$" is defined as "$\neg\Box\neg A$", the rules given by ($\Diamond$) are derivable and their horizontalizations are provable in ***ADhICT***$^1(\to)(\bot)(\Box)$. (In a certain sense conversely, we have that if "$\Box A$" is defined as "$\neg\Diamond\neg A$", the rules given by ($\Box$) are derivable and their horizontalizations are provable in ***ADhICT***$^1(\to)(\bot)(\Diamond)$.)

In virtue of this we can assume that $\dashv$ is always defined, and that in contexts with $\boldsymbol{T}^1$ the operator $\Diamond$ is defined too. So we shall concentrate only on ($\Box$) below.

Now let O be the language of the propositional calculus as in §4 enlarged only with $\Box$. Let $S5\text{p}/D$ be the system

$$\boldsymbol{ADhICT}(\to)(\wedge)(\vee)(\bot)(\top)(\Box)$$

(i.e., $S5\text{p}/D$ is obtained by extending the rules and axiom-schemata given for Cp/D with ($\Box$) and the horizontalizations of the rules given by ($\Box$)). The system $S4\text{p}/D$ will be the proper subsystem of $S5\text{p}/D$ where $\boldsymbol{T}^2$ and $\boldsymbol{h}^2(\boldsymbol{T}^2)$ are replaced by $\boldsymbol{T}^2_{\text{H}}$ and $\boldsymbol{h}^2(\boldsymbol{T}^2_{\text{H}})$.

Note that in $S4\text{p}/D$ the rule $\boldsymbol{T}^2_{\text{K}}$ would have the same effect as $\boldsymbol{T}^2_{\text{H}}$. (Later we shall see that even rejecting $\boldsymbol{T}^2$ completely from $S4\text{p}/D$ would not alter the provable formulae of level 0.)

It can easily be shown that in $S4\text{p}/D$ sequents of level 2 always have a singleton on the right of $\vdash^2$. Hence, $S4\text{p}/D$ has the single-conclusion property for sequents of level 2 (but *not* for sequents of level 1). So, in applying ($\Box$), Σ will always be empty.

In virtue of Lemma 4, $\boldsymbol{D}$ of all levels ≥ 3 is eliminable from $S5\text{p}/D$ and $S4\text{p}/D$.

Consider now the rule

$$(\boldsymbol{nec})\quad \frac{A}{\Box A}$$

and the axiom-schemata

$$(\boldsymbol{l}_1)\quad \Box(A \to B) \to (\Box A \to \Box B),$$

$$(\boldsymbol{l}_2)\quad \Box A \to A,$$

$$(\boldsymbol{l}_3)\quad (\Box A \to \Box B) \to \Box(\Box A \to \Box B),$$

$$(\boldsymbol{l}_4)\quad \Box A \to \Box\Box A.$$

The systems $S5\text{p}/O$ and $S4\text{p}/O$ are obtained by extending the rules and axioms given for Cp/O with, respectively, (***nec***), ($\boldsymbol{l}_1$), ($\boldsymbol{l}_2$) and ($\boldsymbol{l}_3$), and (***nec***), ($\boldsymbol{l}_1$), ($\boldsymbol{l}_2$) and ($\boldsymbol{l}_4$). It can easily be shown that ($\boldsymbol{l}_3$) can be replaced in $S5\text{p}/O$ with the more common axiom-schema

$$\neg\Box A \rightarrow \Box\neg\Box A$$

so that the resulting system has the same theorems.

We shall now demonstrate that $S5\mathrm{p}/D$ and $S4\mathrm{p}/D$ share all their theorems of level 0 with $S5\mathrm{p}/O$ and $S4\mathrm{p}/O$, respectively.

LEMMA 7. 7.1. *The rule* $(\boldsymbol{nec})$ *is derivable, and* $(\boldsymbol{l}_1)$, $(\boldsymbol{l}_2)$ *and* $(\boldsymbol{l}_4)$ *are provable in* $\boldsymbol{ADhIC}(\rightarrow)(\Box)$.

7.2. *The axiom-schema* $(\boldsymbol{l}_3)$ *is provable in* $\boldsymbol{ADhICT}^{\leq 2}(\rightarrow)(\Box)$.

DEMONSTRATION. To facilitate the demonstration we shall first prove the following:

$$(\Box)\uparrow\dfrac{\boldsymbol{A}^2\dfrac{\{\Box A\}\vdash^1\{\Box A\}\qquad \boldsymbol{h}^0(\boldsymbol{I}^0)}{\varnothing\vdash^2\{\{\Box A\}\vdash^1\{\Box A\}\}}}{\{\varnothing\vdash^1\{A\}\}\vdash^2\{\varnothing\vdash^1\{\Box A\}\}.}$$

The schema proved above will be called "(a)". It is the horizontalization of a rule of level 1 corresponding to (*nec*).

7.1. For (*nec*) we have:

$$\dfrac{\boldsymbol{C}^2\dfrac{\dfrac{\dfrac{A}{\boldsymbol{A}^1 \text{ and } \boldsymbol{A}^2}}{\varnothing\vdash^2\{\varnothing\vdash^1\{A\}\}}\qquad \{\varnothing\vdash^1\{A\}\}\vdash^2\{\varnothing\vdash^1\{\Box A\}\}\ \text{(a)}}{\varnothing\vdash^2\{\varnothing\vdash^1\{\Box A\}\}}}{\dfrac{\boldsymbol{D}^2 \text{ and } \boldsymbol{D}^1}{\Box A.}}$$

For $(\boldsymbol{l}_1)$ we have:

①: $\{\varnothing\vdash^1\{A\},\{A\}\vdash^1\{B\}\}\vdash^2\{\varnothing\vdash^1\{B\}\}$ $\quad \boldsymbol{h}^1(\boldsymbol{C}^1)$,

②: $\{\varnothing\vdash^1\{B\}\}\vdash^2\{\varnothing\vdash^1\{\Box B\}\}$ $\quad$ (a),

$$\boldsymbol{C}^2\dfrac{\boldsymbol{C}^2\dfrac{\{\varnothing\vdash^1\{A\rightarrow B\}\}\vdash^2\{\{A\}\vdash^1\{B\}\}\quad h^1((\rightarrow)\uparrow)\qquad ①}{\{\varnothing\vdash^1\{A\rightarrow B\},\varnothing\vdash^1\{A\}\}\vdash^2\{\varnothing\vdash^1\{B\}\}}\qquad ②}{\{\varnothing\vdash^1\{A\rightarrow B\},\varnothing\vdash^1\{A\}\}\vdash^2\{\varnothing\vdash^1\{\Box B\}\}}$$

$$\dfrac{\dfrac{\boldsymbol{D}^2\dfrac{\dfrac{\{\varnothing\vdash^1\{A\rightarrow B\},\varnothing\vdash^1\{A\}\}\vdash^2\{\varnothing\vdash^1\{\Box B\}\}}{2 \text{ applications of } (\Box)\downarrow}}{\varnothing\vdash^2\{\{\Box(A\rightarrow B),\Box A\}\vdash^1\{\Box B\}\}}}{\{\Box(A\rightarrow B),\Box A\}\vdash^1\{\Box B\}}}{\dfrac{2 \text{ applications of } (\rightarrow)\downarrow, \text{ and } \boldsymbol{D}^1}{\Box(A\rightarrow B)\rightarrow(\Box A\rightarrow\Box B).}}$$

For $(\boldsymbol{l}_2)$ we have:

$$(\Box)\downarrow\dfrac{\{\varnothing\vdash^1\{A\}\}\vdash^2\{\varnothing\vdash^1\{A\}\}\quad \boldsymbol{h}^1(\boldsymbol{I}^1)}{\dfrac{\varnothing\vdash^2\{\{\Box A\}\vdash^1\{A\}\}}{\dfrac{\boldsymbol{D}^2, (\rightarrow)\downarrow \text{ and } \boldsymbol{D}^1}{\Box A\rightarrow A.}}}$$

For ($\boldsymbol{l}_4$) we have:

$$\boldsymbol{C}^2\,\dfrac{\{\varnothing\vdash^1\{A\}\}\vdash^2\{\varnothing\vdash^1\{\Box A\}\}\ \text{(a)}\qquad\{\varnothing\vdash^1\{\Box A\}\}\vdash^2\{\varnothing\vdash^1\{\Box\Box A\}\}\ \text{(a)}}{(\Box)\!\downarrow\dfrac{\{\varnothing\vdash^1\{A\}\}\vdash^2\{\varnothing\vdash^1\{\Box\Box A\}\}}{\dfrac{\varnothing\vdash^2\{\{\Box A\}\vdash^1\{\Box\Box A\}\}}{\dfrac{\boldsymbol{D}^2,(\to)\!\downarrow\text{ and }\boldsymbol{D}^1}{\Box A\to\Box\Box A.}}}}$$

7.2. We have:

$$\boldsymbol{C}^2\,\dfrac{\begin{array}{l}\{\{\Box A\}\vdash^1\{\Box B\}\}\vdash^2\{\varnothing\vdash^1\{\Box A\to\Box B\}\}\quad \boldsymbol{h}^1((\to)\!\downarrow)\\ \{\varnothing\vdash^1\{\Box A\to\Box B\}\}\vdash^2\{\varnothing\vdash^1\{\Box(\Box A\to\Box B)\}\}\qquad\text{(a)},\end{array}}{①:\{\{\Box A\}\vdash^1\{\Box B\}\}\vdash^2\{\varnothing\vdash^1\{\Box(\Box A\to\Box B)\}\},}$$

$$②:\{\varnothing\vdash^1\{\Box(\Box A\to\Box B)\}\}\vdash^2\{\varnothing\vdash^1\{\Box A,\Box(\Box A\to\Box B)\}\}\quad \boldsymbol{h}^1(\boldsymbol{T}^1),$$

$$③:\{\varnothing\vdash^1\{\Box A\}\}\vdash^2\{\varnothing\vdash^1\{\Box A,\Box(\Box A\to\Box B)\}\}\quad \boldsymbol{h}^1(\boldsymbol{T}^1),$$

$$\boldsymbol{C}^1\,\dfrac{\boldsymbol{D}^2\,\dfrac{\boldsymbol{C}^2\,\dfrac{\boldsymbol{C}^2\,\dfrac{\boldsymbol{C}^2\,\dfrac{(\Box)\!\downarrow\dfrac{\boldsymbol{T}^2\,\dfrac{\{\varnothing\vdash^1\{A\}\}\vdash^2\{\varnothing\vdash^1\{\Box A\}\}\ \text{(a)}}{\{\varnothing\vdash^1\{A\}\}\vdash^2\{\varnothing\vdash^1\{\Box A\},\varnothing\vdash^1\{\Box B\}\}}}{\varnothing\vdash^2\{\varnothing\vdash^1\{\Box A\},\{\Box A\}\vdash^1\{\Box B\}\}}\quad ①}{\varnothing\vdash^2\{\varnothing\vdash^1\{\Box A\},\varnothing\vdash^1\{\Box(\Box A\to\Box B)\}\},}\quad ②}{\varnothing\vdash^2\{\varnothing\vdash^1\{\Box A\},\varnothing\vdash^1\{\Box A,\Box(\Box A\to\Box B)\}\}}\quad ③}{\varnothing\vdash^2\{\varnothing\vdash^1\{\Box A,\Box(\Box A\to\Box B)\}\}}}{\varnothing\vdash^1\{\Box A,\Box(\Box A\to\Box B)\}}\qquad (\to)\!\uparrow\dfrac{\{\Box A\to\Box B\}\vdash^1\{\Box A\to\Box B\}\,\boldsymbol{h}^0(\boldsymbol{I}^0)}{\{\Box A,\Box A\to\Box B\}\vdash^1\{\Box B\}}}{④:\{\Box A\to\Box B\}\vdash^1\{\Box(\Box A\to\Box B),\Box B\},}$$

$$\dfrac{\boldsymbol{C}^1\,\dfrac{④\qquad (\Box)\!\downarrow\dfrac{\boldsymbol{C}^2\,\dfrac{(\Box)\!\downarrow\dfrac{\boldsymbol{T}^2\,\dfrac{\{\varnothing\vdash^1\{B\}\}\vdash^2\{\varnothing\vdash^1\{\Box B\}\}\ \text{(a)}}{\{\varnothing\vdash^1\{B\},\varnothing\vdash^1\{A\}\}\vdash^2\{\varnothing\vdash^1\{\Box B\}\}}}{\{\varnothing\vdash^1\{B\}\}\vdash^2\{\{\Box A\}\vdash^1\{\Box B\}\}}\quad ①}{\{\varnothing\vdash^1\{B\}\}\vdash^2\{\varnothing\vdash^1\{\Box(\Box A\to\Box B)\}\}}}{\boldsymbol{D}^2\,\dfrac{\varnothing\vdash^2\{\{\Box B\}\vdash^1\{\Box(\Box A\to\Box B)\}\}}{\{\Box B\}\vdash^1\{\Box(\Box A\to\Box B)\}}}}{\{\Box A\to\Box B\}\vdash^1\{\Box(\Box A\to\Box B)\}}}{\dfrac{(\to)\!\downarrow\text{ and }\boldsymbol{D}^1}{(\Box A\to\Box B)\to\Box(\Box A\to\Box B).}}$$

Q.E.D.

Note that no $\boldsymbol{T}$ rule is assumed for the system of Lemma 7.1.

LEMMA 8. 8.1. *If a formula of level* 0 *is provable in* $S5\mathrm{p}/D$, *it is provable in* $S5\mathrm{p}/O$.

8.2. *If a formula of level* 0 *is provable in* $S4\mathrm{p}/D$, *it is provable in* $S4\mathrm{p}/O$.

DEMONSTRATION. Consider the following translation:

$o_3(A)$ is A,

$\hat{o}_3(\Gamma)$ and $\check{o}_3(\Gamma)$ are obtained by substituting everywhere "o_3" for "o_1" in the clauses for $\hat{o}_1(\Gamma)$ and $\check{o}_1(\Gamma)$ (see §4),

$o_3(\Gamma\vdash^1\Delta)$ is $\Box(\hat{o}_3(\Gamma)\to\check{o}_3(\Delta))$,

$o_3(\Gamma\vdash^{n+2}\Delta)$ is $\hat{o}_3(\Gamma)\to\check{o}_3(\Delta)$, where $n\geq 0$.

8.1. We can show by an induction on the length of proof of A^n in $S5\mathrm{p}/D$ that $o_3(A^n)$ is provable in $S5\mathrm{p}/O$ if A^n, where $n\geq 0$, is provable in $S5\mathrm{p}/D$.

8.2. We can show by an induction on the length of proof of A^n in $S4\mathrm{p}/D$ that

$o_3(A^n)$ is provable in $S4\text{p}/O$ if A^n, $0 \leq n \leq 2$, is provable in $S4\text{p}/D$. Here we use Lemma 4 to show that $\boldsymbol{D}^3$ is eliminable. Q.E.D.

From Lemmata 5, 7 and 8, and the eliminability of $\boldsymbol{D}^2$ from Cp/D, we obtain

THEOREM 1. 1.1. *A formula of level* 0 *is provable in* $S5\text{p}/D$ *iff it is provable in* $S5\text{p}/O$.

1.2. *A formula of level* 0 *is provable in* $S4\text{p}/D$ *iff it is provable in* $S4\text{p}/O$.

The system

$$\boldsymbol{ADhICT}_{\mathrm{H}}(\rightarrow)(\wedge)(\vee)(\bot)(\top)(\Box)$$

would be a Heyting propositional modal logic of the $S4$ type. Extending this system with $(\Diamond)$ (provided we have $\Diamond$ in the language O) would destroy the single-conclusion property of sequents of level 1.

§7. **First-order logic.** Let O be the language of the first-order predicate calculus with identity, i.e., O will have a denumerable list of individual constants, a denumerable list of individual variables (without using different letters for free and bound variables), a denumerable list of n-place predicate constants, for every $n \geq 1$, and the constants $\rightarrow, \wedge, \vee, \bot, \top, \forall, \exists$ and $=$. Formulae are defined as usual, but we assume that $\forall xA$ and $\exists xA$ are well-formed only if x occurs free at least once in A. (The results we want to present do not depend essentially on the decision to present predicate logic with this assumption. This decision is motivated by independent reasons concerning the natural understanding of quantifiers.)

In addition to the schemata specified in §1, we shall use a and b as schemata for *both* individual constants *and* individual variables of 0. The schemata x and y will be used *only* for individual variables. Schemata of the form $A^n(x)$ will be used for formulae of D in which *x occurs free at least once*. Schemata of the form $S^x_a A^n(x)|$ will be used for formulae resulting from the substitution of a at the place of every free occurrence of x in $A^n(x)$.

Then we give the following structural rule

Substitution ($\boldsymbol{S}$). $$\frac{A^n(x)}{S^x_a A^n(x)|}, \qquad n \geq 0,$$

provided the following *proviso for substitution* is satisfied:

> if a is an individual variable y, x does not occur free in a subformula of $A^n(x)$ of the form $\forall y B^m(y)$ or $\exists y B^m(y)$.

We shall use the notation "$S^x_a A^n(x)|$" only if the proviso for substitution is satisfied.

Note that no empty applications of $\boldsymbol{S}$ are possible (a substitution would be empty if x did not occur free in $A^n(x)$).

Though $\boldsymbol{S}$ is level-preserving, it will *not* be in the scope of $\boldsymbol{h}$: otherwise we would have the following:

$$\boldsymbol{S}^1 \frac{\{A(x)\} \vdash^1 \{S^x_a A(x)|\}}{\{S^x_b A(x)|\} \vdash^1 \{S^x_a A(x)|\}.} \qquad \boldsymbol{h}^0(\boldsymbol{S}^0)$$

We now give double-line rules for quantifiers:

$$(\forall) \quad \frac{\Gamma \vdash^1 \Delta + \{A(x)\}}{\Gamma \vdash^1 \Delta + \{\forall x A(x)\}},$$

provided x does not occur free in Γ or Δ, and

$$(\exists)\quad \frac{\Gamma + \{A(x)\} \vdash^1 \Delta}{\Gamma + \{\exists x A(x)\} \vdash^1 \Delta},$$

provided x does not occur free in Γ or Δ.

Note that the proviso holds not only for $(\forall)\downarrow$ and $(\exists)\downarrow$, but also for $(\forall)\uparrow$ and $(\exists)\uparrow$. However, in $(\forall)(\exists)\boldsymbol{ADhIC}$ the rules $(\forall)\uparrow$ and $(\exists)\uparrow$ without the proviso are derivable and their horizontalizations are provable. For derivability we have

$$\boldsymbol{C}^1\frac{\Gamma \vdash^1 \Delta + \{\forall x A(x)\} \qquad (\forall)\uparrow\dfrac{\{\forall x A(x)\} \vdash^1 \{\forall x A(x)\} \quad \boldsymbol{h}^0(\boldsymbol{I}^0)}{\{\forall x A(x)\} \vdash^1 \{A(x)\}}}{\Gamma \vdash^1 \Delta + \{A(x)\},}$$

and analogously with $(\exists)\uparrow$. For provability of the horizontalizations we have, setting ①: $\{\forall x A(x)\} \vdash^1 \{A(x)\}$ (provable as above),

$$\boldsymbol{C}^2\frac{\boldsymbol{A}^2\dfrac{①}{\varnothing \vdash^1 \{①\}} \qquad \{\Gamma \vdash^1 \Delta + \{\forall x A(x)\}, ①\} \vdash^2 \{\Gamma \vdash^1 \Delta + \{A(x)\}\} \quad \boldsymbol{h}^1(\boldsymbol{C}^1)}{\{\Gamma \vdash^1 \Delta + \{\forall x A(x)\}\} \vdash^2 \{\Gamma \vdash^1 \Delta + \{A(x)\}\},}$$

and analogously with $(\exists)\uparrow$. (Of course, provability of the horizontalizations entails derivability with $\boldsymbol{A}^2$, $\boldsymbol{D}^2$ and $\boldsymbol{C}^2$.)

Analogously to what we had with $\boldsymbol{S}$, the rules $(\forall)\downarrow$ and $(\exists)\downarrow$ cannot be in the scope of $\boldsymbol{h}$.

Next we give the double-line rule for identity:

$$(=)\quad \frac{S^x_a \Gamma \vdash^1 \Delta|}{\Gamma + \{x = \alpha\} \vdash^1 \Delta},$$

provided x occurs free at least once in $\Gamma \vdash^1 \Delta$.

Analogously to what we had with $\boldsymbol{S}$, $(\forall)\downarrow$ and $(\exists)\downarrow$, the rule $(=)\uparrow$ cannot be in the scope of $\boldsymbol{h}$.

Let C/D be the system

$$\boldsymbol{S}(\forall)(\exists)(=)\boldsymbol{ADhICT}(\rightarrow)(\wedge)(\vee)(\bot)(\top),$$

H/D the system

$$\boldsymbol{S}(\forall)(\exists)(=)\boldsymbol{ADhICT}_{\mathrm{H}}(\rightarrow)(\wedge)(\vee)(\bot)(\top),$$

K/D the system

$$\boldsymbol{S}(\forall)(\exists)(=)\boldsymbol{ADhICT}_{\mathrm{K}}(\rightarrow)(\wedge)(\vee)(\bot)(\top),$$

and RA/D the system

$$\boldsymbol{S}(\forall)(\exists)(=)\boldsymbol{ADhIC}(\rightarrow)(\wedge)(\vee)(\bot)(\top).$$

Each of these systems is a proper subsystem of the preceding one, and they differ only with respect to $\boldsymbol{T}$.

The systems C/O, H/O and K/O will be axiomatizations in O of, respectively, the classical, the Heyting and the Kolmogorov-Johansson ("minimal"), first-order predicate calculi with identity. The system RA/O is obtained by extending the rules and axioms given for RAp/O with

$$\frac{A(x)}{\forall x A(x)},$$

$$\forall x A(x) \to S^x_a A(x)|,$$

$$S^x_a A(x)| \to \exists x A(x),$$

$$\left.\begin{array}{l}\forall x(B \to A(x)) \to (B \to \forall x A(x)),\\ \forall x(A(x) \to B) \to (\exists x A(x) \to B)\end{array}\right\} \begin{array}{l}\text{provided } x \text{ does not}\\ \text{occur free in } B,\end{array}$$

$$a = a,$$

$$a = b \to (S^x_a A(x)| \to S^x_b A(x)|).$$

This rule and these axiom-schemata can be used to extend the rules and axioms of the corresponding propositional calculi in order to get C/O, H/O and K/O.

We can show the following:

LEMMA 9. *A formula of level* 0 *is provable in* C/D *iff it is provable in* C/O.

LEMMA 10. 10.1. *A formula of level* 0 *is provable in* H/D *iff it is provable in* H/O.

10.2. *A formula of level* 0 *is provable in* K/D *iff it is provable in* K/O.

10.3. *A formula of level* 0 *is provable in* RA/D *iff it is provable in* RA/O.

Demonstrations of these lemmata are obtained by enlarging the demonstrations of Lemmata 5 and 6.

To conclude this section we note that $(=)$ should be compared with

$$S^x_a A(x)| \leftrightarrow \forall x(x = a \to A(x)),$$

which can serve as a single axiom-schema for identity, replacing in C/O, H/O, K/O and RA/O the two axiom-schemata for identity given at the end of the list above.

§8. First-order modal logic. Let O be the language of the first-order predicate calculus with identity as in §7 enlarged with $\Box$, and let $S5/D$ be the system

$$\boldsymbol{S}(\forall)(\exists)(=)\boldsymbol{ADhICT}(\to)(\wedge)(\vee)(\bot)(\top)(\Box)$$

and $S4/D$ the proper subsystem of $S5/D$ where $\boldsymbol{T}^2$ and $\boldsymbol{h}^2(\boldsymbol{T}^2)$ are replaced by $\boldsymbol{T}^2_{\mathrm{H}}$ and $\boldsymbol{h}^2(\boldsymbol{T}^2_{\mathrm{H}})$.

The systems $S5/O$ and $S4/O$ are obtained by extending the rules and axioms given for, respectively, $S5\mathrm{p}/O$ and $S4\mathrm{p}/O$, with the rule and axiom-schemata for the quantifiers and identity listed in §7.

We can show

THEOREM 2. 2.1. *A formula of level* 0 *is provable in* $S5/D$ *iff it is provable in* $S5/O$.

2.2. *A formula of level* 0 *is provable in* $S4/D$ *iff it is provable in* $S4/O$.

A demonstration of this theorem is obtained by enlarging the demonstration of Theorem 1.

We note that in $S5/D$ the Barcan formula is provable, whereas in $S4/D$ it is not.

§9. Uniqueness of characterization. Let S be a system for which a constant α is essential, and S^* a system which differs from S only by having in its language a constant α^*, just graphically different from α, where S had α: rules, axioms or axiom-schemata involving α^* in S^* are obtained from those involving α in S by rewriting α as α^*. Let now SS^* be the system in the language which is the union of the languages of S and S^*, for which we assume the rules and axioms of both S and S^*. It seems natural to say that

(A) α is characterized *uniquely* in S iff α and α^* are *synonymous* in SS^*.

If we want (A) to serve as a definition of *uniqueness of characterization*, we must stipulate what we understand by "synonymous" in (A). In this work we shall assume the following

DEFINITION *of synonymity*. The constants α and β are synonymous in a system S iff, for every formula A_α which results from a formula A by substituting α everywhere at the place of a schema ξ, and for every formula A_β which results from A by substituting β everywhere at the place of ξ, A_α is provable in S iff A_β is (i.e., the rules given by

$$\frac{A_\alpha}{\overline{A_\beta}}$$

are admissible in S).

Note that this definition permits both uniform and nonuniform replacements of α and β when we pass from A_α to A_β, and *vice versa*.

It is possible to show that all logical constants are uniquely characterized in the sense of (A) in the systems C/D, H/D, K/D, RA/D, $S5/D$ and $S4/D$. To get an idea how this uniqueness can be demonstrated we shall take the case of $\Box$ in $S5/D$ and $S4/D$. Let S be $S5/D$ or $S4/D$, and let α be $\Box$; then in the system SS^* we have

$$\begin{array}{ll}
 & \{\Box A\} \vdash^1 \{\Box A\} \quad \boldsymbol{h}^0(\boldsymbol{I}^0) \\
\boldsymbol{A}^2 & \overline{\varnothing \vdash^2 \{\{\Box A\} \vdash^1 \{\Box A\}\}} \\
(\Box)\uparrow & \overline{\{\varnothing \vdash^1 \{A\}\} \vdash^2 \{\varnothing \vdash^1 \{\Box A\}\}} \\
(\Box^*)\downarrow & \overline{\varnothing \vdash^2 \{\{\Box^* A\} \vdash^1 \{\Box A\}\}} \\
\boldsymbol{D}^2 & \overline{\{\Box^* A\} \vdash^1 \{\Box A\},}
\end{array}$$

and we proceed analogously to prove

$$\{\Box A\} \vdash^1 \{\Box^* A\}.$$

It can be shown that this is enough to guarantee the synonymity of $\Box$ and $\Box^*$ in SS^*.

That the uniqueness of characterization so achieved is not a trivial property is shown by the following:

LEMMA 11. *The constant* $\rightarrow$ *is not uniquely characterized in the implicational fragments of* Cp/O, Hp/O *and* RAp/O (*the implicational fragments of* Hp/O *and* Kp/O *are identical*).

DEMONSTRATION. If $\rightarrow$ were uniquely characterized in S, where S is one of the systems of the lemma, then $(A \rightarrow B) \rightarrow (A \rightarrow^* B)$ would be provable in SS^*. But we can show that this formula is not provable in SS^*. It is enough to show this when S is the implicational fragment of Cp/O; *a fortiori*, $(A \rightarrow B) \rightarrow (A \rightarrow^* B)$ cannot then be provable in any weaker system.

Take the matrices

$\to$	1	2	3	4
1	1	2	3	4
2	1	1	1	1
3	1	1	1	1
4	1	1	1	1

$\to^*$	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

which always give the value 1 to the theorems of SS^*. But we have

$$\begin{array}{ccccccc} (A & \to & B) & \to & (A & \to^* & B). \\ 3 & 1 & 2 & 2 & 3 & 2 & 2 \end{array}$$

Hence, this formula is not provable in SS^*. Q.E.D.

Connected with this lemma is the fact that the deduction theorem fails for the systems SS^*, where S is one of the systems of the lemma and α is $\to$.

By giving the value 4 to $\perp$ and using the standard definitions of the other constants in terms of $\to$ and $\perp$, the demonstration above can easily be extended to show that $\to$ is not uniquely characterized in Cp/O. Let now S be C/O. It is known that S is a conservative extension of its implicational fragment Si. Assuming that SS^* is a conservative extension of $SiSi^*$, where α is $\to$, it would be possible to show that $\to$ is not uniquely characterized in C/O, and analogously with H/O, K/O and RA/O.

We can also show

LEMMA 12. *The constant* $\Box$ *is not uniquely characterized in* $S5p/O$ *and* $S4p/O$.

DEMONSTRATION. If $\Box$ were uniquely characterized in S, where S is one of the systems of the lemma, then $\Box^*A \to \Box A$ would be provable in SS^*. But we can show that this formula is not provable in SS^*. It is enough to show this when S is S5p/O.

Take the matrices

$\to$	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

	$\Box$
1	1
2	4
3	4
4	4

	$\Box^*$
1	1
2	2
3	3
4	4

and assign to $\perp$ the value 4. The matrices for the other constants are obtained by defining these constants in terms of $\to$ and $\perp$. These matrices always give the value 1 to the theorems of SS^*. But we have

$$\begin{array}{ccccc} \Box^* & A & \to & \Box & A. \\ 2 & 2 & 3 & 4 & 2 \end{array}$$

Hence, this formula is not provable in SS^*. Q.E.D.

Let now S be $S5/O$ or $S4/O$, and let Sp be $S5$p/O or $S4$p/O. Assuming that SS^* is a conservative extension of $SpSp^*$, where α is $\Box$, it would be possible to show that $\Box$ is not uniquely characterized in $S5/O$ and $S4/O$.

All logical constants except $\to$, $\Box$ and $\perp$ are uniquely characterized in the systems C/O, H/O, K/O, RA/O, $S5/O$ and $S4/O$. The constant $\perp$ is uniquely characterized in all of these systems except K/O and RA/O, where it is not characterized at all. By, so to speak, "climbing" one level, and giving rules of level 1 for $\to$ in the systems in D, we have uniquely characterized also this constant. But the case with $\Box$ is different. It is possible to show that rules of level 1 like those given by

$$(\Box_{S5})\quad \frac{\Gamma \vdash^1 \Delta + \{A\}}{\Gamma \vdash^1 \Delta + \{\Box A\}},$$

provided that if $A_1 \in \Gamma$, A_1 is $\Box B$ for some B, and if $A_2 \in \Delta$, A_2 is $\Box C$ for some C, and by

$$(\Box_{S4})\quad \frac{\Gamma \vdash^1 \{A\}}{\Gamma \vdash^1 \{\Box A\}},$$

provided that if $A_1 \in \Gamma$, A_1 is $\Box B$ for some B, which when added to C/D deliver all the theorems of $S5/O$ and $S4/O$, cannot characterize $\Box$ uniquely. To obtain uniqueness of characterization we need something like $(\Box_{S5})$ with the proviso

if $A_1 \in \Gamma$, A_1 is $\Box B$ or $\Box^* B$ for some B,
and if $A_2 \in \Delta$, A_2 is $\Box C$ or $\Box^* C$ for some C,

and a corresponding form of $(\Box^*_{S5})$ where "$\Box^* A$" replaces "$\Box A$". We could proceed analogously with $(\Box_{S4})$.

Thus we had to "climb" a further level, and assume the rules of level 2 given by $(\Box)$, in order to characterize $\Box$ uniquely. Using the same metaphor, we could say that the nonuniqueness of $\Box$ at level 0 is "deeper" than the nonuniqueness of $\to$.

§10. Conclusion. We shall now try to present briefly, and not very precisely, some views to which the results of this work might be leading.

Let us interpret the sequent $\Gamma \vdash^{n+1} \Delta$ as saying that there is a *deduction* in which the formulae in Γ are premises and the formulae in Δ conclusions. When Δ has more than one member, we can understand the deduction in question as a multiple-conclusion deduction in the style of Shoesmith and Smiley [1978], where the conclusions are taken alternatively. When Γ is empty, the deduction in question would not depend on any premise; if then moreover $\Delta = \{A^n\}$, this would mean that A^n is a theorem. When Δ is empty, the deduction in question could be understood as a refutation of one of the premises in Γ.

Then the double-line rules for logical constants could show that logical constants serve, so to speak, as punctuation marks for some structural features of deductions. If we understand by "logic" the science of *formal* deductions, and if we take it that basic formal deductions are structural deductions, all other formal deductions introducing only constants as punctuation marks for some structural features of deductions, we can formulate the thesis that this punctuation function is a criterion for being a logical constant.

We shall briefly survey what punctuation function belongs to the logical constants we have considered.

Implication is up to a point a substitute for the turnstile at level 0: it can reduce a deduction of level 1 to a formula of level 0. Conjunction and disjunction serve to economize: they reduce to one deduction two deductions which differ only at one place in the conclusions or in the premises. The constant $\bot$ is a substitute for the empty collection of conclusions, and a substitute for the empty collection of premises.

First-order quantifiers serve to represent some features of deductions involving the presence of *arbitrary* singular terms. The proviso for the double-line rule ($\forall$) and ($\exists$) serves to guarantee that the expression in question is arbitrary. Roughly speaking, all quantifiers express something about "any": if "any" is prefixed to a consequent, or an asserted proposition, it becomes "every", and if it is prefixed to an antecedent, it becomes "some". Identity serves to indicate substitution possibilities: what holds for a holds also for whatever is assumed to be identical with a.

The necessity operator differs from the constants mentioned above by having a punctuation function of level 2: it can reduce a deduction of level 2, in which all the premises assert that a formula of O is a theorem, to a deduction of level 1.

We have seen that the alternative logics we have mentioned differ only in their assumptions on structural deductions—more specifically, only in their assumptions concerning *Thinning*. Logical constants were always given with the same rules. If indeed a logic is completely determined by its assumptions on structural deductions, logical constants performing always a determined punctuation function for these deductions, to obtain an alternative logic with the *same* constants we can only change assumptions on structural deductions. This situation, which obtains between classical logic and various intuitionistic logics we have mentioned, and between $S5$ and $S4$, could serve to corroborate the thesis that a criterion for two logics being alternative is that they differ only in their assumptions on structural deductions. (Modal logic is not an alternative to classical logic, but a supplement.)

We have seen that an intuitionistic restriction on *Thinning* applied to *Thinning* of level 2 produces $S4$ out of $S5$. It is plausible to suppose that this fact is not foreign to the existence of the well-known translations from classical and Heyting logic into $S5$ and $S4$, respectively. Let us take it that formulae of level 1 of D constitute an object language O_1, and then let us introduce logical constants into O_1. With the rule $\boldsymbol{T}^2$ unrestricted, the logic of O_1 will be classical; with $\boldsymbol{T}^2$ restricted to $\boldsymbol{T}^2_{\mathrm{H}}$, it will be intuitionistic. We know that at level 0 with these forms of $\boldsymbol{T}^2$ we should obtain $S5$ and $S4$, respectively (where $\boldsymbol{T}^1$ is unrestricted, making the logic of level 0 classical). We could then say that $S5$ represents at level 0 a classical logic of level 1, and $S4$ represents at level 0 an intuitionistic logic of level 1. That is, with $S5$ we represent in classical logic the principles of a classical deductive metalogic, and with $S4$ we represent in classical logic the principles of an intuitionistic deductive metalogic. With the translations we would connect the formulae of $S5$ and $S4$ with what they represent.

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