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SEQUENTIAL COMPLETENESS AND {0, 1}-SEQUENTIAL COMPLETENESS ARE DIFFERENT

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Assuming $\varkappa = 2^\omega$ (a set theoretical assumption weaker than Martin's axiom), we construct a $\{0, 1\}$-sequentially regular Fréchet space which is sequentially complete but fails to be $\{0, 1\}$-sequentially complete. This solves (conditionally) Problem 1.23 in [3] concerning the classification of sequential convergence spaces. The space is of the form $N \cup \mathcal{N}$ ($\mathcal{N}$ is an almost disjoint family of infinite subsets of N) and has the corresponding nice topological properties (e.g., it is Hausdorff, separable, first countable, locally compact, totally disconnected, 0-dimensional).

1. INTRODUCTION

Recall that a Fréchet space is a topological space in which, whenever a point belongs to the closure of a set, then there is a sequence in the set converging to this point. First countable spaces are Fréchet and Fréchet spaces are sequential convergence spaces in which the closure is idempotent. For sequential convergence spaces the notion of sequential regularity and $\{0, 1\}$-sequential regularity play analogous roles as complete regularity and 0-dimensionality, respectively, for topological spaces (cf. [4]). Namely, sequential regularity or $\{0, 1\}$-sequential regularity means that, in the space in question, a sequence (x_n) convergences to a point x iff for each real-valued or $\{0, 1\}$-valued, respectively, continuous function f we have $f(x) = \lim f(x_n)$, i.e., the sequential convergence structure of the space is projectively generated by the corresponding class of functions. Clearly, $\{0, 1\}$-sequential regularity implies sequential regularity. Note that a completely regular Fréchet space is sequentially regular. Moreover, if it is also 0-dimensional, then it is $\{0, 1\}$-sequentially regular. To a certain extent, the notion of sequential completeness or $\{0, 1\}$-sequential completeness plays a similar role for sequential convergence spaces as realcompactness or $\{0, 1\}$-compactness, respectively, for topological spaces (cf. [2]). A sequential convergence space is said to be sequentially complete or $\{0, 1\}$-sequentially complete, if for each sequence (x_n) no subsequence of which converges there is, respectively, a real-valued or $\{0, 1\}$-valued continuous function f such that the sequence $(f(x_n))$ fails to converge (in the real line). More information about sequential completeness and $\{0, 1\}$-sequential completeness and their mutual relationship, as well as further references on the topic, can be found in [3].

The title of the present paper might be misleading, viz., the real line is a trivial example of a sequentially complete space which fails to be $\{0, 1\}$-sequentially complete. The point is that the real line is not $\{0, 1\}$-sequentially regular. Our aim is to show that the two notions mentioned in the title are different in the class of $\{0, 1\}$-sequentially regular spaces.

Let N be an infinite countable set and let $\mathcal{N}$ be an almost disjoint family of infinite subsets of N. Define a topology for $N \cup \mathcal{N}$ as follows: all points $n \in N$ are isolated; for each $A \in \mathcal{N}$, sets $\{A\} \cup A'$, where $A' \subset A$ and $A \setminus A'$ is finite, form a neighborhood base at A. In addition to other nice topological properties, $N \cup \mathcal{N}$ is a Hausdorff 0-dimensional (and hence $\{0, 1\}$-sequentially regular) Fréchet space. Clearly, a one-to-one sequence (x_n) convergences in $N \cup \mathcal{N}$ to a point $A \in \mathcal{N}$ iff $x_n \in A$ for all but finitely many n.

As a rule, each ordinal number is identified with the set of its predecessors. Let us denote by ω the first infinite ordinal. In [1] the following cardinal invariant $\varkappa$ has been introduced. Let $\varkappa$ be the smallest cardinal number for which there is a system $(\mathcal{U}_\alpha)_{\alpha \in \varkappa}$ such that:

1. For all $\alpha \in \varkappa$, $\mathcal{U}_\alpha = (U_{\alpha,\beta})_{\beta \in 2^\omega}$ is a maximal disjoint system of clopen sets in $\omega^* = \beta\omega \setminus \omega$;
2. There is no function $g : \varkappa \to 2^\omega$ such that $\bigcap_{\alpha \in \varkappa} U_{\alpha, g(\alpha)}$ has a nonempty interior.

The assumption $\varkappa = 2^\omega$ is weaker than Martin's axiom (cf. [1]) and hence, in particular, than the continuum hypothesis.

Dealing with one-to-one sequences and their subsequences the following convention will be used. We shall make no distinction between a one-to-one sequence (x_n) and the set $\{x_n;\ n \in \omega\}$. Further, if S is an infinite subset of ω, then by $(x_n)_{n \in S}$ we understand the subsequence of (x_n) the k-th term of which is x_{n_k}, where n_k is the k-th element of the ordered set S (a subset of ω).

Under the assumption $\varkappa = 2^\omega$, we are going to construct a space X of the type $N \cup \mathcal{N}$ which is sequentially complete but fails to be $\{0, 1\}$-sequentially complete. Let $T = \{\exp(2\pi i t);\ t \in [0, 1)\}$ be the unit circle and let $T_Q = \{\exp(2\pi i t);\ t \in [0, 1) \cap \cap Q\}$ be the set of all "rational" points of T. The space X can be visualized in the form of a cylinder. Put $X = (T \times 2^\omega) \setminus ((T \setminus T_Q) \times (\omega + 1))$. First, we identify the countable set $T_Q \times \omega$ with N via a one-to-one correspondence. Second, for each $(r, \alpha) \in X \setminus N$ we define an infinite countable subset $N(r, \alpha)$ of N so that for $(r, \alpha) \neq \neq (s, \beta)$ the intersection $N(r, \alpha) \cap N(s, \beta)$ is finite and the topology for X defined via the space $N \cup \mathcal{N}$ (where the set $\mathcal{N} = \{N(r, \alpha);\ (r, \alpha) \in X \setminus N\}$ is canonically identified with $X \setminus N$) has the desired properties. For $(r, \omega) \in T_Q \times \{\omega\}$ define $N(r, \omega) = \{(r, \alpha);\ \alpha \in \omega\}$. For $r \in T$ and $\alpha \in 2^\omega \setminus \omega + 1$ the set $N(r, \alpha)$ consists of points (s_n, α_n), where (s_n) is a one-to-one sequence in T_Q converging in T to r and (α_n)

is a suitable sequence in ω. To define the sets $N(r, \alpha)$ for $\alpha \in 2^{\omega} \setminus \omega + 1$ precisely, we need several auxiliary results.

As already stated, X is a $\{0, 1\}$-sequentially regular Fréchet space. We shall show that X is sequentially complete. In fact, we shall show that if A is a closed discrete infinite countable subset of X, then there is a real-valued continuous function f on X having on A at least two accumulation points. Finally, we shall prove that if the countable closed discrete set $T_Q \times \{\omega\} \subset X$ is decomposed into two infinite subsets Y and Z, then for each neighborhood U of Y and each neighborhood V of Z we have $\mathrm{cl}_X U \cap \mathrm{cl}_X V \neq \emptyset$. Consequently, each continuous $\{0, 1\}$-valued function on X is on $T_Q \times \{\omega\}$ almost constant. Thus X fails to be $\{0, 1\}$-sequentially complete.

For notational reasons, it will be convenient to fix $J = 2^{\omega} \setminus \omega + 1$. Throughout the paper we assume $\varkappa = 2^{\omega}$. The assumption is stressed each time we use it.

2. AUXILIARY RESULTS

Let $\mathscr{B}$ be the set of all mappings of T_Q into ω equipped with the quasi-order "modulo finite", i.e., for $g, h \in \mathscr{B}$ put $g < h$ whenever the set $\{q \in T_Q;\ g(q) \geqq h(q)\}$ is finite. Clearly, $\mathscr{B}$ is isomorphic to the Baire space ${}^{\omega}\omega$. As shown in [1], if $\varkappa = 2^{\omega}$, then each unbounded subset in ${}^{\omega}\omega$ has cardinality 2^{ω} and, in turn, this implies the existence of a scale (i.e. a well-ordered dominating family in ${}^{\omega}\omega$). We shall identify mappings in $\mathscr{B}$ with their graphs in $T_Q \times \omega$. Note that for each $(r, \alpha) \in T \times J$ the set $N(r, \alpha)$ will be a subset of the graph of a mapping $g_\alpha \in \mathscr{B}$, i.e., a restriction of g_α to a certain subset of T_Q.

Lemma 1. *Assuming $\varkappa = 2^{\omega}$, let $(g_\alpha)_{\alpha \in 2^{\omega}}$ be a scale in $\mathscr{B}$. Let (α_n) be a strictly increasing sequence of ordinal numbers smaller then 2^{ω}. Then there is a sequence (h_n), $h_n \subset g_{\alpha_n}$, such that:*

(i) *for each $n \in \omega$, $g_{\alpha_n} \setminus h_n$ is a finite set;*
(ii) *for each $k, n \in \omega$, $h_k \cap h_n = \emptyset$;*
(iii) *for each $\beta \in 2^{\omega} \setminus \{\alpha_n;\ n \in \omega\}$, $g_\beta \cap (\bigcup_{n \in \omega} h_n)$ is a finite set.*

Proof. Denote $\alpha = \sup\{\alpha_n;\ n \in \omega\}$ and $h_n = \{(q; g_{\alpha_n}(q));\ q \in T_Q$ and $g_{\alpha_k}(q) < g_{\alpha_n}(q) < g_\alpha(q)$ for all $k \in n\}$. Clearly, for each $n \in \omega$, $h_n \subset g_{\alpha_n}$ and $g_{\alpha_n} \setminus h_n$ is a finite set. To prove (ii), choose $\beta \in 2^{\omega}$. If $\beta > \alpha$, then $g_\beta \cap (\bigcup_{n \in \omega} h_n) = \bigcup_{n \in \omega} (g_\beta \cap h_n) \subset \{(q, g_\beta(q));\ q \in T_Q$ and $g_\beta(q) < g_\alpha(q)\}$, where the last set is finite. If $\alpha_k < \beta < \alpha_{k+1}$ for some $k \in \omega$, then $g_\beta \cap (\bigcup_{n \in \omega} h_n) = ((\bigcup_{n \leqq k} h_n) \cap g_\beta) \cup ((\bigcup_{n > k} h_n) \cap g_\beta) \subset \bigcup_{n \leqq k} (g_{\alpha_n} \cap g_\beta) \cup \{(q, g_\beta(q)),\ q \in T_Q$ and $g_{\alpha_{k+1}}(q) \leqq g_\beta(q)\}$, where the both sets in the union are finite. The case $\beta < \alpha_0$ easily follows from the previous one. This proves (iii).

Lemma 2. *Let r be a real number and let $\alpha \in 2^{\omega}$. Let f_β, $\beta \in \alpha$, be arbitrary functions on $(r - 1, r)$ into $[0, 1]$. Let (r_n) be a one-to-one sequence of real numbers* *in $(r-1, r)$ converging to r. Then there is an infinite subset S of ω such that for each $\beta \in \alpha$ the sequence $(f_\beta(r_n))_{n\in S}$ converges in $[0, 1]$.*

Proof. For each $\beta \in \alpha$, let $\mathscr{U}_\beta = (U_{\beta,\gamma})_{\gamma\in 2^\omega}$ be a maximal almost disjoint system of infinite subsets of ω (it is not contained property in any almost disjoint system of infinite subsets of ω) having the following property: the sequence $(f_\beta(r_n))_{n\in U_{\beta,\gamma}}$ converges whenever $U_{\beta,\gamma} \in \mathscr{U}_\beta$. Now, $\varkappa = 2^\omega$ implies the existence of an infinite subset S of ω such that for each $\beta < \alpha$ there exists an ordinal number $\gamma \in 2^\omega$ such that $S \setminus U_{\beta,\gamma}$ is a finite set. Consequently, the sequence $(f_\beta(r_n))_{n\in S}$ converges for each $\beta < \alpha$.

Lemma 3. *Let r be a real number and let $\alpha \in 2^\omega$. Let f_β, $\beta \in \alpha$ be continuous functions on $(r-1, r)$ into $[0, 1]$. Let (r_n) be an increasing sequence of real numbers converging to r. Then there is a sequence (U_n) of open subsets of $(r-1, r)$ such that: (i) $r_n \in U_n$ for each $n \in \omega$; (ii) if (s_n) is a sequence of real numbers such that $s_n \in U_n$, $n \in \omega$, and $\beta \in \alpha$, then the sequence $(f_\beta(r_n))$ converges iff the sequence $(f_\beta(s_n))$ converges.*

Proof. For each $n \in \omega$, let $(U_{n,k})_{k\in\omega}$ be a monotone fundamental system of open neighborhoods at r_n, where $U_{n,k} \subset (r-1, r)$. Let $\beta \in \alpha$. Then there is a mapping h_β of ω into ω such that for each sequence (s_n), $s_n \in U_{n,h_\beta(n)}$, the sequence $(f_\beta(s_n))$ converges iff the sequence $(f_\beta(r_n))$ converges. Let h be a mapping of ω into ω such that for each $\beta \in \alpha$ the set $\{n \in \omega;\ h_\beta(n) \geqq h(n)\}$ is finite (the existence of h is guaranteed by the assumption $\varkappa = 2^\omega$). Put $U_n = U_{n,h(n)}$, $n \in \omega$. The sequence (U_n) has the required properties.

3. CONSTRUCTION

For each $r \in T$, denote by T_r the set of all points $s \in T$ such that the distance between r and s (in the complex plane) is less than 1. Denote by I_r, D_r the sets of all "strictly increasing" and "strictly decreasing", respectively, sequences in $T_r \cap T_Q$ converging to r. Denote by P_r the set of all pairs (A_r, B_r), where A_r and B_r belong to I_r and, considered as sets, A_r and B_r are disjoint. The cardinality of P_r is clearly 2^ω. Let $(A_{r,\alpha}, B_{r,\alpha})_{\alpha\in J}$ be a one-to-one mapping of J onto P_r.

For each $r \in T$, let $(f_{r,\alpha})_{\alpha\in J}$ be a transfinite sequence of continuous functions $f_{r,\alpha} : T \setminus \{r\} \to [0, 1]$ such that

$f_{r,\alpha}[A_{r,\alpha}] = 0$,

$f_{r,\alpha}[B_{r,\alpha}] = 1$,

$f_{r,\alpha}$ is "symmetric about r", and

$f_{r,\alpha}$ is linear "between neighboring points of $A_{r,\alpha} \cup B_{r,\alpha}$".

As we shall see, functions $f_{r,\alpha}$, $(r, \alpha) \in T \times J$, play a fundamental role in the construction of our space X.

Lemma 4. *Let $r \in T$ and let (r_n) be a "strictly monotone" sequence in T_r converging* *to r. Then there is an ordinal number $\alpha \in J$ such that the sequence $(f_{r,\alpha}(r_n))$ does not converge.*

Proof. Since functions $f_{r,\alpha}$ are symmetric about r, we consider only increasing sequences. Passing to a suitable subsequence, the general case can be reduced to the following two. First, $(r_n) \in I_r$. Then the assertion follows directly from the construction of functions $f_{r,\alpha}$. Second, all r_n belong to $T_r \setminus T_Q$ (i.e. all r_n are "irrational"). Choose a sequence (s_n) in $T_r \cap T_Q$ such that $r_n < s_n < r_{n+1}$ for all $n \in \omega$. Let A be the subsequence of (s_n) consisting of those s_n for which $n = 4k$ or $n = 4k + 1$, $k \in \omega$ and let B be the subsequence of (s_n) consisting of the remaining terms. Then, for some $\alpha \in J$ we have $(A, B) = (A_{r,\alpha}, B_{r,\alpha})$. Hence the sequence $(f_{r,\alpha}(r_n))$ has at least two accumulation points, one of which is 0 and the other 1.

Definition. Let $r \in T$ and let $A = (r_n)$ be a "strictly monotone" sequence in T_r converging to r. Denote by $o(A)$ the smallest ordinal number α, $\alpha \in J$, for which the sequence $(f_{r,\alpha}(r_n))$ does not converge. It will be called the order of A.

The next assertion follows directly from Lemma 2.

Lemma 5. *Let $r \in T$ and let A be a "strictly monotone" sequence in T_r converging to r. Then for each ordinal number $\alpha \in J$ there is a subsequence B of A such that $o(B) > \alpha$.*

Lemma 6. *Let Y and Z be infinite disjoint subsets of T_Q such that $Y \cup Z = T_Q$. Then there is a point $r \in T$ such that for each ordinal number $\alpha \in J$ there are sequences (p_n) in Y and (q_n) in Z, both belonging to $I_r \cup D_r$, such that $o((p_n)) = o((q_n)) > \alpha$.*

Proof. Fitst, suppose that both Y and Z are dense in T. Then, using Lemma 5, choose $r \in T$ and, for a given $\alpha \in J$, choose an increasing sequence (r_n) in T_r converging to r such that $o((r_n)) > \alpha$. Since Y and Z are dense in T, it follows from Lemma 3 that there are sequences (p_n) in Y and (q_n) in Z, both belonging to I_r, such that $o((p_n)) = o((q_n)) = o((r_n)) > \alpha$. Second, one of the sets, say Y, is not dense in T. Then there is an interval $[r, s] \subset T$ such that $[r, s] \cap Y$ is a finite set and one of the endpoints, say r, is an accumulation point of Y. By Lemma 5, for a given $\alpha \in J$, there is a sequence $(p_n) \in I_r$ such that $p_n \in Y$ for all $n \in \omega$ and $o((p_n)) > \alpha$. Let (r_n) be the sequence which is symmetric to (p_n) about r. Then $o((p_n)) = o((r_n))$. It follows from Lemma 3 that there is a sequence (q_n) in Z, $(q_n) \in D_r$, such that $o((r_n)) = o((q_n))$.

Clearly, for each $r \in T$ there are 2^ω partitions of T_Q into two infinite subsets Y, Z such that for each $\alpha \in J$ there are sequences (p_n) in Y and (q_n) in Z, both in $I_r \cup D_r$, for which $o((p_n)) = o((q_n)) > \alpha$. Let $(Y_{r,\beta}, Z_{r,\beta})_{\beta \in J}$ be a one-to-one mapping of J onto the set of all such partitions.

Finally, let u be a bijection of $J = 2^\omega \setminus \omega + 1$ onto $J \times J$, let $v : J \times J \to J$ be the canonical projection onto the first factor, and let $w = v \circ u : J \to J$ be their composition. Then w is a transfinite sequence of ordinal numbers in J and each $\alpha \in J$ occurs in $(w(\alpha))_{\alpha \in J}$ 2^{ω}-times.

Now we are ready to define the sets $N(r, \alpha)$ for $r \in T$ and $\alpha \in J$. Assuming $\varkappa = 2^{\omega}$, let $(g_\beta)_{\beta \in J}$ be a scale in $B(\approx^{\omega}\omega)$. Choose sequences (p_n) in $Y_{r,w(\alpha)}$ and (q_n) in $Z_{r,w(\alpha)}$, both in $I_r \cup D_r$, such that $o((p_n)) = o((q_n)) > \alpha$. Define a sequence $S(r, \alpha) = (s_n(r, \alpha))$ as follows: $s_{2n}(r, \alpha) = p_n$ and $s_{2n+1}(r, \alpha) = q_n$, $n \in \omega$. Then $S(r, n)$ is a sequence in T_Q converging to r and $o(S(r, \alpha)) > \alpha$. Put $N(r, \alpha) = g_\alpha \mid S(r, \alpha) = \{(s_n(r, \alpha), g_\alpha(s_n(r, \alpha)));$ $n \in \omega\}$. It follows readily that $\{N(r, \alpha);\ r \in T,\ \alpha \in J\}$ is an almost disjoint family of infinite subsets of $T_Q \times \omega$. Recall that for $\alpha = \omega$ and $r \in T_Q$ we have $N(r, \omega) =$ $= \{(r, n);\ n \in \omega\}$. Hence $\mathcal{N} = \{N(r, \alpha);\ (r, \alpha) \in X \setminus (T_Q \times \omega)\}$ is an almost disjoint family of infinite subsets of $N = T_Q \times \omega$.

4. SEQUENTIAL COMPLETENESS VERSUS {0, 1}-SEQUENTIAL COMPLETENESS

Recall that the space $N \cup \mathcal{N}$, and hence also X, is a $\{0, 1\}$-sequentially regular Fréchet space. Our final task is to verify that X is sequentially complete but fails to be $\{0, 1\}$-sequentially complete.

Proposition 1. *The space X is sequentially complete.*

Proof. Let (x_n) be a sequence in X no subsequence of which converges. We have to construct a real-valued continuous function f on X such that the sequence $(f(x_n))$ does not converge in the real line. Clearly, we can assume that (x_n) is one-to-one. Since X is a Fréchet space, $A = (x_n)$ is a closed discrete subset of X. It suffices to consider three cases.

1. $A \subset T_Q \times \omega$. Define a function f on X as follows: $f(x_{2n}) = 1$ for all $n \in \omega$ and $f(x) = 0$ otherwise. Since A is closed and discrete, f has the desired properties.

2. $A \subset T_Q \times \{\omega\}$. Then there is a one-to-one sequence (q_n) in T_Q such that $x_n = (q_n, \omega)$, $n \in \omega$. Let $r \in T$ be an accumulation point of (q_n). Then for some $\alpha \in J$, the function $f_{r,\alpha} : T \setminus \{r\} \to [0, 1]$ oscillates on (q_n), i.e., sets $\{q_n;\ f_{r,\alpha}(q_n) = 0\}$ and $\{q_n;\ f_{r,\alpha}(q_n) = 1\}$ are infinite. So it suffices to show that there is a continuous function $f : X \to [0, 1]$ such that $f((q_n, \omega)) = f_{r,\alpha}(q_n)$ for all $n \in \omega$. Define

$f((q, \omega)) = f_{r,\alpha}(q)$ for $q \in T_Q \setminus \{r\}$,
$f((q, n)) = f_{r,\alpha}(q)$ for $q \in T_Q \setminus \{r\}$ and $n > g_\alpha(q)$,
$f((s, \beta)) = f_{r,\alpha}(s)$ for $s \in T$, $s \neq r$, and $\beta > \alpha$,
$f((r, \beta)) = \lim f_{r,\alpha}(s_n(r, \beta))$ for $\beta > \alpha$, where $(s_n(r, \beta)) = S(r, \beta)$ and $N(r, \beta) =$ $= g_\beta \mid S(r, \beta)$ (note that $\lim f_{r,\alpha}(s_n(r, \beta))$ is well-defined since $o(S(r, \beta)) > \beta > \alpha$, and
$f(x) = 0$ otherwise.

It follows from the construction of f that f is a continuous function on X.

3. $A \subset T \times J$. Then there are two possibilities.

3.1. There is a strictly increasing sequence (α_n) in J and a sequence (r_n) in T such that $((r_n, \alpha_n))$ is a subsequence of A. Then, by Lemma 1, there is a sequence (h_n),

$h_n \subset g_{\alpha_n}$, such that: (i) for each $n \in \omega$, $g_{\alpha_n} \setminus h_n$ is a finite set, (ii) for each $k, n \in \omega$, $h_k \cap h_n = \emptyset$, (iii) for each $\beta \in J \setminus \{\alpha_n; n \in \omega\}$, $g_\beta \cap (\bigcup_{n \in \omega} h_n)$ is a finite set. Arrange T_Q into a one-to-one sequence (q_n). Define a function f on X as follows:

$f((q_k, h_{2n}(q_k))) = 1$ for $n \in \omega$ and $k > n$,
$f((r, \alpha_{2n})) = 1$ for $r \in T$ and $n \in \omega$, and
$f(x) = 0$ otherwise.

Then f is a continuous function on X, $f[\bigcup_{n \in \omega} (r_{2n}, \alpha_{2n})] = 1$ and $f[\bigcup_{n \in \omega} (r_{2n+1}, \alpha_{2n+1})] = 0$. Clearly, sets $\{x_n; f(x_n) = 0\}$ and $\{x_n; f(x_n) = 1\}$ are infinite.

3.2. There is an ordinal number $\alpha \in J$ such that $x_n \in T \times \{\alpha\}$ for all but finitely many $n \in \omega$. Then there exists $r \in T$ and a strictly monotone, say decreasing, sequence (r_n) in T_r converging to r such that $((r_n, \alpha))$ is a subsequence of A. Consider sets $S(r_n, \alpha)$, $n \in \omega$ and $S(r, \alpha)$. They form an almost disjoint family in T_Q. It is easy to see that there are subsets $S'(r_n, \alpha)$ of $S(r_n, \alpha)$ such that $S(r_n, \alpha) \setminus S'(r_n, \alpha)$ are finite sets, $S(r, \alpha) \cap S'(r_n, \alpha) = \emptyset$ for all $n \in \omega$ and $S'(r_k, \alpha) \cap S'(r_1, \alpha) = \emptyset$ whenever $k \neq 1$. Define a function f on X as follows:

$f[g_\alpha \mid S'(r_{2n}, \alpha)] = f((r_{2n}, \alpha)) = 1$ for $n \in \omega$, and
$f(x) = 0$ otherwise.

Then f is a continuous function on X and $(f(x_n))$ fails to converge in the real line. This completes the proof.

Proposition 2. *The space X fails to be $\{0, 1\}$-sequentially complete.*

Proof. Arrange T_Q into a one-to-one sequence (q_n). Then $T_Q \times \{\omega\} = ((q_n, \omega))$ is a sequence in X no subsequence of which converges. It suffices to prove that for each $\{0, 1\}$-valued continuous function f on X, one of the sets $Y = \{q_n \in T_Q; f((q_n, \omega)) = 0\}$, $Z = \{q_n \in T_Q; f((q_n, \omega)) = 1\}$ is finite. Clearly, then $(f((q_n, \omega)))$ is a convergent sequence.

Suppose that, on the contrary, both Y and Z are infinite. It follows from Lemma 6 that for some $r \in T$ and $\beta \in J$ we have $(Y, Z) = (Y_{r,\beta}, Z_{r,\beta})$. Observe that if A is a subset of $T_Q \times \{\omega\}$, then sets $V(A, h) = A \cup \{(q, n) \in T_Q \times \omega; q \in A \text{ and } n > h(q)\}$, $h \in \mathscr{B}$, form an open base of A. Choose an element $h \in \mathscr{B}$ such that $f((q, n)) = f((q, \omega))$ whenever $n \in \omega$ and $n > h(q)$. Since the set $\{\alpha \in J; w(\alpha) = \beta\}$ is cofinal in J, there exists $\alpha(h) \in J$ such that $w(\alpha(h)) = \beta$ and $g_{\alpha(h)} > h$. Hence $(r, \alpha(h))$ belongs to the closure of $V(Y, h)$ and at the same time to the closure of $V(Z, h)$. Since $f[V(Y, h)] = 0$ and $f[V(Z, h)] = 1$, we have a contradiction.

Remark 1. Consider the subspace $T_Q \times \{\omega\}$ of X. Using a slight modification of the proof of Proposition 2, it can be verified that no continuous $\{0, 1\}$-valued function f on $T_Q \times \{\omega\}$ for which both sets $\{q_n \in T_Q; f((q_n, \omega)) = 0\}$ and $\{q_n \in T_Q; f((q_n, \omega)) = 1\}$ are infinite can be extended to a continuous real-valued function on X.

Remark 2. Consider the set $X^* = X \cup \{p\}$ equipped with the following topology:

X is an open subspace of X^* and a neighborhood base at p consists of sets $\{p\} \cup A$, where A is a subset of $T_Q \times \{\omega\}$ such that $(T_Q \times \{\omega\}) \setminus A$ is finite. It can be easily checked that X is a $\{0, 1\}$-sequential envelope of X and, since X is sequentially complete, X is a sequential envelope of itself (cf. [2]). Thus X is another example of a $\{0, 1\}$-sequentially regular space the sequential envelope and the $\{0, 1\}$-sequential envelope of which are different (cf. [3]). Note that X is a sequential space but fails to be Fréchet.

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