SEQUENTIAL CONFIDENCE INTERVAL FOR THE REGRESSION COEFFICIENT BASED ON KENDALL'S TAU

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SUMMARY

The object of the present investigation is to consider a robust procedure for the problem of providing a bounded-length confidence interval for the regression coefficient (in a simple regression model) based on Kendall's (1955) tau. The problem of estimating the difference in the location parameters in the two-sample case may be viewed as a special case of our problem. It is shown that the estimate of the regression coefficient based on Kendall's tau [cf. Sen (1968)], as extended here in the sequential case, possesses certain desirable properties. Comparison with the procedure based on the least squares estimator [considered by Gleser (1965) and Albert (1966)] is also made.

1. INTRODUCTION

Consider a sequence $\{X_1, X_2, \ldots.\}$ of independent (real valued) random variables with (absolutely) continuous cumulative distribution functions (cdf) $F_1(x), F_2(x), \ldots,$ where

$$F_i(x) = F(x-\alpha-\beta t_i), \quad i=1,2,\ldots; \tag{1.1}$$

the t_i are known regression constants, β is the regression coefficient and α is a nuisance parameter. It is desired to determine a confidence interval

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$I_n = \{\beta: \hat{\beta}_{L,n} \leq \beta \leq \hat{\beta}_{U,n}\}$ such that (i) $P\{\beta \epsilon I_n\} = 1-\alpha$, the desired confidence coefficient and (ii) $0 \leq \hat{\beta}_{U,n} - \hat{\beta}_{L,n} \leq 2d$, (d predetermined). Since the form F( ) is not known, we are not in a position to prescribe any fixed-sample size procedure valid for all F.

Sequential procedures for such a problem, based on the classical least squares estimators of α and β, are due to Gleser (1965) and Albert (1966). These procedures, like ours (to follow), are based on the method suggested by Anscombe (1952) and Chow and Robbins (1965). However, being based on the least squares estimates, these procedures are vulnerable to gross errors or outliers, and may be quite inefficient for distributions with heavy tails (e.g., the Cauchy, logistic or the double exponential distribution). For this reason, we consider here an alternative robust procedure based on the nonparametric estimate of β, considered in Sen (1968). The procedure is explained in Section 2. Section 3 deals with the main results of the paper. Section 4 is devoted to the study of the asymptotic relative efficiency (A.R.E.) of the different procedures. In Section 5, we consider the two-sample location problem, which is a particular case of (1.1) when the t_i can be either 0 or 1. The appendix is devoted to the asymptotic linearity of a stochastic process involving Kendall's tau; this result is used repeatedly in Section 3.

2. THE PROPOSED SEQUENTIAL PROCEDURE

For every real $b(-\infty<b<\infty)$, define $Z_i(b) = X_i - bt_i$, $i=1,2,\ldots$, and let

$$U_n(b) = \binom{n}{2}^{-1} \sum_{1\leq i<j\leq n} c(t_j-t_i)c(Z_j(b)-Z_i(b)), \tag{2.1}$$

where $c(u)$ is equal to 1, 0 or -1 according as $u>$, $=$ or <0. Note that by definition, $U_n(\beta)$ is distribution-free and its distribution (symmetric about zero) is tabulated (for small values of n) by Kendall (1955) and Smid (1956). If we let

$$V_n = (1/18)\{n(n-1)(2n+5)-\sum_{j=1}^{a_n} u_{nj}(u_{nj}-1)(2u_{nj}+5)\}, \tag{2.2}$$

[where among $(t_1,\ldots,t_n)$ there are $a_n (\geq 2)$ distinct values with frequencies u_{nj}, $j=1,\ldots,a_n$], then for large n, $\binom{n}{2}U_n(\beta)/V_n^{\frac{1}{2}}$ has approximately a normal distribution with mean 0 and variance 1 [cf. Hoeffding (1948) and Kendall (1955)]. As such, for each n and $\underset{\sim}{t}_n = (t_1,\ldots,t_n)$, we can find an $U^*_{n,\underset{\sim}{t}_n}$ such that

$$P\{-U^*_{n,\underset{\sim}{t}_n} \leq U_n(\beta) \leq U^*_{n,\underset{\sim}{t}_n} | \beta\} = 1-\alpha_n, \tag{2.3}$$

where $\alpha_n \to \alpha$ as $n \to \infty$. [For large n, $U^*_{n,\underset{\sim}{t}_n} \simeq \tau_{\alpha/2} V_n^{\frac{1}{2}}/\binom{n}{2}$, where $\tau_{\alpha/2}$ is the upper $100\alpha/2\%$ point of the standard normal distribution.] As in Sen (1968), we use (2.3) to derive the following (fixed-sample) confidence interval for β. Let

$$Y_{ij} = (X_j-X_i)/(t_j-t_i), \quad (i,j)\varepsilon S_n, \tag{2.4}$$

where $S_n = \{(i,j): t_i \neq t_j, 1 \leq i < j \leq n\}$. We denote by n^* the number of elements of S_n (i.e., the number of distinct pairs (t_i,t_j) in $\underset{\sim}{t}_n$), and denote the n^* ordered values in (2.4) by

$$Y_{n(1)} \leq Y_{n(2)} \leq \cdots \leq Y_{n(n^*)}, \quad n \geq 2, \tag{2.5}$$

Also, let

$$M_n^{(1)} = \tfrac{1}{2}[n^* - \tbinom{n}{2} U^*_{n,\underset{\sim}{t}_n}], \quad M_n^{(2)} = \tfrac{1}{2}[n^* + \tbinom{n}{2} U^*_{n,\underset{\sim}{t}_n}]. \tag{2.6}$$

Then, as in Sen (1968), we have

$$P\{Y_{n(M_n^{(1)})} < \beta < Y_{n(M_n^{(2)}+1)} | \beta\} = 1-\alpha_n. \tag{2.7}$$

Now, in order to obtain a confidence interval for β of length $\leq 2d$, we define a stopping variable $N=N(d)$ to be the first integer $\geq n_o$ (the initial sample size, may be 2 or more), for which $Y_{n(M_n^{(2)}+1)} - Y_{n(M_n^{(1)})} \leq 2d$. Then, our proposed (sequential) confidence interval for β is

$$I_{N(d)} = \{\beta\colon Y_{N(d)(M_{N(d)}^{(1)})} < \beta < Y_{N(d)(M_{N(d)}^{(2)}+1)}\}. \tag{2.8}$$

In principle, our procedure is similar to that of Anscombe (1952) and of Chow and Robbins (1965). However, they used the sample mean square due to error to set up an appropriate confidence interval, whereas we use Kendall's tau to derive such an interval. The main results are stated in the next section.

3. PROPERTIES OF THE CONFIDENCE INTERVAL $I_{N(d)}$

In the remainder of the paper, we shall stick to the following notations and assumptions. Let

$$T_n^2 = \sum_{i=1}^n (t_i - \bar{t}_n)^2, \text{ where } \bar{t}_n = n^{-1}\sum_{i=1}^n t_i, \tag{3.1}$$

$$\rho_n = \{\sum_{i=1}^n (t_i - \bar{t}_n)(i - \tfrac{n+1}{2})\}/\{A_n T_n\}; \tag{3.2}$$

$$A_n^2 = (1/12)\{n(n^2-1) - \sum_{j=1}^{a_n} u_{nj}(u_{nj}^2 - 1)\}. \tag{3.3}$$

Then, concerning $\{\underset{\sim}{t}_n\}$, we assume that

$$\text{(i) } \max_{1\leq i\leq n} |t_i - \bar{t}_n|/T_n \to 0 \text{ as } n\to\infty, \tag{3.4}$$

$$\text{(ii) } \liminf_n n^{-1} T_n^2 > c > 0 \ (\Rightarrow \lim_{n\to\infty} T_n^2 = \infty), \tag{3.5}$$

$$\text{(iii) } \liminf_n [\sum_{j=1}^{a_n} u_{nj}(n - u_{nj})/n^2] > 0, \tag{3.6}$$

(iv) $\lim_{n\to\infty} \rho_n = \rho$, where $|\rho|>0$, (3.7)

(v) $T_n^2 = Q(n)$ is a strictly increasing function of n, such that for every $a>0$

$$\lim_{n\to\infty} Q(a_n n)/Q(n) = s(a) \text{ whenever } \lim_{n\to\infty} a_n = a, \tag{3.8}$$

where $s(a)$ is strictly monotonically increasing and continuous in a, and $s(1)=1$.

Note that our (ii) and (v) are less restrictive than similar conditions of Gleser (1965), who requires that $\lim_{n\to\infty} n^{-1}T_n^2 = c>0$. If $t_i=i$, $i=1,2,\ldots$, (i)-(v) hold, but Gleser's does not. Note that by (3.3) and (3.6), we have

$$n^{-3}A_n^2 = O(1). \tag{3.9}$$

Concerning $F(x)$ in (1.1), we assume that $f(x) = F'(x)$ exists (a.e.), and is continuous in x (a.e.); also assume that

$$\int_{-\infty}^{\infty} f^2(x)dx = B(F)<\infty. \tag{3.10}$$

Our main theorem of the paper is the following.

Theorem 1. Under the assumptions made above, $N(=N(d))$ is a non-increasing function of $d(>0)$, $N(d)$ is finite almost surely (a.s.), $E[N(d)]<\infty$ for all $d>0$, $\lim_{d\to 0} N(d)=\infty$ a.s., and $\lim_{d\to 0} E[N(d)]=\infty$. Further,

$$\lim_{d\to 0} N(d)/Q^{-1}(\nu(d)) = 1 \text{ a.s.}, \tag{3.11}$$

$$\lim_{d\to 0} P\{\beta\varepsilon I_{N(d)}\} = 1-\alpha \quad \text{for all } F, \tag{3.12}$$

$$\lim_{d\to 0} E[N(d)]/Q^{-1}(\nu(d)) = 1, \tag{3.13}$$

where

$$\nu(d) = \tau_{\alpha/2}^2[12\rho^2 d^2 B^2(F)]^{-1}. \tag{3.14}$$

Proof. The proof of the theorem is completed in several steps. First, we let $\hat{\beta}_{L,n} = Y_{n(M_n^{(1)})}$ and $\hat{\beta}_{U,n} = Y_{n(M_n^{(2)}+1)}$, which are defined as in (2.6) and (2.7). Then we have the following.

Lemma 3.1. For every $\delta>0$ there exists an $n_o(\delta)$, such that for $n\geq n_o(\delta)$,

$$P\{\hat{\beta}_{U,n} > \beta+(\log n)/(\rho_n T_n)\} \leq n^{-(1+\delta)}, \tag{3.15}$$

$$P\{\hat{\beta}_{L,n} < \beta-(\log n)/(\rho_n T_n)\} \leq n^{-(1+\delta)}. \tag{3.16}$$

We shall only prove (3.15) as (3.16) follows on the same line. By definition of $\hat{\beta}_{U,n}$, $U_n(\hat{\beta}_{U,n}) > -U^*_{n,\underset{\sim}{t}_n}$. Hence,

$$\begin{aligned} P\{\hat{\beta}_{U,n} > \beta+(\log n)/(\rho_n T_n)\} &= P\{U_n(\beta+(\log n)/\rho_n T_n) \geq -U^*_{n,\underset{\sim}{t}}\} \\ &= P\{U_n(\beta+(\log n)/\rho_n T_n) - E[U_n(\beta+(\log n)/\rho_n T_n)] \\ &\qquad \geq - U^*_{n,\underset{\sim}{t}_n} -E[U_n(\beta+(\log n)/\rho_n T_n)]\}. \end{aligned} \tag{3.17}$$

Using the fact that for large n, $U^*_{n,\underset{\sim}{t}_n} \doteqdot \tau_{\alpha/2} V_n^{\frac{1}{2}}/\binom{n}{2} = O(n^{-\frac{1}{2}})$, and $E[U_n(\beta+(\log n)/\rho_n T_n)] = - 4A_n(\log n)\binom{n}{2}^{-1}[B(F)+o(1)] = O(n^{-\frac{1}{2}} \log n)$, we may write $-U^*_{n,\underset{\sim}{t}_n} -E[U_n(\beta+(\log n)/\rho_n T_n)]$ as

$$4A_n(\log n)\binom{n}{2}^{-1}B(F)[1+o(1)+O(\log n)^{-1}]. \tag{3.18}$$

Now, we make use of a result by Hoeffding (1963; (5.7)) on the deviation of a

U-statistic and obtain from (3.17) and (3.18) that for n adequately large

$$P\{\hat{\beta}_{U,n} > \beta+(\log n)/\rho_n T_n\} \leq \exp[-[\tfrac{n}{2}]t_n^2/4]$$
$$\leq \exp[-(n-2)t_n^2/8], \tag{3.19}$$

where $t_n = 4A_n(\log n)\binom{n}{2}^{-1}B(F) = O(n^{-\frac{1}{2}}\log n)$. Thus, $(n-2)t_n^2/8$ can be made greater than $(1+\delta)\log n$, for any $\delta>0$, when n is made greater than $n_o(\delta)$. Hence, for $n\geq n_o(\delta)$, the right hand side of (3.19) can be made less than $\exp[-(1+\delta)\log n] = n^{-(1+\delta)}$. Q.E.D.

Lemma 3.2. *There exists two positive numbers* δ_1 *and* δ_2 *and an* n, *say* $n_o(\delta_1,\delta_2)$, *such that for all* $n \geq n_o(\delta_1,\delta_2)$,

$$P\{|\sqrt{3}\, B(F)\rho_n T_n(\hat{\beta}_{U,n}-\hat{\beta}_{L,n})/\tau_{\alpha/2}-1| \geq O(n^{-\delta_1}\log n)\} \leq O(n^{-\delta_2-1}). \tag{3.20}$$

The proof of the lemma directly follows from lemma 3.1 and the following theorem whose proof is sketched in the Appendix.

Theorem 3.3. *There exists two positive* δ_1,δ_2 *and an* $n_o(\delta_1,\delta_2)$ *such that for* $n \geq n_o(\delta_1,\delta_2)$,

$$P\{\sup_{|a|<\log n}|\binom{n}{2}\{U_n(\beta+a/\rho_n T_n) - U_n(\beta)\} + 4aB(F)A_n|/V_n^{\frac{1}{2}}$$
$$\geq O(n^{-\delta_1}\log n)\} \leq O(n^{-1-\delta_2}). \tag{3.21}$$

Lemma 3.4. *For every real* $x(-\infty<x<\infty)$

$$\lim_{n\to\infty} P\{\sqrt{12}\, B(F)\rho_n T_n(\hat{\beta}_{U,n}-\beta)-\tau_{\alpha/2}\leq x\} = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}dt. \tag{3.22}$$

The proof is given in Sen (1968). For the asymptotic normality of $\rho_N T_N(\hat{\beta}_{U,N}-\beta)$, we require, as in Anscombe (1952), the "uniform continuity in probability" of $\{\hat{\beta}_{U,n}\}$ with respect to $\rho_n T_n$. For this, we have the following lemma.

Lemma 3.5. *For every positive* ε *and* η *there exists a* $\delta(>0)$, *such that as* $n\to\infty$

$$P\{\sup_{|n-n'|<\delta n} |\rho_n T_n(\hat{\beta}_{U,n}-\hat{\beta}_{U,n'})| > \eta\} < \varepsilon, \tag{3.23}$$

and a similar statement holds for $\{\hat{\beta}_{L,n}\}$.

Proof. By virtue of lemma 3.1 and theorem 3.3, and the definition of $\hat{\beta}_{U,n}$,

$$\begin{aligned}
\rho_n T_n(\hat{\beta}_{U,n'}-\hat{\beta}_{U,n}) &= \rho_{n'}T_{n'}(\hat{\beta}_{U,n'}-\beta)(\rho_n T_n/\rho_{n'}T_{n'})-\rho_n T_n(\hat{\beta}_{U,n}-\beta) \\
&= \{\tbinom{n'}{2}/\sqrt{12}B(F)V_{n'}^{\frac{1}{2}}\}\{U_n(\hat{\beta}_{U,n'})-U_n(\beta)+o(1)\}\{\rho_n T_n/\rho_{n'}T_{n'}\} \\
&\quad - \{\tbinom{n}{2}/\sqrt{12}\, B(F)V_n^{\frac{1}{2}}\}\{U_n(\hat{\beta}_{U,n})-U_n(\beta)+o(1)\} \\
&= \{\tau_{\alpha/2}/\sqrt{12}\, B(F)\}\{(\rho_n T_n)/(\rho_{n'}T_{n'})-1\} + \\
&\quad [\sqrt{12}B(F)]^{-1}\{[(\rho_n T_n)/(\rho_{n'}T_{n'})]\tbinom{n'}{2}/V_{n'}^{\frac{1}{2}}U_{n'}(\beta)-\tbinom{n}{2}V_n^{-\frac{1}{2}}U_n(\beta)\}+o(1) \\
&= [\sqrt{12}B(F)]^{-1}\{[\rho_n T_n\tbinom{n'}{2}/\rho_{n'}T_{n'}V_{n'}^{\frac{1}{2}}]-\tbinom{n}{2}/V_n^{\frac{1}{2}}\}U_{n'}(\beta) \\
&\quad + [\tbinom{n}{2}/V_n^{\frac{1}{2}}][U_{n'}(\beta)-U_n(\beta)] + o(1).
\end{aligned} \tag{3.24}$$

Since $U_n(\beta)$ is a U-statistic in the independent and identically distributed random variables $Z_i(\beta)$, $1\le i\le n$, it follows from Berk (1966) that $\{U_n(\beta)\}$ forms a reverse martingale sequence. Hence, if we write $n_o=n\delta$, and let $W_j = U_{n+n_o-j}(\beta) - U_{n+n_o-j+1}(\beta)$, $1\le j\le 2n_o$, $W_o = U_{n+n_o}(\beta)$, it follows that $\{W_0,W_1,\ldots,W_{2n_o}\}$ possesses the properties of a martingale difference sequence. Hence, by using the well-known

Kolmogorov inequality [cf. Loève (1963, p. 386)], we have

$$P\{\max_{1\le j\le 2n_o}|W_1+\ldots+W_j|>t_n\} \le t_n^{-2}\, E(W_1+\ldots+W_{2n_o})^2 \tag{3.25}$$

But, $W_1+\ldots+W_j = U_{n+n_o-j}(\beta)-U_{n+n_o}(\beta)$, $1\le j\le 2n_o$, and the right hand side of (3.25) is equal to $t_n^{-2}\{V[U_{n-n_o}(\beta)]-V[U_{n+n_o}(\beta)]\} = (4/9)(2n_o t_n^{-2}/(n^2-n_o^2))(1+O(n^{-1}))$. Hence, letting $t_n = \eta n^{-\frac{1}{2}}$, we have

$$P\{\max_{1\le j\le 2n_o} n^{\frac{1}{2}}|U_{n+n_o-j}(\beta)-U_{n+n_o}(\beta)|>\eta\} \le (8\delta/9\eta^2)(1+O(n^{-1})). \tag{3.26}$$

This immediately leads to (by proper choice of δ)

$$P\{\max_{|n-n'|<\delta n} n^{\frac{1}{2}}|U_{n'}(\beta)-U_n(\beta)|>\eta'\} \le \varepsilon', \tag{3.27}$$

for n sufficiently large. Also, by (5.7) of Hoeffding (1963),

$$P\{|U_n(\beta)|>K_{\varepsilon''}\}\le\varepsilon'', \text{ where } K_{\varepsilon''}<\infty. \tag{3.28}$$

Now, we write $[\rho_n T_n\binom{n'}{2}/\rho_{n'}T_{n'}V_{n'}^{\frac{1}{2}} - \binom{n}{2}/V_n^{\frac{1}{2}}]$ as $(\rho_n T_n/\rho_{n'}T_{n'}-1)\binom{n'}{2}/V_{n'}^{\frac{1}{2}} + [\binom{n'}{2}-\binom{n}{2}]/V_{n'}^{\frac{1}{2}} + \binom{n}{2}[V_{n'}^{-\frac{1}{2}}-V_n^{-\frac{1}{2}}]$. Note that for $|n'-n|\le\delta n$, $\rho_n T_n/\rho_{n'}T_{n'} \to s(n'/n) \in [s(1-\delta),\ s(1+\delta)]$, and hence, differs from 1 by an arbitrarily small quantity, as $n\to\infty$ (by (iv) and (v)), $n\binom{n}{2}/V_n = 9/2 + O(n^{-1})$ (by (iii)), $|\binom{n'}{2}-\binom{n}{2}|\le\delta(2+\delta)n^2$, and $|V_{n'}^{-\frac{1}{2}}-V_n^{-\frac{1}{2}}| = |V_{n'}-V_n|/\{|V_{n'}^{\frac{1}{2}}+V_n^{\frac{1}{2}}|V_n^{\frac{1}{2}}V_{n'}^{\frac{1}{2}}\} = O(n^{-3})|V_{n'}-V_n| = O(n^{-3})[(2/9)\{n^3(1+3\delta+O(\delta^2))-n^3\} + O(\delta)n^3] = O(\delta)$, where the last order follows by using (iii) on the second term of the right hand side of (2.2). Thus, (3.23) follows from (3.24), (3.27), (3.28) and the above discussion. Hence the lemma.

Returning now to the proof of theorem 1, we note that by lemma 3.2 and the definition of N(d), for all d>0, N(d) is finite a.s., and it is non-increasing in d. To show that $E[N(d)]<\infty$ for every d>0, we write

$$E[N(d)] = \Sigma nP\{N(d)=n\} = \Sigma_{n\geq 0}P\{N(d)>n\}. \tag{3.29}$$

Hence, it suffices to show that for every $d>0$, $\Sigma_{n\geq 0}P\{N(d)>n\}$ converges; a sufficient condition for this is to show that as $n\to\infty$,

$$P\{N(d)>n\} \leq O(n^{-1-\gamma}),\text{ where } \gamma>0. \tag{3.30}$$

Now, by definition

$$\begin{aligned} P\{N(d)>n\} &= P\{(\hat{\beta}_{U,k}-\hat{\beta}_{L,k})>2d, \forall\, k\leq n\} \\ &\leq P\{(\hat{\beta}_{U,n}-\hat{\beta}_{L,n})>2d\} = P\{\sqrt{3}\, B(F)\rho_n T_n(\hat{\beta}_{U,n}-\hat{\beta}_{L,n})>2d\sqrt{3}\, B(F)\rho_n T_n\} \\ &\leq O(n^{-1-\delta_2}),\ \delta_2>0, \end{aligned} \tag{3.31}$$

as by (3.20), $\sqrt{3}\, B(F)\rho_n T_n(\hat{\beta}_{U,n}-\hat{\beta}_{L,n})-1 = O(n^{-\delta_1}\log n)$ with probability $\geq 1-O(n^{-1-\delta_2})$, and by (ii) and (iv) $\rho_n T_n\to+\infty$ as $n\to\infty$. Thus, $E[N(d)]<\infty$, for every $d>0$. Using the fact that $\hat{\beta}_{U,n}-\hat{\beta}_{L,n}>0$ (a.s.) for each n, we have $\lim_{d\to 0} N(d) = \infty$. Now, from the Monotone Convergence Theorem, $\lim_{d\to 0} E[N(d)]=\infty$.

(3.11) is a direct consequence of (3.20), the definition of $\nu(d)$ in (3.14), (iv) in (3.7) and (v) after (3.7). (3.12) follows readily from theorem 1 of Anscombe (1952) along with our lemma 3.2, lemma 3.4 and lemma 3.5. To prove (3.13), we write

$$E[N(d)]/Q^{-1}[\nu(d)] = \{Q^{-1}[\nu(d)]\}^{-1}\Sigma_1+\Sigma_2+\Sigma_3\ nP\{N(d)=n\}, \tag{3.32}$$

where the summation Σ_1 extends over all $n<n_1(d)$, Σ_2 over all $n_1(d)\leq n\leq n_2(d)$ and Σ_3 over all $n>n_2(d)$, and where

$$Q(n_i(d)) = \nu(d)\{1+(-1)^i\varepsilon + (1+(-1)^i)/2\},\ i=1,2. \tag{3.33}$$

Since $\lim_{d\to 0} \nu(d) = \infty$ and $\lim_{d\to 0} N(d)/Q^{-1}(\nu(d)) = 1$ a.s., and $Q(\ )$ is , for every $\varepsilon>0$, there exists a value of d, say $d_\varepsilon(>0)$, such that for all $0<d\leq d_\varepsilon$, $P\{n_1(d) \leq N(d) \leq n_2(d)\} \geq P\{|N(d)/Q^{-1}(\nu(d))-1|<\varepsilon\} \geq 1-\eta$, where η is arbitrarily small. Hence, for $d\leq d_\varepsilon$,

$$\{Q^{-1}[\nu(d)]\}^{-1}\Sigma_1 nP\{N(d)=n\} \leq (1-\varepsilon)P\{N(d)<n_1(d)\} \leq \eta(1-\varepsilon). \tag{3.34}$$

Also, proceeding along the same line as in (3.31), we have

$$\begin{aligned}\{Q^{-1}[\nu(d)]\}^{-1}\Sigma_3 nP\{N(d)=n\} &= \{Q^{-1}(\nu(d))\}^{-1}n_2(d)P\{N(d)=n_2(d)\} \\ &+ \{Q^{-1}(\nu(d))\}^{-1}\Sigma_3 P\{N(d)>n\} \\ &\leq \{Q^{-1}(\nu(d))\}^{-1}[\Sigma_3 P\{N(d)>n\} + n_2(d)P\{N(d)>n_2(d)-1\}] \\ &= \{Q^{-1}[\nu(d)]\}^{-1}[O(n_2(d))^{-\delta_2} + O(n_2(d))O([n_2(d)]^{-1-\delta_2})] \\ &= O[(n_2(d))^{-\delta_2}] \to 0 \text{ as } d\to 0,\end{aligned} \tag{3.35}$$

since $Q^{-1}[\nu(d)]\to\infty$ as $d\to 0$ and hence, $n_2(d)\to\infty$ as $d\to 0$. Finally, elementary computations yield that

$$\begin{aligned}&|\{Q^{-1}[\nu(d)]\}^{-1}\ \Sigma_2 nP\{N(d)=n\}-1| \\ &\leq \varepsilon\Sigma_2 P\{N(d)=n\}+\eta \ \leq \varepsilon+\eta.\end{aligned} \tag{3.36}$$

Hence, (3.13) follows from (3.34), (3.35) and (3.36). Q.E.D.

4. ASYMPTOTIC RELATIVE EFFICIENCY

For two procedures A and B for determining (sequentially) bounded-length confidence intervals for β (with the same bound 2d), we define the A.R.E. of A with respect to B as

$$e_{A,B} = \lim_{d\to 0} \{(EN_B(d))/(EN_A(d))\}, \tag{4.1}$$

where $N_A(d)$ and $N_B(d)$ stand for the respective stopping variables.

It may be noted that though Gleser (1965) considered the case where $n^{-1}T_n^2 \to c > 0$ as $n\to\infty$, his results can be easily extended to the case where our (ii) and (v) of section 3 hold. As such, if $N_L(d)$ stands for the stopping variable for his procedure (based on the least squares estimators), it follows that our theorem 1 also holds for $N_L(d)$ with the only change that $\nu(d)$ has to be replaced by $\nu_L(d) = \sigma^2\tau^2_{\alpha/2}/d^2$, where σ^2 is the variance of the distribution $F(x)$ in (1.1). Hence, using the result in (3.13) for both the cases and writing $N_K(d)$ for the proposed procedure, we have

$$e_{K,L} = \lim_{d\to 0} \{Q^{-1}(\nu_L(d))/Q^{-1}(\nu(d))\}. \tag{4.2}$$

Note that, by definition $\nu_L(d)/\nu(d) = 12\sigma^2\rho^2B^2(F) = e$, is independent of d, and let us write $e = s(e^*)$, so that $e^* = s^{-1}(e)$ is monotonic in e, with $e^*=1$ when $e=1$. Also, let $\phi_d = Q^{-1}(\nu_L(d))/Q^{-1}(\nu(d))$, $Q^{-1}(\nu_L(d)) = \nu_L^*(d)$ and $Q^{-1}(\nu(d)) = \nu^*(d)$. Note that by (v) of section 3, $\nu_L^*(d)$ and $\nu^*(d)$ both tend to ∞ as $d\to 0$. Hence, we have

$$\begin{aligned} s(e^*) = e = \nu_L(d)/\nu(d) &= \lim_{d\to 0} \{\nu_L(d)/\nu(d)\} \\ &= \lim_{d\to 0} \{Q(\nu_L^*(d))/Q(\nu^*(d))\} \\ &= \lim_{d\to 0} \{Q(\phi(d)\nu^*(d))/Q(\nu^*(d))\}. \end{aligned} \tag{4.3}$$

Using (3.8) and proving by contradiction, it follows that

$$\lim_{d\to 0} \phi(d) = e^* = s^{-1}(e). \tag{4.4}$$

Hence, from (4.2) and (4.4), it follows that

$$e_{K,L} = s^{-1}(e) = s^{-1}(12\sigma^2\rho^2[\int_{-\infty}^{\infty} f^2(x)dx]^2). \tag{4.5}$$

In Sen (1968), various bounds for $e = 12\sigma^2\rho^2(\int_{-\infty}^{\infty} f^2(x)dx)^2$ are studied, with special reference to the bounds for ρ^2. As we shall see in the next section, for the two-sample problem, $\rho=1$, and $s(e) = e$, so that (4.5) reduces to $12\sigma^2(\int_{-\infty}^{\infty} f^2(x)dx)^2$, the usual A.R.E. of the Wilcoxon two-sample test with respect to the Student's t-test. Since $12\sigma^2(\int_{-\infty}^{\infty} f^2(x)dx)^2 \geq 0.864$ for all continuous F, it follows that

$$\inf_{\{F\}} e_{K,L} = s^{-1}(\rho^2[0.864]). \tag{4.6}$$

Consider now the case of equispaced regression line, where $t_i=i$, $i=1,2,\ldots$, so that $T_n^2 = n(n^2-1)/12$. In this case, we get at once, that $\rho=1$ and

$$e_{K,L} = (12\sigma^2(\int_{-\infty}^{\infty} f^2(x)dx)^2)^{\frac{1}{3}} \tag{4.7}$$

For the case of normal F, this reduces to $(0.95)^{1/3} \simeq .985$, while the infimum in (4.6) is given by $(0.864)^{1/3} \simeq .953$. This clearly indicates the robustness and efficiency of the proposed procedure. For many non-normal cdf (e.g., double exponential, logistic, Cauchy, etc.) the proposed procedure is more efficient than the least squares procedure.

5. TWO-SAMPLE LOCATION PROBLEM

Consider the special design, where the t_i can either be 1 or 0. Thus, we have only two different distributions $F(x-\alpha)$ and $F(x-\alpha-\beta)$, where β denotes the difference in the location parameters of the two distributions. If at the n-th stage, we have m_n of the t_i equal to 1 and the rest equal to zero, we obtain that

$$T_n^2 = m_n(n-m_n)/n \le n/4, \text{ for all } n\ge 1. \tag{5.1}$$

Looking at the definition of $T_n^2 = Q(n)$, (3.11) and (3.13), we observe that an optimum choice of m_n is $[\frac{1}{2}n]$, the integral part of $\frac{1}{2}n$. Thus, among all designs for obtaining a bounded-length confidence interval for β, in this problem, an optimum design (which minimizes the expected value of N(d) for small d) consists in taking every alternative observations for the two distributions. Here also, $\rho_n=\rho=1$, and $n^{-1}T_n^2 \to 1/4$, and hence, by (4.5), the A.R.E. reduces to $12\sigma^2(\int_{-\infty}^{\infty} f^2(x)dx)^2$, various bounds for which are well-known. Looking at (2.4)-(2.7), we observe that $n^* = m_n(n-m_n) \sim \frac{1}{4}n^2$, and $U^*_{n,\underset{\sim}{t}_n}$ can be computed, for small values of n, from the extensive tables given in Owen (1963). For large n, note that V_n, defined by (2.2), reduces to $(1/18)\{2[n^3-m_n^3-(n-m_n)^3] + 3[n^2-m_n^2-(n-m_n)^2]\} \simeq \frac{1}{12}n^3+O(n^2)$, and hence, $U^*_{n,\underset{\sim}{t}_n}$ can be computed by reference to the usual normal probability tables.

6. APPENDIX: PROOF OF THEOREM 3.3

We may assume without any loss of generality that $\beta=0$ (as otherwise, we may set $b=\beta+b'$ and work with b'). Also, we shall explicitly consider the case of $a\varepsilon[0, \log n]$ as the case of $a\varepsilon[-\log n, 0]$ follows on the same line. Let $b_n \sim n^{\delta_1}$ (as $n\to\infty$), $\delta_1>0$, and let $\eta_{r,n} = (r/b_n)\log n$, $r=1,\dots,b_n$. Since, $U_n(b)\downarrow$ in b, for $a\varepsilon[\eta_{r-1,n},\eta_{r,n}]$, we have

$$U_n(\eta_{r,n}/\rho_nT_n) \le U_n(a/\rho_nT_n) \le U_n(\eta_{r-1,n}/\rho_nT_n) \tag{6.1}$$

and similar inequalities involving their expectations. Then, we get after a few simple steps,

$$\sup_{a\varepsilon[\eta_{r-1,n},\eta_{r,n}]} |U_n(0)-U_n(a/\rho_nT_n) + EU_n(a/\rho_nT_n)|$$

$$\leq \max_{j=r-1,r} |W_{n,j}| + E[U_n(\eta_{r-1,n}/\rho_nT_n) - U_n(\eta_{r,n}/\rho_nT_n)], \tag{6.2}$$

where $W_{n,j} = U_n(0)-U_n(\eta_{j,n}/\rho_nT_n) + EU_n(\eta_{j,n}/\rho_nT_n)$, $0\leq j\leq b_n$, $1\leq r\leq b_n$ (note that $\eta_{0,n}=0$, $EU_n(0)=0$). Hence,

$$\sup_{0\leq a\leq \log n} [\binom{n}{2}/V_n^{\frac{1}{2}}] |U_n(0) - U_n(a/\rho_nT_n) + EU_n(a/\rho_nT_n)|$$

$$\leq \max_{0\leq j\leq b_n} [\binom{n}{2}/V_n^{\frac{1}{2}}] |W_{n,j}| + \max_{1\leq j\leq b_n} [\binom{n}{2}/V_n^{\frac{1}{2}}] E[U_n(\eta_{j-1,n}/\rho_nT_n)-U_n(\eta_{j,n}/\rho_nT_n)]. \tag{6.3}$$

Using (3.4), (3.11), $\eta_{j,n}-\eta_{j-1,n} = \log n/b_n$ and $A_n^2/V_n \to 3/4$ as $n\to\infty$ [cf. Sen (1968)], we get,

$$\{\binom{n}{2}/V_n^{\frac{1}{2}}\} E[U_n(\eta_{j-1,n}/\rho_nT_n) - U_n(\eta_{j,n}/\rho_nT_n)]$$

$$= (\eta_{j,n}-\eta_{j-1,n})(A_n/V_n^{\frac{1}{2}})[B(F)+o(1)] \tag{6.4}$$

$$= O(n^{-\delta_1}\log n), \text{ for all } j=1,2,\ldots,b_n.$$

Let

$$Y_{nij} = X_i-(\eta_{j,n}/\rho_nT_n)t_i (1\leq i\leq n,\ 1\leq j\leq b_n),$$

$$\phi(Y_{nij},Y_{nkj}) = c(t_k-t_i)[c(Y_{nkj}-Y_{nij}+(\eta_{j,n}/\rho_nT_n)(t_k-t_i))-c(Y_{nkj}-Y_{nij})],$$

$1\leq i\neq k\leq n$, $1\leq j\leq b_n$. Write, $W_{n,j} = n^{-1}(n-1)^{-1} \sum\sum_{1\leq i\neq k\leq n} [\phi(Y_{nij},Y_{nkj})-E\phi(Y_{nij},Y_{nkj})]$, $1\leq j\leq b_n$ ($W_{n,0}=0$). Note that $W_{n,j}$ is a U-statistic minus its expectation (see e.g., Hoeffding (1948)). We can write, $W_{n,j} = 2(\bar{Z}_{nj}-\mu_{nj}) + R_{nj}$, where, $\bar{Z}_{nj} = n^{-1}\sum_{i=1}^n Z_{nij} = (n-1)^{-1}\sum_{k=1(\neq i)}^n c(t_k-t_i)\ [F(X_i-\alpha)-F(X_i-\alpha-(\eta_{j,n}/\rho_nT_n)(t_k-t_i))]$; $\mu_{nj} = E\,\bar{Z}_{nj}$, $1\leq j\leq b_n$. Then,

$$P[|W_{n,j}|[\binom{n}{2}/V_n^{\frac{1}{2}}] > c_1 n^{-\delta_1} \log n]$$

$$\leq P[|\bar{Z}_{n,j}-\mu_{nj}| > \tfrac{1}{4}t_n] + P[|R_{nj}| > \tfrac{1}{2}t_n], \tag{6.5}$$

where $c_1(>0)$ is not depending on n, $t_n = c_1 V_n^{\frac{1}{2}}\binom{n}{2}^{-1} n^{-\delta_1} \log n = O(n^{-\frac{1}{2}-\delta_1}\log n)$, as $n\to\infty$, since, $V_n = O(n^3)$, (as $n\to\infty$), (from (3.9) and the fact that $A_n^2/V_n \to 3/4$ as $n\to\infty$). Also, from (3.4), (3.9) and (3.10), it follows after some algebraic manipulations that $\mu_{nj} = 4\eta_{j,n}(A_n/[n(n-1)])\ (B(F)+o(1)) = O(n^{-\frac{1}{2}}\log n)$, uniformly in $j=1,2,\ldots,b_n$. Hence, there exists a positive integer n_1 such that for $n\geq n_1$, $0<t_n'<\tfrac{1}{2}\mu_{nj}<1-\tfrac{1}{2}\mu_{nj}<1$, uniformly in $j=1,2,\ldots,b_n (t_n'=\frac{1}{8}t_n)$. Also, $\bar{Z}_{nj}$ is the average of n independent random variables each assuming the values 0 and 2. Now, using the inequality (2.1) of Hoeffding (1963), we get, for $n\geq n_1$,

$$P[\tfrac{1}{2}|\bar{Z}_{nj}-\mu_{nj}|>t_n']$$

$$\begin{aligned}&\leq [\{\mu_{nj}'/(\mu_{nj}'+t_n')\}^{n(\mu_{nj}'+t_n')}\{(1-\mu_{nj}')/(1-\mu_{nj}'-t_n')\}^{n(1-\mu_{nj}'-t_n')}]\\ &+ [\{\mu_{nj}'/(\mu_{nj}'-t_n')\}^{n(\mu_{nj}'-t_n')}\{(1-\mu_{nj}')/(1-\mu_{nj}'+t_n')\}^{n(1-\mu_{nj}'+t_n')}]\end{aligned} \tag{6.6}$$

where $\mu_{nj}' = \tfrac{1}{2}\mu_{nj}$. It follows after some algebraic simplifications that as $n\to\infty$,

$$\begin{aligned}&n[(\mu_{nj}'\pm t_n')\log(\frac{\mu_{nj}'}{\mu_{nj}'\pm t_n'}) + (1-\mu_{nj}'\mp t_n')\log(\frac{1-\mu_{nj}'}{1-\mu_{nj}'\mp t_n'})]\\ &= -\ nt_n'^2/\{2(\mu_{nj}'\pm t_n')(1-\mu_{nj}'\mp t_n')\} + o(\frac{nt_n'^2}{2(\mu_{nj}'\pm t_n')}).\\ &= O(\log n^{-c_1^2/4}).\end{aligned} \tag{6.7}$$

Also,

$$P[|R_{nj}|>\tfrac{1}{2}t_n] \leq E(R_{nj}^2)/(\tfrac{1}{4}t_n^2) \tag{6.8}$$

But,

$$E(R_{nj}^2) = \frac{2}{n^2(n-1)^2} \sum_{1\leq i\neq k\leq n}\sum V[\phi(Y_{nij},Y_{nkj})]$$

$$- \frac{4}{n^2(n-1)^2} \sum_{1\leq i\neq k\leq n}\sum E[E\{\phi(Y_{nij},Y_{nkj})|Y_{nij}\}-E\phi(Y_{nij},Y_{nkj})]^2 \tag{6.9}$$

$$\leq \frac{2}{n^2(n-1)^2} \sum_{1\leq i\neq k\leq n}\sum V[\phi(Y_{nij},Y_{nkj})] \leq \frac{4}{n(n-1)}\mu_{nj} = O(n^{-5/2}\log n).$$

It follows now from (6.5)-(6.9) and the Bonferroni inequality that

$$\begin{aligned} P\{\max_{1\leq j\leq b_n}|W_{nj}|>t_n\} &\leq b_n\, O(n^{-c_1^2/4}) + \sum_{j=1}^{b_n} E(R_{nj}^2)/(\tfrac{1}{4}t_n^2) \\ &= O(n^{\delta_1-c_1^2/4}) + O(n^{-3/2+3\delta_1}/(\log n)^{\frac{1}{2}}). \end{aligned} \tag{6.10}$$

First make a proper choice δ_1 and subsequently of $c_1(>0)$ to make the right hand side of (6.10) less than or equal to $[(\text{const})(n^{-1-\delta_2})]$ for any given $\delta_2>0$. The result now follows from (6.3), (6.4) and the Borel-Cantelli lemma.

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