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SEQUENTIAL CONVERGENCES ON MV-ALGEBRAS

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The notion of an MV-algebra was introduced by Chang [2]. Various systems of axioms and various notation for MV-algebras have been applied; we shall use those from [4]; cf. also [13].

We investigate sequential convergences on MV-algebras. The definition is analogous to that studied for lattice ordered groups (cf. [6], [8]), Boolean algebras [9], [11] or lattices [12].

Let $\mathscr{A}$ be an MV-algebra and let G be a lattice ordered group. We denote by $\operatorname{Conv}\mathscr{A}$ and $\operatorname{Conv}G$ the set of all sequential convergences on $\mathscr{A}$ or on G, respectively. Next, let $\operatorname{Conv}^{\mathrm{b}}G$ be the set of all bounded sequential convergences on G; this notion has been dealt with in [10]. All the sets $\operatorname{Conv}\mathscr{A}$, $\operatorname{Conv}G$ and $\operatorname{Conv}^{\mathrm{b}}G$ are partially ordered by inclusion.

Mundici [14] proved that for each MV-algebra $\mathscr{A}$ there exists an abelian lattice ordered group G with a strong unit u such that $\mathscr{A}$ can be constructed by means of G. In this construction, the underlying set A of $\mathscr{A}$ is the interval $[0,u]$ of G.

We shall prove that the partially ordered set $\operatorname{Conv}\mathscr{A}$ is isomorphic to $\operatorname{Conv}^{\mathrm{b}}G$. From this we deduce that each interval of $\operatorname{Conv}\mathscr{A}$ is a complete Bouwerian lattice. The lattice $\operatorname{Conv}\mathscr{A}$ has a greatest element if and only if $\operatorname{Conv}G$ has a greatest element.

It will be shown that if $[0,u]$ is a Boolean algebra, then the relation $\operatorname{Conv}\mathscr{A} = \operatorname{Conv}\mathscr{B}$ is valid (where $\mathscr{B}$ is the Boolean algebra under consideration, and $\operatorname{Conv}\mathscr{B}$ is as in [9]).

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The following definition of an MV-algebra is recalled from [4].

1.1. Definition. An *MV-algebra* is a system $\mathscr{A} = \langle A; \oplus, *, \neg, 0, 1\rangle$ (where $\oplus, *$ are binary operations, $\neg$ is a unary operation and 0, 1 are nulary operations) such that the following identities are satisfied:

(m_1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(m_2) $x \oplus 0 = x$;
(m_3) $x \oplus y = y \oplus x$;
(m_4) $x \oplus 1 = 1$;
(m_5) $\neg\neg x = x$;
(m_6) $\neg 0 = 1$;
(m_7) $x \oplus \neg x = 1$;
(m_8) $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$;
(m_9) $x * y = \neg(\neg x \oplus \neg y)$.

Let $\mathbb{N}$ be the set of all positive integers and for each $n \in \mathbb{N}$ let $A_n = A$. The direct product of sets A_n $(n \in \mathbb{N})$ will be denoted by $A^{\mathbb{N}}$. The elements of $A^{\mathbb{N}}$ are denoted by $(a_n)_{n\in\mathbb{N}}$ or simply by (a_n); they will be called sequences in $\mathscr{A}$. The notion of a subsequence of a sequence in $\mathscr{A}$ has the usual meaning. If $(x_n) \in A^{\mathbb{N}}$ and $x \in A$ such that $x_n = x$ for each $n \in \mathbb{N}$, then we write $(x_n) = \operatorname{const} x$.

If $K \subseteq A^{\mathbb{N}} \times A$, then a relation of the form $((x_n), x) \in K$ will be denoted also by writing $x_n \longrightarrow_K x$.

For each $x \in A$ and $y \in A$ we put

$$x \vee y = (x * \neg y) \oplus y,$$
$$x \wedge y = \neg(\neg x \vee \neg y).$$

Let us consider the structure $\mathscr{L}(\mathscr{A}) = \langle A; \vee, \wedge\rangle$. Then we have

1.2. Proposition. (Cf., e.g., [4].) *$\mathscr{L}(\mathscr{A})$ is a distributive lattice with the least element* 0 *and the greatest element* 1.

The partial order induced on A by means of the lattice $\mathscr{L}(\mathscr{A})$ will be denoted by $\leqslant$. When considering a partial order on the set A we always mean the partial order $\leqslant$.

1.3. Definition. A subset K of $A^{\mathbb{N}} \times A$ will be said to be a *sequential convergence in $\mathscr{A}$* if the following conditions are satisfied:

(i) If $x_n \longrightarrow_K x$ and (y_n) is a subsequence of (x_n), then $y_n \longrightarrow_K x$.

(ii) If $(x_n) \in A^{\mathbb{N}}$, $x \in A$ and if for each subsequence (y_n) of (x_n) there is a subsequence (z_n) of (y_n) such that $z_n \longrightarrow_K x$, then $x_n \longrightarrow_K x$.

(iii) If $(x_n) \in A^n$, $x \in A$, $(x_n) = \operatorname{const} x$, then $x_n \longrightarrow_K x$.

(iv) If $x_n \longrightarrow_K x$ and $x_n \longrightarrow_K y$, then $x = y$.

(v) If $x_n \longrightarrow_K x$ and $y_n \longrightarrow_K y$, then $x_n \oplus y_n \longrightarrow_K x \oplus y$, $x_n * y_n \longrightarrow_K x * y$ and $\neg x_n \longrightarrow_K \neg x$.

(vi) If $x_n \leqslant y_n \leqslant z_n$ is valid for each $n \in \mathbb{N}$ and if $x_n \longrightarrow_K x$, $z_n \longrightarrow_K x$, then $y_n \longrightarrow_K x$.

In what follows we shall say "convergence" instead of "sequential convergence". We denote by $\operatorname{Conv} \mathscr{A}$ the set of all convergences in $\mathscr{A}$. The set $\operatorname{Conv} \mathscr{A}$ is partially ordered by inclusion.

Let $K(0)$ be the set consisting of all elements $((x_n), x)$ of $A^{\mathbb{N}} \times A$ such that there is $m \in \mathbb{N}$ with $x_n = x$ for each $n \geqslant m$. Then we obviously have

1.4. Lemma. *$K(0)$ is the least element of $\operatorname{Conv} \mathscr{A}$.*

The notion of convergence in a lattice was defined in [12]. It is defined as follows (we apply analogous notation as above).

1.5. Definition. Let $\mathscr{L} = (L; \wedge, \vee)$ be a lattice. A subset K of $L^{\mathbb{N}} \times L$ is a *convergence in $\mathscr{L}$* if the conditions (i)–(iv), (vi) from 1.3 are satisfied and if, moreover, the following condition holds:

(v(1)) If $x_n \longrightarrow_K x$ and $y_n \longrightarrow_K y$, then $x_n \wedge y_n \longrightarrow_K x \wedge y$ and $x_n \vee y_n \longrightarrow_K x \vee y$.

From the definition of the lattice $\mathscr{L}(\mathscr{A})$ and from 1.3 we immediately obtain

1.6. Lemma. *Let $K \in \operatorname{Conv} \mathscr{A}$. Then K is a convergence on the lattice $\mathscr{L}(\mathscr{A})$.*

If $\{K_i\}_{i \in I}$ is a nonempty system of elements in $\operatorname{Conv} \mathscr{A}$, then 1.3 yields that $\bigcap\limits_{i \in I} K_i$ also belongs to $\operatorname{Conv} \mathscr{A}$. Hence we have

1.7. Lemma. *The partially ordered set $\operatorname{Conv} \mathscr{A}$ is a $\wedge$-semilattice. If $K \in \operatorname{Conv} \mathscr{A}$, then the interval $[K(0), K]$ is a complete lattice. Hence if $\operatorname{Conv} \mathscr{A}$ has a greatest element, then $\operatorname{Conv} \mathscr{A}$ is a complete lattice.*

For lattice ordered groups we apply the same notation as in [1]. The following theorems 1.8 and 1.9 are due to Mundici [14] (for the case of linearly ordered MV-algebras cf. Chang [11]).

1.8. Theorem. *Let G be an abelian lattice ordered group with a strong unit u. Let A be the interval $[0, u]$ of G. For each a and b in A we put*

$$a \oplus b = (a + b) \wedge u, \quad \neg a = u - a, \quad 1 = u.$$

Next, let the binary operation $*$ *on* A *be defined by* (m_9). *Then* $\mathscr{A} = \langle A; \oplus, *, \neg, 0, 1\rangle$ *is an* MV*-algebra.*

If G and $\mathscr{A}$ are as in 1.8 then we put $\mathscr{A} = \mathscr{A}_0(G, u)$.

1.9. Theorem. *Let* $\mathscr{A}$ *be an* MV*-algebra. Then there exists an abelian lattice ordered group* G *with a strong unit* u *such that* $\mathscr{A} = \mathscr{A}_0(G, u)$.

1.10. Lemma. *Let* $\mathscr{A}$ *and* G *be as in* 1.9. *Let* $x, y \in A$, $x \leqslant y$. *Then*

$$y - x = \neg(x \oplus \neg y).$$

Proof. According to 1.8 we have

$$x \oplus \neg y = (x + (u - y)) \wedge u = (u - (y - x)) \wedge u = u - (y - x),$$

hence

$$\neg(x \oplus \neg y) = u - (u - (y - x)) = y - x.$$

□

1.11. Lemma. *Let* $\mathscr{A}$ *be an* MV*-algebra,* $x, y, z \in A$, $x \leqslant y \leqslant z$. *Then*

$$\neg(x \oplus \neg z) = \neg(x \oplus \neg y) \oplus \neg(y \oplus \neg z).$$

Proof. According to 1.10 and 1.8 we have

$$\begin{aligned}\neg(x \oplus \neg z) &= z - x = (x - y) + (y - x) = (z - y) \oplus (y - x)\\ &= \neg(y \oplus \neg z) \oplus (x \oplus \neg y).\end{aligned}$$

□

1.12. Lemma. *Let* $\mathscr{A}$ *be an* MV*-algebra,* $x, y \in A$, $x \leqslant y$. *Then*

$$x = \neg(\neg y \oplus \neg(x \oplus \neg y)).$$

Proof. In view of 1.10 we have

$$\begin{aligned}\neg(\neg y \oplus \neg(x \oplus \neg y)) &= \neg(\neg y \oplus (y - x)) = \neg((u - y) \oplus (y - x))\\ &= \neg(((u - y) + (y - x)) \wedge u) = \neg((u - x) \wedge u)\\ &= \neg(u - x) = \neg\neg x = x.\end{aligned}$$

□

2. The systems $\operatorname{Conv}_0 G$ and $\operatorname{Conv}_0 \mathscr{A}$

All lattice ordered groups considered in the present paper are assumed to be abelian. If G is a lattice ordered group then its underlying set will be also denoted by the same symbol G.

2.1. Definition. Let G be a lattice ordered group and let K be a subset of $G^{\mathbb{N}} \times G$. The set K is said to be a *convergence in* G if the conditions (i)–(iv), (v(1)), (vi) above are satisfied and if, moreover, the following condition holds:

(v(2)) If $x_n \longrightarrow_K x$ and $y_n \longrightarrow_K y$ then $x_n + y_n \longrightarrow_K x + y$ and $-x_n \longrightarrow_K -y$.

We denote by $\operatorname{Conv} G$ the set of all convergences in G; this set is partially ordered by inclusion.

For each K in $\operatorname{Conv} G$ we put

$$K^0 = \{(x_n) \in G^{\mathbb{N}} : x_n \longrightarrow_K 0 \text{ and } x_n \geqslant 0 \text{ for each } n \in \mathbb{N}\},$$
$$\operatorname{Conv}_0 G = \{K^0 \colon K \in \operatorname{Conv} G\}.$$

We can regard $G^{\mathbb{N}}$ as the direct product $\prod_{n\in\mathbb{N}} G_n$, where $G_n = G$ for $n \in \mathbb{N}$. Hence $G^{\mathbb{N}}$ is a lattice ordered group. For each lattice ordered group H the symbol H^+ denotes the positive cone of H; thus H^+ is a lattice ordered semigroup.

2.2. Lemma. (Cf. [6], 1.2 and 1.3.) *Let K^1 be a subset of $G^{\mathbb{N}}$. Then K^1 belongs to $\operatorname{Conv}_0 G$ if and only if K^1 is a convex subsemigroup of the semigroup $(G^{\mathbb{N}})^+$ such that the following conditions are satisfied:*

(I) *If $(g_n) \in K^1$ then each subsequence of (g_n) belongs to K^1.*

(II) *Let $(g_n) \in (G^{\mathbb{N}})^+$. If each subsequence of (g_n) has a subsequence belonging to K^1, then $(g_n) \in K^1$.*

(III) *Let $g \in G$. Then $\operatorname{const} g$ belongs to K^1 if and only if $g = 0$.*

The set $\operatorname{Conv}_0 G$ is partially ordered by inclusion. The following lemma is easy to verify. (Cf. also [6].)

2.3. Lemma. (i) *For each K in $\operatorname{Conv} G$ put $\varphi_1(K) = K^0$. Then φ_1 is an isomorphism of $\operatorname{Conv} G$ onto $\operatorname{Conv}_0 G$.*

(ii) *Let $K^1 \in \operatorname{Conv}_0 G$. Put $K = \{((x_n), x) \in G^{\mathbb{N}} \times G \colon |x_n - x| \in K^1\}$. Then $K \in \operatorname{Conv} G$ and $\varphi_1(K) = K_1$.*

Direct products of MV-algebras have been investigated in [2] and [13].

Let $\mathscr{A}$ be an MV-algebra. Similarly as in the case of lattice ordered groups above we denote by $A^{\mathbb{N}}$ the direct product $\prod_{n\in\mathbb{N}} A_n$, where $A_n = A$ for each $n \in \mathbb{N}$.

Next, for each $K \in \operatorname{Conv} \mathscr{A}$ we denote

$$K^0 = \{(x_n) \in A^{\mathbb{N}} : x_n \longrightarrow_K 0\},$$
$$\operatorname{Conv}_0 \mathscr{A} = \{K^0 \colon K \in \operatorname{Conv} \mathscr{A}\}.$$

In view of the definition of $\operatorname{Conv}_0 \mathscr{A}$ and according to 1.3 we have

2.4. Lemma. *Let $K^1 \in \operatorname{Conv}_0 \mathscr{A}$. Then*

(i) *K^1 satisfies the conditions* (I), (II) *and* (III) *from* 2.2;
(ii) *K^1 is a convex subset of the lattice $(A^{\mathbb{N}}; \vee, \wedge)$;*
(iii) *K^1 is closed with respect to the operation $\oplus$.*

The following two lemmas 2.5 and 2.6 will show that the set $\operatorname{Conv} \mathscr{A}$ can be reconstructed from $\operatorname{Conv}_0 \mathscr{A}$.

Let $K(1)$ be a nonempty subset of $A^{\mathbb{N}}$. For $((x_n), x) \in A^{\mathbb{N}} \times A$ we consider the following condition:

$(*)$ There exist $(u_n), (v_n) \in A^{\mathbb{N}}$ such that
(i) $u_n \leqslant x$, $u_n \leqslant x_n$ for each $n \in \mathbb{N}$ and $(\neg(u_n \oplus \neg x)) \in K(1)$;
(ii) $v_n \geqslant x$, $v_n \geqslant x_n$ for each $n \in \mathbb{N}$ and $(\neg(x \oplus \neg v_n)) \in K(1)$.

We denote by $K(2)$ the set of all $((x_n), x) \in A^{\mathbb{N}} \times A$ such that the condition $(*)$ is valid. If $((x_n), x)$ belongs to $K(2)$ then we write $x_n \longrightarrow_{K(2)} x$.

2.5. Lemma. *Let $K(1)$ be a nonempty subset of $A^{\mathbb{N}}$ satisfying the conditions* (i), (ii) *and* (iii) *from* 2.4. *Let $K(2)$ be defined as above. Then $K(2) \in \operatorname{Conv} \mathscr{A}$.*

Proof. We shall verify that $K(2)$ satisfies the conditions (i)–(vi) from 1.3.

(i): The validity of (i) is obvious.

(ii): Let G be as in 1.9. In view of the assumption (cf. 2.4 (i)) $K(1)$ satisfies the conditions (I), (II) and (III) from 2.2. Thus 2.2 yields that $K(1)$ belongs to $\operatorname{Conv}_0 G$.

Let $(x_n) \in A^{\mathbb{N}}$, $x \in A$. Suppose that the assumptions of the condition (ii) of 1.3 hold, where K is replaced by $K(2)$.

Then for a sequence (z_n) as in (ii) of 1.3 we have $z_n \longrightarrow_{K(2)} x$. Thus there are $(u_n), (v_n) \in A^{\mathbb{N}}$ such that the conditions (i) and (ii) from $(*)$ are valid, where x_n is replaced by z_n.

According to 1.10 we have

$$\neg(u_n \oplus \neg x) = x - u_n,$$
$$\neg(x \oplus \neg v_n) = v_n - x,$$

hence $(x-u_n) \in K(1)$ and $(v_n-x) \in K(1)$. Since $K(1) \in \operatorname{Conv}_0 G$ and $u_n \leqslant z_n \leqslant v_n$ for each $n \in \mathbb{N}$, by applying 2.3 we obtain that in the lattice ordered group G we have $(|z_n - x|) \in K(1)$ and thus (cf. 2.1) we get $(|x_n - x|) \in K(1)$. This implies that

$$(x - (x_n \wedge x)) \in K(1), \quad ((x_n \vee x) - x) \in K(1),$$

whence $(x_n, x) \in K(2)$. Therefore the condition (ii) from 1.3 is valid for $K(2)$.

(iii): Under the assumptions as in (iii) it suffices to put $u_n = v_n = x$ for each $n \in \mathbb{N}$ and then according to the definition of $K(2)$ the relation $x_n \longrightarrow_{K(2)} x$ holds.

(iv): Since (iv) is a consequence of (v) it suffices to deal with (v).

(v): Let $x_n \longrightarrow_{K(2)} x$. Hence there are (u_n) and (v_n) in $A^{\mathbb{N}}$ such that (∗) is valid. Thus for each $n \in \mathbb{N}$ we have

$$\neg u_n \geqslant \neg x, \quad \neg u_n \geqslant \neg x_n, \quad \neg v_n \leqslant \neg x, \quad \neg v_n \leqslant \neg x_n.$$

We have also

$$\begin{aligned} \neg(\neg v_n \oplus \neg\neg x) &= \neg(x \oplus \neg v_n) \in K(1), \\ \neg(\neg x \oplus \neg\neg u_n) &= \neg(u_n \oplus \neg x) \in K(1), \end{aligned}$$

whence $\neg x_n \longrightarrow_{K(2)} \neg x$.

Next, let $x_n \longrightarrow_{K(2)} x$ and $x'_n \longrightarrow_{K(2)} x'$. Let (u_n) and (v_n) be as in (∗); further let (u'_n) and (v'_n) have analogous meanings (with respect to x'_n and x'). Denote $x'' = x \oplus x'$, $u''_n = u_n \oplus u'_n$, $v''_n = v_n \oplus v'_n$. Hence $u''_n \leqslant x''$, $u''_n \leqslant x''_n$, $v''_n \geqslant x''$ and $v''_n \geqslant x''_n$.

We also have

$$u''_n = u_n \oplus u'_n \leqslant x \oplus u'_n \leqslant x \oplus x' = x''.$$

In view of 1.9 there are $p_n, q_n \in A^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$

$$u''_n + p_n = u''_n \oplus p_n = x \oplus u'_n, \quad (x \oplus u'_n) + q_n = x \oplus u'_n \oplus q_n = x''.$$

Then

$$\begin{aligned} 0 \leqslant p_n &= (x \oplus u'_n) - (u_n \oplus u'_n) = \\ &= ((x + u'_n) \wedge u) - ((u_n + u'_n) \wedge u). \end{aligned}$$

Whenever a, b and c are elements of a lattice ordered group with $a \leqslant b$, then it is easy to verify that

$$(b \wedge u) - (a \wedge u) \leqslant b - a.$$

Hence

$$0 \leqslant p_n \leqslant (x + u'_n) - (u_n + u'_n) = x - u_n.$$

Since $(x - u_n) \in K(1)$ we obtain that (p_n) belongs to $K(1)$. Similarly we prove that (q_n) belongs to $K(1)$ as well. Therefore $(x'' - u''_n) \in K(1)$ (cf. also 1.11). In another notation

$$(\neg(u''_n \oplus \neg x'')) \in K(1).$$

By analogous steps we can verify that

$$(\neg(x'' \oplus \neg v''_n)) \in K(1).$$

Hence according to the definition of $K(2)$ we obtain

$$x_n \oplus x'_n \longrightarrow_{K(2)} x \oplus y.$$

(vi): Let the assumptions from (vi) be fulfilled. Then there are $(u_n), (v_n) \in A^{\mathbb{N}}$ such that $x_n \leqslant x \leqslant v_n$, $u_n \leqslant x_n$, $z_n \leqslant v_n$ for each $n \in \mathbb{N}$ and

$$\neg(u_n \oplus \neg x) \in K(1), \quad \neg(x \oplus \neg v_n) \in K(1).$$

Then $u_n \leqslant y_n \leqslant v_n$ for each $n \in \mathbb{N}$. Hence $y_n \longrightarrow_{K(2)} x$.

The relation $x_n * y_n \longrightarrow_{K(2)} x * y$ is a consequence of the above results and of (m_9). □

2.6. Lemma. *Let $K(1)$ and $K(2)$ be as above. Then $(K(2))^0 = K(1)$.*

Proof. Let $(x_n) \in (K(2))^0$, $x_n \longrightarrow_{K(2)} 0$. in view of the definition of $K(2)$ there is $(v_n) \in A^{\mathbb{N}}$ such that $v_n \geqslant x_n$ for each $n \in \mathbb{N}$ and

$$(\neg(0 \oplus \neg v_n)) \in K(1).$$

Thus $(v_n) \in K(1)$. Put $u_n = 0$ for each $n \in \mathbb{N}$. In view of the convexity of $K(1)$ we obtain that (x_n) belongs to $K(1)$. Hence $(K(2))^0 \subseteq K(1)$.

Conversely, let $(x_n) \in K(1)$. If we put $u_n = v_n = x_n$ for each $n \in \mathbb{N}$, then in view of the definition of $K(2)$ we get $x_n \longrightarrow_{K(2)} 0$, whence $(x_n) \in (K(2))^0$. □

2.7. Corollary. $\mathrm{Conv}_0\,\mathscr{A}$ *is the system of all subsets of $A^{\mathbb{N}}$ which satisfy the conditions* (i), (ii) *and* (iii) *from* 2.4.

If $K \in \mathrm{Conv}\,\mathscr{A}$, then we put $f_1(K) = K^0$. Next, for $K(1) \in \mathrm{Conv}_0\,\mathscr{A}$ we set $f_2(K(1)) = K(2)$ (under the notation as above).

Whenever K and K' belong to $\operatorname{Conv}\mathscr{A}$ and $K \subseteq K'$, then $f_1(K) \subseteq f_1(K')$. Similarly, if K_1 and K_2 are elements of $\operatorname{Conv}_0\mathscr{A}$ with $K_1 \subseteq K_2$, then $f_2(K_1) \subseteq f_2(K_2)$.

2.8. Lemma. *Let $K \in \operatorname{Conv} A$. Then $f_2(K^0) = K$.*

Proof. Put $f_2(K^0) = K(2)$. Let $((x_n), x) \in K(2)$. Hence there exist $(u_n), (v_n) \in A^{\mathbb{N}}$ such that $(*)$ holds, where $K(1) = K^0$. Thus

$$(\neg(x \oplus \neg v_n)) \in K^0.$$

According to 1.10 $(v_n - x) \in K^0$, hence $((v_n - x), 0) \in K$, i.e., $v_n - x \longrightarrow_K 0$. Then $(v_n - x) \oplus x \longrightarrow_K x$. Clearly $(v_n - x) \oplus x = (v_n - x) + x = v_n$, hence $v_n \longrightarrow_K x$.

Next, $(\neg(u_n \oplus \neg x)) \in K^0$, whence in view of 1.10, $(x - u_n) \in K^0$, i.e., $x - u_n \longrightarrow_K 0$. According to 1.12,

$$u_n = \neg(\neg x \oplus \neg(u_n \oplus \neg x)).$$

Thus by applying 1.10 $u_n = \neg(\neg x \oplus (x - u_n))$ and hence

$$u_n \longrightarrow_K \neg(\neg x \oplus 0) = x.$$

Then by 1.3 (vi) we obtain that $x_n \longrightarrow_K x$. Therefore $K(2) \subseteq K$.

Conversely, let $((x_n), x) \in K$. Put $u_n = x_n \wedge x$ and $v_n = x_n \vee x$ for each $n \in \mathbb{N}$. Then $u_n \longrightarrow_K x$ and $v_n \longrightarrow_K x$, hence

$$\neg(u_n \oplus \neg x) \longrightarrow_K \neg(x \oplus \neg x) = \neg u = 0,$$
$$\neg(x \oplus \neg v_n) \longrightarrow_K \neg(x \oplus \neg x) = 0.$$

Consequently,

$$(\neg(u_n \oplus \neg x)) \in K^0, \quad (\neg(x \oplus \neg v_n)) \in K^0.$$

Therefore $((x_n), x) \in K(2)$. Summarizing, we conclude $K(2) = K$. □

2.9. Theorem. *f_2 is an isomorphism of the partially ordered set $\operatorname{Conv}_0\mathscr{A}$ onto $\operatorname{Conv}\mathscr{A}$ and $f_1 = f_2^{-1}$.*

Proof. This is a consequence of 2.6, 2.7, 2.8 and of the fact that both f_1 and f_2 are monotone. □

3. The relations between $\operatorname{Conv}_0 \mathscr{A}$ and $\operatorname{Conv} G$

Again, let $\mathscr{A}$ be an MV-algebra. Next, let G be as in 1.9.

First we shall investigate the relations between the partially ordered sets $\operatorname{Conv}_0 \mathscr{A}$ and $\operatorname{Conv}_0 G$.

For each $K \in \operatorname{Conv}_0 G$ we put

$$g_1(K) = A^{\mathbb{N}} \cap K.$$

3.1. Lemma. *If $K \in \operatorname{Conv}_0 G$, then $g_1(K) \in \operatorname{Conv}_0 \mathscr{A}$.*

Proof. Let $K_1 \in \operatorname{Conv}_0 G$. Then there is $K \in \operatorname{Conv} G$ such that $K_1 = K_0$. Hence if $(x_n), (y_n) \in K_1$, then $(x_n \vee y_n), (x_n \wedge y_n)$ and $(x_n + y_n)$ belong to K_1. Thus in view of 2.2 and 2.7 we obtain that $g_1(K_1) \in \operatorname{Conv}_0 \mathscr{A}$. □

As an immediate consequence of the definition of g_1 we get

3.2. Lemma. *Let $K_1, K_2 \in \operatorname{Conv} G$, $K_1 \subseteq K_2$. Then $g_1(K_1) \subseteq g_1(K_2)$.*

3.3. Lemma. *Let $a_1, a_2, \ldots, a_n \in A$, $n \geqslant 2$. Then $a_1 \oplus a_2 \oplus \ldots \oplus a_n = (a_1 + a_2 + \ldots + a_n) \wedge u$.*

Proof. By obvious induction. □

A nonempty subset X of $(G^{\mathbb{N}})^+$ is said to be regular if there exists $K \in \operatorname{Conv}_0 G$ such that $X \subseteq K$.

3.4. Lemma. *Let X be a nonempty subset of $(G^{\mathbb{N}})^+$. Then the following conditions are equivalent:*

(i) *X is not regular.*

(ii) *There exist $(x_n^1), (x_n^2), \ldots, (x_n^m) \in X$, subsequences (y_n^k) of (x_n^k) ($k = 1, 2, \ldots, m$) and an element $0 < g \in G$ such that $g \leqslant y_n^1 + y_n^2 + \ldots + y_n^m$ is valid for each $n \in \mathbb{N}$.*

Proof. This is a consequence of Lemma 2.5 in [10]. □

3.5. Lemma. *Let $X \in \operatorname{Conv}_0 \mathscr{A}$. Then the set X is regular.*

Proof. By way of contradiction, assume that X is not regular. Hence the condition (ii) from 3.4 holds. Then

(1) $0 \leqslant \wedge u = (y_n^1 + y_n^2 + \ldots + y_n^m) \wedge u = y_n^1 \oplus y_n^2 \oplus \ldots \oplus y_n^m.$

According to the definition of $\operatorname{Conv}_0 \mathscr{A}$ there exists $K \in \operatorname{Conv} \mathscr{A}$ such that $X = K^0$. Hence $x_n^k \longrightarrow_K 0$ and thus $y_n^k \longrightarrow_K 0$ for each $k \in \{1, 2, \dots, m\}$. Therefore

$$y_n^1 \oplus y_n^2 \oplus \dots \oplus y_n^m \longrightarrow_K 0 ;$$

in view of (1) we have arrived at a contradiction.

According to 3.5 for each $X \in \operatorname{Conv}_0 \mathscr{A}$ there exists a uniquely determined element Y of $\operatorname{Conv}_0 G$ such that (i) $X \subseteq Y$, and (ii) whenever $Y_1 \in \operatorname{Conv}_0 G$ and $X \subseteq Y_1$, then $Y \subseteq Y_1$. We denote $Y = g_2(X)$ and $F = g_2$ $(\operatorname{Conv}_0 \mathscr{A})$. □

3.6. Lemma. *Let $X_1, X_2 \in \operatorname{Conv}_0 \mathscr{A}$, $X_1 \subseteq X_2$. Then $g_2(X_1) \subseteq g_2(X_2)$.*

Proof. This is an immediate consequence of the definition of g_2. □

The set $g_2(X)$ can be constructively defined as follows.

Let δX be the system of all subsequences of sequences belonging to X. The convex closure (in $G^{\mathbb{N}}$) of the system $\{\text{const}\, 0\} \cup X$ will be denoted by $[X]$. Next, let $\langle X \rangle$ be the subgroup of $(G^{\mathbb{N}})^+$ generated by the set X. The symbol X^* will denote the set of all sequences in G^+ each subsequence of which has a subsequence belonging to X.

Then we have

3.7. Lemma. (Cf. [5] or [10], 2.2.) *Let $\emptyset \neq X \subseteq \operatorname{Conv}_0 \mathscr{A}$. Then $g_2(X) = [\langle \delta X \rangle]^*$.*

3.8. Lemma. *Let $X_1, X_2 \in \operatorname{Conv}_0 \mathscr{A}$, $X_1 \not\subseteq X_2$. Then $g_2(X_1) \not\subseteq g_2(X_2)$.*

Proof. There exists $(x_n) \in X_1 \setminus X_2$. By way of contradiction, suppose tha $g_2(X_1) \subseteq g_2(X_2)$. Since $X_1 \subseteq g_2(X_1)$, we obtain that $(x_n) \in g_2(X_2)$. Thus in view of 3.7, $(x_n) \in [\langle \delta X_2 \rangle]^*$. Since $X_2 \in \operatorname{Conv}_0 \mathscr{A}$, $\delta X_2 = X_2$. Also, $(A^{\mathbb{N}})^* \supseteq A^{\mathbb{N}}$ and $(x_n) \in A^{\mathbb{N}}$, thus $(x_n) \in [\langle \delta X_2 \rangle]^* \cap (A^{\mathbb{N}})^* = ([\langle \delta X_2 \rangle] \cap A^{\mathbb{N}})^* = ([\langle X_2 \rangle] \cap A^{\mathbb{N}})^*$.

Let $(z_n) \in [\langle X_2 \rangle] \cap A^{\mathbb{N}}$. Thus there is $(v_n) \in \langle X_2 \rangle$ such that $(z_n) \leqslant (v_n)$. Hence $z_n \leqslant v_n \in A$ for each $n \in \mathbb{N}$. There are $(t_n^1), \dots, (t_n^m)$ in X_2 such that $v_n = t_n^1 + \dots + t_n^m$ for each $n \in \mathbb{N}$. Hence

$$z_n \leqslant (t_n^1 + \dots + t_n^m) \wedge u = t_n^1 \oplus t_n^2 \oplus \dots \oplus t_n^m .$$

Because $X_2 \in \operatorname{Conv}_0 \mathscr{A}$ we get $(t_n^1 \oplus \dots \oplus t_n^m) \in X_2$ and hence $(z_n) \in X_2$. Next, X_2 satisfies Urysohn's condition (cf. the condition (ii) in 1.3); this yields that $(x_n) \in X_2$, which is a contradiction. □

3.9. Lemma. *Let $X \in \operatorname{Conv}_0 \mathscr{A}$ and $Z \subseteq [-mu, mu]^{\mathbb{N}}$ for some $m \in \mathbb{N}$. Then the following conditions are valid:*

(i) *$\delta X = X$ and $[X] = X$.*

(ii) *If $(z_n) \in A^{\mathbb{N}}$ and $(z_n) \in \langle X \rangle$, then $(z_n) \in X$.*

(iii) *If $(t_n) \in Z^*$, then there is $k \in \mathbb{N}$ such that $t_n \in [-mu, mu]$ for each $n \geqslant k$.*

Proof. The conditions (i) and (ii) follow from the definition of $\operatorname{Conv}_0 \mathscr{A}$ (cf. also 2.4). The validity of (iii) is obvious. □

For $K \in \operatorname{Conv}_0 G$ let K^b be the set of all $(x_n) \in K$ such that (x_n) is a bounded sequence in G. We denote by $\operatorname{Conv}_0^b G$ the system $\{K \in \operatorname{Conv}_0 G \colon K = K^b\}$; this system has been investigated in [10]. For each $K \in \operatorname{Conv}_0 G$, K^b belongs to $\operatorname{Conv}_0^b G$.

There exist examples for which $K \neq K^b$. Clearly $g_1(K) = g_1(K^b)$ for each $K \in \operatorname{Conv}_0 G$. Hence the mapping g_1 fails to be a monomorphism.

3.10. Lemma. *Let $X \in \operatorname{Conv}_0 \mathscr{A}$. Then $g_2(X)$ is bounded.*

Proof. In view of 3.7, $g_2(X) = [\langle \delta X \rangle]^*$. Next, according to 3.9 (i) we have $\delta X = X$. Hence for each $(y_n) \in \langle \delta X \rangle$ with $y_n \geqslant 0$ for each $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $(z_n^1), (z_n^2), \ldots, (z_n^m)$ in X such that

$$y_n = z_n^1 + z_n^2 + \ldots + z_n^m \quad \text{for each } n \in \mathbb{N}.$$

Thus for each $(v_n) \in [\langle \delta X \rangle]$ there are $m \in \mathbb{N}$ and $(a_n^i), (b_n^i) \in X$ $(i = 1, 2, \ldots, m)$ such that

$$a_n^1 + \ldots + a_n^m \leqslant v_n \leqslant b_n^1 + \ldots + b_n^m.$$

Therefore $v_n \in [-mu, mu]$ for each $n \in \mathbb{N}$. Thus according to 3.9, (iii), for each $(t_n) \in g_2(X)$ there is $k \in \mathbb{N}$ such that $t_n \in [-mu, mu]$ for each $n \geqslant k$. This yields that each sequence belonging to $g_2(X)$ is bounded. □

3.11. Lemma. *Let $Y \in \operatorname{Conv}_0 G$ and assume that Y is bounded. Put $g_1(Y) = X$. Then $g_2(X) = Y$.*

Proof. The relation $g_1(Y) = X$ gives that $X \subseteq Y$. Hence

$$g_2(X) = [\langle \delta X \rangle]^* \subseteq [\langle \delta Y \rangle]^*.$$

Since $Y \in \operatorname{Conv}_0 G$ we get $[\langle \delta Y \rangle]^* = Y$ and thus $g_2(X) \subseteq Y$.

Let $(y_n) \in Y$, $y_n \geqslant 0$ for each $n \in \mathbb{N}$. Since (y_n) is bounded there is $m \in \mathbb{N}$ such that $0 \leqslant y_n \leqslant u_1 + u_2 + \ldots + u_m$, where $u_i = u$ for $i = 1, 2, \ldots, m$. Thus there are elements z_n^i in G $(n \in \mathbb{N},\ i = 1, 2, \ldots, m)$ with

$$y_n = z_n^1 + \ldots + z_n^m, \quad 0 \leqslant z_n^i \leqslant u_i.$$

This yields that $z_n^i \leqslant y_n$ for each $n \in \mathbb{N}$ and each $i \in \{1, 2, \dots, m\}$. Thus $(z_n^i) \in Y$ and, at the same time, $(z_n^i) \in A^{\mathbb{N}}$, hence $(z_n^i) \in X$ for $i = 1, 2, \dots, k$. Thus $(y_n) \in \langle X \rangle$ and hence $(y_n) \in g_2(X)$. From this we easily deduce that $Y \subseteq g_2(X)$. Summarizing, we conclude $Y = g_2(X)$. □

3.12. Corollary. $g_2(\operatorname{Conv}_0 \mathscr{A}) = \operatorname{Conv}_0^b G$.

3.13. Theorem. *g_2 is an isomorphism of the partially ordered set $\operatorname{Conv}_0 \mathscr{A}$ onto $\operatorname{Conv}_0^b G$.*

Proof. This is a consequence of 3.2, 3.6, 3.8 and 3.12. □

An element K of $\operatorname{Conv} G$ will be called bounded if, whenever $((x_n), x) \in K$, then the sequence (x_n) is bounded in G. We denote by $\operatorname{Conv}^b G$ the set of all elements of $\operatorname{Conv} G$ which are bounded. It is easy to verify that $\operatorname{Conv}^b G$ is a convex subset of $\operatorname{Conv} G$ and contains the least element of $\operatorname{Conv} G$.

3.14. Theorem. *The partially ordered set $\operatorname{Conv} \mathscr{A}$ is isomorphic to $\operatorname{Conv}^b G$.*

Proof. Let f_1 be as in 2.9 and let g_2 be as above. Since f_1 and g_2 are isomorphisms, from

$$\operatorname{Conv} \mathscr{A} \xrightarrow{f_1} \operatorname{Conv}_0 \mathscr{A} \xrightarrow{g_2} \operatorname{Conv}_0^b G$$

we obtain an isomorphism of $\operatorname{Conv} \mathscr{A}$ onto $\operatorname{Conv}_0^b G$. The isomorphism φ_1 from 2.3 gives an isomorphism

$$\operatorname{Conv}_0 G \xrightarrow{\varphi_1^{-1}} \operatorname{Conv} G.$$

We obviously have

$$\varphi_1^{-1}(\operatorname{Conv}_0^b G) = \operatorname{Conv}^b G.$$

Thus there is an isomorphism of $\operatorname{Conv} \mathscr{A}$ onto $\operatorname{Conv}^b G$. □

3.15. Theorem. *Each interval of the partially ordered set $\operatorname{Conv} \mathscr{A}$ is a complete Brouwerian lattice.*

Proof. In view of [6] each interval of $\operatorname{Conv} G$ is a complete Brouwerian lattice. Now it suffices to apply 3.14. □

3.16. Theorem. *The following conditions are equivalent:*

(i) *$\operatorname{Conv} \mathscr{A}$ is a complete lattice.*
(ii) *$\operatorname{Conv} G$ is a complete lattice.*

Proof. This follows from 3.14 and 3.15. □

The following example shows that $\operatorname{Conv}^b G$ need not be equal to $\operatorname{Conv} G$.

Let G be the set of all bounded real functions defined on the set $\mathbb{R}$ of all reals; the operation $+$ and the partial order on G have the usual meaning. Let $u \in G$ be such that $u(t) = 1$ for each $t \in \mathbb{R}$. Consider the MV-algebra $\mathscr{A} = \mathscr{A}_0(G, u)$.

For each $n \in \mathbb{N}$ let $x_n \in G$ be defined as follows:

$$x_n(n) = n \quad \text{and} \quad x_n(t) = 0 \quad \text{whenever} \quad t \in \mathbb{R} \setminus \{n\}.$$

Thus $x_{n(1)} \wedge x_{n(2)} = 0$ whenever $n(1)$ and $(n(2)$ are distinct positive integers. There is $K \in \operatorname{Conv}_0 G$ such that $(x_n) \in K$. It is easy to verify that whenever $K(1) \in \operatorname{Conv}_0 \mathscr{A}$ then $g_2(K(1)) \neq K$. Hence $K \notin \operatorname{Conv}_0^b G$ and thus $\operatorname{Conv}_0^b G \neq \operatorname{Conv}_0 G$. Therefore $\operatorname{Conv}^b G \neq \operatorname{Conv} G$.

We shall apply the following definition of higher degrees of distributivity (it has been applied for the case of lattice ordered groups in [7]; cf. also [8] and [11]).

Let L be a lattice and let $\alpha > 0$, $\beta > 0$ be cardinals. L is called (α, β)-distributive if

(i) whenever T and S are sets with $\operatorname{card} T \leqslant \alpha, \operatorname{card} S \leqslant \beta$, then the relation

$$(1) \quad \bigwedge_{t \in T} \bigvee_{s \in S} x_{t,s} = \bigvee_{\varphi \in S^T} \bigwedge_{t \in T} x_{t,\varphi(t)}$$

is valid if all joins and meets standing in (1) do exist in L, and

(ii) the condition dual to (i) is also valid.

Next, L is called α-distributive if it is (α, α)-distributive. L is completely distributive if it is α-distributive for each cardinal α.

It is easy to verify that a lattice ordered group is (α, β)-distributive if and only if it satisfies one of the conditions (i) or (ii) above.

Again, let G and $\mathscr{A}$ be as above. In what follows we assume that $\operatorname{card} A > 1$.

3.17. Lemma. *Let α, β be cardinals. Then the following conditions are equivalent:*

(i) *G is not (α, β)-distributive.*

(ii) *There exists $x \in G$ with $0 < x$ such that, whenever $y \in G$, $0 < y \leqslant x$, then the interval $[0, y]$ of G is not (α, β)-distributive.*

Proof. It is obvious that (ii)$\Longrightarrow$(i). Let (i) be valid. Then according to 1.3 and 1.3.1 in [7] there are elements $x_{t,s}$ and x in G $(t \in T, s \in S, \operatorname{card} T \leqslant \alpha, \operatorname{card} S \leqslant \beta)$ such that $x_{t,s} \in [0, x]$ for each $t \in T$, $s \in S$ and

$$\text{(a)} \quad \bigwedge_{t \in T} \bigvee_{s \in S} x_{t,s} = x, \quad \bigvee_{\varphi \in S^T} \bigwedge_{t \in T} x_{t,\varphi(t)} = 0.$$

Let $y \in G$, $0 < y \leqslant x$. Put $x'_{t,s} = x_{t,s} \wedge y$ for each $t \in T$ and $s \in S$. Since G is infinitely distributive, from (a) we obtain

$$y = y \wedge x = \bigwedge_{t \in T} \bigvee_{s \in S} x'_{t,s}, \quad 0 = 0 \wedge y = \bigvee_{\varphi x^T} \bigwedge_{t \in T} x'_{t,\varphi(t)}.$$

Hence the interval $[0, y]$ is not (α, β)-distributive. □

3.18. Lemma. *Let α, β be cardinals. Then the following conditions are equivalent:*

(i) *G is not (α, β)-distributive.*
(ii) *$\mathscr{A}$ is not (α, β)-distributive.*

Proof. Let (i) be valid. Then in view of 3.17 the condition (ii) from 3.17 holds. Put $y = u \wedge x$. Hence the interval $[0, y]$ of G is not (α, β)-distributive. Since $[0, y]$ is, at the same time, an interval in $\mathscr{A}$ we infer that $\mathscr{A}$ is not (α, β)-distributive. Conversely, suppose that $\mathscr{A}$ is not (α, β)-distributive. Since $A = [0, u]$ and A is a closed sublattice of G, we infer that G is not (α, β)-distributive. □

3.19. Theorem. *Let $\mathscr{A}$ be $(\aleph_0, 2)$-distributive. Then* $\operatorname{Conv} \mathscr{A}$ *possesses a greatest element.*

Proof. In view of 3.18, G is $(\aleph_0, 2)$-distributive. Hence according to [11] $\operatorname{Conv} G$ has a greatest element. Therefore 3.16 yields that $\operatorname{Conv} \mathscr{A}$ has a greatest element. □

4. Convergences on the lattice $[0, u]$

For a lattice L we apply Definition 1.5. Let $\operatorname{Conv} L$ be the system of all convergences on L; this system is partially ordered by inclusion.

The symbol $\operatorname{Conv}_c L$ will denote the set of all $K \in \operatorname{Conv} L$ which satisfy the following condition:

(c) If (x_n) is a sequence in L such that for each $n \in \mathbb{N}$ the element x_n possesses a complement x'_n, then

$$x_n \longrightarrow_K 0 \iff x'_n \longrightarrow_K u.$$

4.1. Lemma. *Let $\mathscr{A}, G$ be as above and let L be the interval $[0, u]$ of G. Let $K \in \operatorname{Conv} \mathscr{A}$. Then $K \in \operatorname{Conv}_c L$.*

Proof. According to 1.6, $K \in \operatorname{Conv} L$. Suppose that (x_n) is a sequence in L such that for each $n \in \mathbb{N}$, x'_n is a complement of x_n in L. It is easy to verify that for each $n \in \mathbb{N}$, $x'_n = \neg x_n$. Hence if $x_n \longrightarrow_K 0$, then $\neg x_n \longrightarrow_K \neg 0 = u$. Similarly we can verify that if $x'_n \longrightarrow_K u$, then $x_n \longrightarrow_K 0$. Thus $K \in \operatorname{Conv}_c L$. □

If a lattice L is bounded, distributive and complemented (i.e., if it is a Boolean algebra) then we have to distinguish between convergences on L considered as a lattice (cf. Definition 1.5) and convergences on L considered as a Boolean algebra; namely, we can apply the following definition (cf. [9]).

4.2. Definition. Let B be a Boolean algebra; the corresponding lattice (where the unary operation $'$ of complementation is not taken into account) will be denoted by B_ℓ. The system $\operatorname{Conv} B$ is defined as the set of all $K \in \operatorname{Conv} B_\ell$ such that

$$x_n \longrightarrow_K x \Longrightarrow x'_n \longrightarrow_K x'.$$

4.3. Lemma. *Let B be a Boolean algebra. Then* $\operatorname{Conv} B = \operatorname{Conv}_c B_\ell$.

Proof. The greatest element of B will be denoted by u. According to the definition of $\operatorname{Conv} B$ the relation $\operatorname{Conv} B \subseteq \operatorname{Conv}_c B_\ell$ is valid. Let $K \in \operatorname{Conv}_c B_\ell$. Assume that $x_n \longrightarrow_K x$. Then

$$x_n \vee x \longrightarrow_K x, \quad x_n \wedge x \longrightarrow_K x.$$

From the former of the above relations we obtain

$$(x_n \vee x) \wedge x' \longrightarrow_K 0.$$

Then by applying the condition (c)

$$((x_n) \vee x) \wedge x')' \longrightarrow_K u,$$

hence

$$(x_n \vee x)' \vee x \longrightarrow_K u,$$
$$(x'_n \wedge x') \vee x \longrightarrow_K u,$$

$$x'_n \vee x \longrightarrow_K u.$$

Therefore $(x'_n \vee x) \wedge x' \longrightarrow_K x'$ and so

$$x'_n \wedge x' \longrightarrow_K x'.$$

Analogously we obtain that

$$x'_n \vee x' \longrightarrow_K x'.$$

Since $x'_n \wedge x' \leqslant x'_n \leqslant x'_n \vee x'$ we get $x'_n \longrightarrow_K x'$. Thus $K \in \operatorname{Conv} B$ and hence $\operatorname{Conv}_c B_\ell \subseteq \operatorname{Conv} B$. □

Again, let $L = [0, u]$ be as above.

4.4. Lemma. *Assume that $L = B_\ell$, where B is a Boolean algebra. Then $a \oplus b = a \vee b$ for each $a, b \in L$.*

Proof. Put $a \wedge b = v$, $a - v = a_1$, $b - v = b_1$. Then $a_1 \wedge b_1 = 0$, hence $a_1 + b_1 = a_1 \vee b_1$. Thus we have also $a_1 \oplus b_1 = a_1 \vee b_1$. Therefore

$$a \oplus b = (v \oplus a_1) \oplus (v \oplus b_1) = (v \oplus v) \oplus (a_1 \oplus b_1).$$

Since $L = B_\ell$, according to [2], Theorem 1.17, we have $v \oplus v = v$ and so

$$a \oplus b = v \oplus (a_1 \vee b_1) = (v \oplus a_1) \vee (v \oplus b_1) = a \vee b.$$

□

4.5. Lemma. *Let L be as in 4.4.. Let $K \in \operatorname{Conv} L$, $x_n \longrightarrow_K x$ and $y_n \longrightarrow_K y$. Then $x_n \oplus y_n \longrightarrow_K x + y$.*

Proof. We have $x_n \vee y_n \longrightarrow_K x \vee y$ and now it suffices to apply 4.4. □

4.6. Theorem. *Let $\mathscr{A}$ and L be as above. Assume that $L = B_\ell$, where B is a Boolean algebra. Then $\operatorname{Conv} \mathscr{A} = \operatorname{Conv}_c L$.*

Proof. In view of 4.1, $\operatorname{Conv} \mathscr{A} \subseteq \operatorname{Conv}_c L$. Next, according to 4.5 and by the definition of $\operatorname{Conv}_c L$ we obtain that $\operatorname{Conv}_c L \subseteq \operatorname{Conv} \mathscr{A}$. □

Let us remark that if $\operatorname{Conv} \mathscr{A} = \operatorname{Conv}_c L$, then there need not exist a Boolean algebra B with $B_\ell = L$.

Example. Let G be the additive group of all integers with the natural linear order. Put $u = 2$ and consider the MV-algebra $\mathscr{A} = \mathscr{A}_0(G, u)$. Then $\operatorname{card} A = 3$, hence $B_\ell \neq L = [0, u]$ for each Boolean algebra B. Next, $\operatorname{Conv} \mathscr{A} = \operatorname{Conv} L = \operatorname{Conv}_c L = \{K(0)\}$, where $K(0)$ is the least element of $\operatorname{Conv} \mathscr{A}$. □

4.7. Definition. Let L be as above and let $K \in \operatorname{Conv} L$. The lattice L is called *strongly nondiscrete* with respect to K if for each $0 < a \in L$ there exists a sequence (x_n) in L such that $0 < x_n < a$ for each $n \in \mathbb{N}$ and $x_n \longrightarrow_K 0$.

The following question remains open:

Let $\mathscr{A}$ and L be as above. Assume that

(i) $\operatorname{Conv} \mathscr{A} = \operatorname{Conv}_c L$;

(ii) if $K \in \operatorname{Conv} \mathscr{A}$ and $K \neq K(0)$, then L is strongly nondiscrete with respect to K.

Does there exist a Boolean algebra B with $L = B_\ell$?

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