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SEQUENTIAL ESTIMATION OF THE RATIO OF SCALE PARAMETERS IN THE EXPONENTIAL TWO-SAMPLE PROBLEM

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We consider a sequential point estimation of the ratio of two exponential scale parameters. For a fully sequential sampling scheme, second order approximations are obtained to the expected sample size and the risk of the sequential procedure. We also propose a bias-corrected procedure to reduce the risk.

Key words and phrases: Doob's maximal inequality, fully sequential procedure, regret, second order approximation, Wald's lemma.

1. Introduction

Let $X_1, X_2, \dots$ and $Y_1, Y_2, \dots$ be independent observations from the populations Π_1 and Π_2, respectively, where Π_i is according to an exponential distribution having the probability density function (pdf)

$$f_{\sigma_i}(x) = \sigma_i^{-1} \exp(-x/\sigma_i), \quad x > 0 \tag{1.1}$$

with $0 < \sigma_i < \infty$ for $i = 1, 2$. We assume that the scale parameters σ_1 and σ_2 are both unknown. We want to estimate the ratio σ_1/σ_2 of scale parameters. Taking samples of sizes n and m from Π_1 and Π_2, respectively, we estimate $\theta = \sigma_1/\sigma_2$ by

$$\hat{\theta}_{(n,m)} = \overline{X}_n / \overline{Y}_m$$

where $\overline{X}_n = n^{-1} \sum_{i=1}^n X_i$ and $\overline{Y}_m = m^{-1} \sum_{i=1}^m Y_i$. As the loss function, we consider

$$L\left(\hat{\theta}_{(n,m)}\right) = \left(\hat{\theta}_{(n,m)} - \theta\right)^2 + c(n+m)$$

where $c > 0$ is the known cost per unit sample in each population, and the risk is given by $R(\hat{\theta}_{(n,m)}) = E\{L(\hat{\theta}_{(n,m)})\}$ which is finite if $m > 2$.

As for two-sample cases, the sequential estimation of the difference of the means under the above loss structure has been investigated in the literature. For instance, Ghosh and Mukhopadhyay (1980) and Mukhopadhyay and Chattopadhyay (1991) considered the normal and the exponential cases, respectively and gave second order approximations to the risks as $c \to 0$. Mukhopadhyay and Purkayastha (1994) and Uno and Isogai (2000) treated the

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same problem in the case of unspecified distributions. It is interesting to estimate the ratio of scale parameters in a two-sample problem. The estimation of the ratio of two normal variances is especially important. However, sequential procedures for estimating the ratio of scale parameters have not been proposed so far. Hence, in this paper we propose a sequential procedure for estimating the ratio of two exponential scale parameters. Our sequential procedure can be applied to the estimation of the ratio of two normal variances, which will be pointed out in Remark 1 below. In Section 2, we present a fully sequential procedure and give second order asymptotic expansions for the expected sample size and the regret of the sequential procedure. A bias-corrected procedure is also proposed to reduce the risk and it is compared with the original one by simulation experiments. All proofs of the results are given in Section 3.

2. Main results

In this section, we propose a fully sequential procedure and investigate second order asymptotic properties of the procedure. Let $m > 2$. Estimating $\theta = \sigma_1/\sigma_2$ by $\hat{\theta}_{(n,m)}$, the risk is given by

$$R\left(\hat{\theta}_{(n,m)}\right) = E\left(\overline{X}_n/\overline{Y}_m - \theta\right)^2 + c(n+m) = \left(\frac{1}{n} + \frac{1}{m}\right)\theta^2 + r_{n,m}\theta^2 + c(n+m),$$

where $r_{n,m} = (\frac{1}{n} + \frac{1}{m})\frac{3m-2}{(m-1)(m-2)} + \frac{2}{(m-1)(m-2)}$. Since $r_{n,m} = O\left((\frac{1}{n} + \frac{1}{m})^2\right)$ as n and m tend to infinity, we have

$$R\left(\hat{\theta}_{(n,m)}\right) = \left(\frac{1}{n} + \frac{1}{m}\right)\theta^2 + c(n+m) + O\left(\left(\frac{1}{n} + \frac{1}{m}\right)^2\right).$$

If we ignore the order term above, then the risk $R(\hat{\theta}_{(n,m)})$ is (approximately) minimized by taking

$$n = m = c^{-1/2}\theta = n^* \quad \text{(say)} \tag{2.1}$$

(in practice, one of the two integers closest to this value) with $R(\hat{\theta}_{(n^*,n^*)}) \approx 4cn^*$ for sufficiently small c. But σ_1 and σ_2 are unknown, so is n^*. Takada (1986, 1998) gave details of the nonexistence of fixed sample size procedures. Since fixed sample size procedures are not available, we propose the following sequential sampling procedure motivated by (2.1). As the starting sample sizes, we take $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ from Π_1 and Π_2, respectively, where $k > 2$. If $k < c^{-1/2}\overline{X}_k/\overline{Y}_k$, then we take one observation in addition from each population, that is, X_{k+1} and Y_{k+1} are taken from Π_1 and Π_2, respectively. The resulting stopping time is defined by

$$N = N_c = \inf\{n \geq k : \ n \geq c^{-1/2}\overline{X}_n/\overline{Y}_n\}. \tag{2.2}$$

Then, by the strong law of large numbers, $P(N < \infty) = 1$ for all $c > 0$. Once the sampling stops, using the total $2N$ samples $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$, we estimate $\theta = \sigma_1/\sigma_2$ by $\hat{\theta}_N \equiv \hat{\theta}_{(N,N)} = \overline{X}_N/\overline{Y}_N$. The risk $R(\hat{\theta}_N)$ associated with $\hat{\theta}_N$ is

$$R\left(\hat{\theta}_N\right) = E\left(\overline{X}_N/\overline{Y}_N - \theta\right)^2 + cE(2N).$$

The performance of the sequential procedure is assessed by the regret $R(\hat{\theta}_N) - 4cn^*$.

We shall now give the main results concerning second order approximations to the expected sample size and the risk of the procedure.

THEOREM 1. (i) *If $k > 3$, then as $c \to 0$,*

$$E(N) = n^* + \rho - 1 + o(1),$$

where ρ is a constant given in (3.11) *and* $0 \le \rho \le \frac{3}{2}$.
(ii) *If $k > 12$, then as $c \to 0$,*

$$R(\hat{\theta}_N) - 4cn^* = 4c + o(c).$$

We shall propose another procedure to reduce the risk. The following theorem concerns the bias of the sequential procedure $\hat{\theta}_N$.

THEOREM 2. *If $k > 6$, then as $c \to 0$,*

$$E\left(\hat{\theta}_N\right) - \theta = -\sqrt{c} + o\left(\sqrt{c}\right).$$

Taking account of Theorem 2, we propose a bias-corrected procedure

$$\hat{\theta}_N^* = \overline{X}_N/\overline{Y}_N + \sqrt{c}.$$

Then, from Theorem 2, if $k > 6$, $E(\hat{\theta}_N^*) = \theta + o(\sqrt{c})$ as $c \to 0$. The risk associated with $\hat{\theta}_N^*$ is given by $R(\hat{\theta}_N^*) = E(\hat{\theta}_N^* - \theta)^2 + cE(2N)$ and its second order asymptotic expansion is given below.

THEOREM 3. *If $k > 12$, then as $c \to 0$,*

$$R\left(\hat{\theta}_N^*\right) - 4cn^* = 3c + o(c).$$

We have, from Theorems 1 (ii) and 3, if $k > 12$, then $R(\hat{\theta}_N^*) - R(\hat{\theta}_N) = -c + o(c)$ as $c \to 0$, which says that the risk of the bias-corrected procedure $\hat{\theta}_N^*$ is asymptotically less than that of the original procedure $\hat{\theta}_N$ by one cost.

For two exponential populations Π_1 and Π_2, Mukhopadhyay and Chattopadhyay (1991) considered sequential point estimation of the difference $\sigma_1 - \sigma_2$ and showed that the regret of their sequential procedure was $4c + o(c)$ as $c \to 0$. Thus, from Theorem 1 (ii), our procedure $\hat{\theta}_N$ and the procedure by Mukhopadhyay and Chattopadhyay (1991) are equal in the regret. Furthermore, from Theorem 3, our bias-corrected procedure $\hat{\theta}_N^*$ is superior in the regret to the procedure by Mukhopadhyay and Chattopadhyay (1991).

Remark 1. For the exponential one-sample problem, Starr and Woodroofe (1972) proposed a sequential procedure for estimating the scale parameter, which could be applied to the estimation of the normal variance. For two normal populations, it is interesting to estimate the ratio of the variances. Our procedure can also be applied to the estimation of the ratio of two normal variances by means of the transformation given in Lemma 10.1 of Woodroofe (1982).

Simulation. We shall give brief simulation results for the cases when $(\sigma_1,\ \sigma_2) = (2,\ 1)$ and $(1,\ 2)$. The cost c is chosen such that $n^* = \theta/\sqrt{c} = 40,\ 80$ and set the pilot sample size $k = 13$ for each population. The simulation results in Table 1 are based on 1,000,000 repetitions by means of the stopping rule N defined by (2.2). It looks from Table 1 that the bias-corrected procedure $\hat{\theta}_N^*$ betters the regret of the original procedure $\hat{\theta}_N$. As $c \to 0$ ($n^* = 80$), Table 1 seems to support Theorems 1 (ii) and 3.

Table 1. Comparison between $\hat{\theta}_N$ and $\hat{\theta}_N^*$.

$k = 13$		$\sigma_1 = 2,\ \sigma_2 = 1$, $\theta = 2$		$\sigma_1 = 1,\ \sigma_2 = 2$, $\theta = 0.5$	
		$n^* = 40$	$n^* = 80$	$n^* = 40$	$n^* = 80$
	$\sqrt{c}$	0.05	0.025	0.0125	0.00625
	$4cn^*$	0.4	0.2	0.025	0.0125
	$E(N)$	39.955470	80.039878	39.977044	80.051275
	$E(\hat{\theta}_N)$	1.944155	1.973843	0.486314	0.493539
	$E(\hat{\theta}_N^*)$	1.994155	1.998843	0.498814	0.499789
regret	$R(\hat{\theta}_N) - 4cn^*$	$5.350729\,c$	$4.649074\,c$	$5.137294\,c$	$4.865760\,c$
	$R(\hat{\theta}_N^*) - 4cn^*$	$4.116933\,c$	$3.556518\,c$	$3.947593\,c$	$3.798148\,c$

3. Proofs

We shall prove all results given in Section 2. Throughout this section, let $U_i = X_i/\sigma_1$ and $V_i = Y_i/\sigma_2$ for $i = 1, 2, \ldots$ and M be a generic positive constant. Further, let $c_0 > 0$ be chosen such that $n^* \geq 1$ and $E(N^2) < \infty$ for all $c \in (0,\ c_0]$ by Proposition 2 of Aras and Woodroofe (1993) and Lemma 4 below. We use the following notation:

$$D_n = \sum_{i=1}^n (U_i - 1), \quad Q_n = \sum_{i=1}^n (V_i - 1), \quad \overline{U}_n = \frac{1}{n}\sum_{i=1}^n U_i \quad \text{and} \quad \overline{V}_n = \frac{1}{n}\sum_{i=1}^n V_i.$$

The stopping variable N defined by (2.2) is written in the form

$$N = N_c = \inf\{n \geq k\,(> 2) :\ Z_n \geq n^*\},$$

where

$$Z_n = n\frac{\overline{V}_n}{\overline{U}_n} = n - D_n + Q_n + \xi_n \tag{3.1}$$

and by Taylor's Theorem,

$$\xi_n \equiv Z_n - (n - D_n + Q_n) = -n(\overline{U}_n - 1)(\overline{V}_n - 1) + n\overline{V}_n(\overline{U}_n - 1)^2\eta_n^{-3}, \tag{3.2}$$

in which η_n is a random variable lying between 1 and $\overline{U}_n$. We shall give four lemmas which are needed to prove Theorems 1 and 2.

Lemma 1. *Let $q > 0$. Then $\sup_{c>0} E(\overline{U}_N)^q \le E\left\{\sup_{n\ge1}(\overline{U}_n)^q\right\} \le M$, and if $k > q$ then $\sup_{c>0} E(\overline{U}_N)^{-q} \le E\left\{\sup_{n\ge k}(\overline{U}_n)^{-q}\right\} \le M$. These assertions hold for $\overline{V}_N$ instead of $\overline{U}_N$.*

Proof. From (1.1), U_1 is according to a standard exponential distribution with pdf $f_1(x)$. Hence, for a real number s,

$$E\left(\overline{U}_k\right)^s = \frac{\Gamma(k+s)}{k^s\Gamma(k)} < \infty \qquad \text{if} \quad k > -s, \tag{3.3}$$

where $\Gamma(x)$ is the gamma function. For $q > 1$, from the Doob's maximal inequality,

$$\sup_{c>0} E\left(\overline{U}_N\right)^q \le E\left\{\sup_{n\ge1}\left(\overline{U}_n\right)^q\right\} \le \left(\frac{q}{q-1}\right)^q E(U_1)^q < \infty. \tag{3.4}$$

For $0 < q \le 1$, we have from the Hölder inequality, for $q' > 1$, $E(\overline{U}_N)^q \le \{E(\overline{U}_N)^{q'}\}^{q/q'}$ which is finite from (3.4). Thus, the first assertion holds. We shall show the second assertion. For $q > 1$, from the Doob's maximal inequality and (3.3),

$$\begin{aligned}\sup_{c>0} E\left(\overline{U}_N\right)^{-q} \le E\left\{\sup_{n\ge k}\left(\overline{U}_n\right)^{-q}\right\} &\le \left(\frac{q}{q-1}\right)^q E\left(\overline{U}_k\right)^{-q} \\ &< \infty \qquad \text{if} \quad k > q.\end{aligned} \tag{3.5}$$

For $0 < q \le 1$, it follows from the Hölder inequality and (3.5) that for $1 < q' < 2$, $E(\overline{U}_N)^{-q} \le \{E(\overline{U}_N)^{-q'}\}^{q/q'} < \infty$ if $k > q'$, for which $k > 2$ is sufficient. Hence, the second assertion holds. The last assertion is clear because U_i and V_i are the same in distribution. □

Lemma 2. *Let $q \ge 1$.*
(i) *$\{(N/n^*)^{-q},\, c > 0\}$ is uniformly integrable if $k > q$.*
(ii) *$\{(N/n^*)^{q},\, 0 < c \le c_0\}$ is uniformly integrable if $k > q$.*

Proof. From the definition (3.1) of N, we have $(N/n^*)^{-q} \le (\overline{V}_N/\overline{U}_N)^q$. Thus, for $a > 1$, from the Hölder inequality with $u > 1$ and $u^{-1} + v^{-1} = 1$,

$$E(N/n^*)^{-aq} \le \left\{E\left(\overline{V}_N\right)^{aqu}\right\}^{1/u}\left\{E\left(\overline{U}_N\right)^{-aqv}\right\}^{1/v}.$$

Hence, from Lemma 1, $\{(N/n^*)^{-q},\, c>0\}$ is uniformly integrable if $k>q$. So (i) holds. For (ii), observe that $(N-1)\overline{V}_{N-1}/\overline{U}_{N-1}<n^*$ on $\{N>k\}$, so that for c_0,

$$\begin{aligned} N/n^* &\le \left\{\left(\overline{U}_{N-1}/\overline{V}_{N-1}\right)+(1/n^*)\right\} I_{\{N>k\}}+(k/n^*)I_{\{N=k\}} \\ &\le \left(\overline{U}_{N-1}/\overline{V}_{N-1}\right) I_{\{N>k\}}+(k+1), \end{aligned}$$

where $I_{\{\,\cdot\,\}}$ denotes the indicator function. Therefore, by c_r-inequality (see Loève (1977), p. 157), for $0<c\le c_0$,

$$\begin{aligned} (N/n^*)^q &\le \left\{\left(\overline{U}_{N-1}/\overline{V}_{N-1}\right) I_{\{N>k\}}+(k+1)\right\}^q \\ &\le M\left\{\left(\overline{U}_{N-1}/\overline{V}_{N-1}\right)^q I_{\{N>k\}}+(k+1)^q\right\}. \end{aligned}$$

For $a>1$, from the Hölder inequality with $u>1$ and $u^{-1}+v^{-1}=1$,

$$\begin{aligned} E\left\{\left(\overline{U}_{N-1}/\overline{V}_{N-1}\right)^q I_{\{N>k\}}\right\}^a &\le \left\{E\left(\overline{U}_{N-1}\right)^{aqu} I_{\{N>k\}}\right\}^{1/u} \\ &\quad\times\left\{E\left(\overline{V}_{N-1}\right)^{-aqv} I_{\{N>k\}}\right\}^{1/v} \\ &\le \left[E\left\{\sup_{n\ge k}\left(\overline{U}_n\right)^{aqu}\right\}\right]^{1/u}\left[E\left\{\sup_{n\ge k}\left(\overline{V}_n\right)^{-aqv}\right\}\right]^{1/v}, \end{aligned}$$

which, together with Lemma 1, proves (ii). □

From Theorem 2 of Chow *et al.* (1979), we have the next lemma.

Lemma 3. *For $q\ge 1$, if $\{(N/n^*)^q,\, 0<c\le c_0\}$ is uniformly integrable for some $c_0>0$, then $\{\,(n^{*-\frac{1}{2}}|D_N|)^q,\, 0<c\le c_0\,\}$ and $\{\,(n^{*-\frac{1}{2}}|Q_N|)^q,\, 0<c\le c_0\,\}$ are uniformly integrable.*

Let $\boldsymbol{W}=(\zeta_1,\,\zeta_2)$ be distributed according to a bivariate normal distribution with mean vector $(0,0)$ and covariance matrix $\Sigma=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. In the notation of Aras and Woodroofe (1993), letting

$$\text{(3.6)}\qquad \boldsymbol{X}_i=(U_i-1,\,V_i-1),\quad \boldsymbol{S}_n=(D_n,\,Q_n)\quad\text{and}\quad \boldsymbol{c}=(-1,\,1),$$

we have the following lemma.

Lemma 4. *If $k>3$, then the conditions* (C1)–(C6) *of Aras and Woodroofe* (1993) *are satisfied with $p=3$.*

Proof. Clearly, (C1) holds for $p=3$. From Proposition 4 of Aras and Woodroofe (1993), (C4) is satisfied, (C5) holds for all $\alpha\ge 3/2$ and (C6) holds with $\xi=\zeta_1^2-\zeta_1\zeta_2$. We shall show (C2) with $p=3$. Let $0<\varepsilon<\frac{1}{2}$. Since $Z_n-(n/\varepsilon)=n\{(\overline{V}_n/\overline{U}_n)-\varepsilon^{-1}\}\le 0$ on $\{\overline{V}_n/\overline{U}_n\le 1/\varepsilon\}$, we have for some

$s > 3$,

$$\begin{aligned}
&E\left\{\left(Z_n - \frac{n}{\varepsilon}\right)^+\right\}^s \\
&= E\left[\left(Z_n - \frac{n}{\varepsilon}\right)^s I_{\{\overline{V}_n/\overline{U}_n > 1/\varepsilon\}}\right] \le n^s\, E\left[\left(\overline{V}_n/\overline{U}_n\right)^s I_{\{\overline{V}_n/\overline{U}_n > 1/\varepsilon\}}\right] \\
&= n^s\, E\left[\left(\overline{V}_n/\overline{U}_n\right)^s I_{\{\overline{V}_n/\overline{U}_n > 1/\varepsilon,\ \overline{U}_n < 1-\varepsilon\}}\right] \\
&\quad + n^s\, E\left[\left(\overline{V}_n/\overline{U}_n\right)^s I_{\{\overline{V}_n/\overline{U}_n > 1/\varepsilon,\ \overline{U}_n \ge 1-\varepsilon\}}\right] \\
&= \quad J_1(n) \quad + \quad J_2(n), \qquad \text{say.}
\end{aligned}$$

By the independency of $\overline{U}_n$ and $\overline{V}_n$ and Lemma 1, we have, for $u > 1$ and $u^{-1} + v^{-1} = 1$,

$$\begin{aligned}
J_1(n) &\le n^s\, E\left[\left(\overline{V}_n/\overline{U}_n\right)^s I_{\{\overline{U}_n < 1-\varepsilon\}}\right] \\
&\le n^s \left\{E\left(\overline{V}_n\right)^s\right\} \left\{E\left(\overline{U}_n\right)^{-su}\right\}^{1/u} \left\{P\left(\overline{U}_n < 1-\varepsilon\right)\right\}^{1/v} \\
&\le M\, n^s \left\{P\left(\overline{U}_n - 1 < -\varepsilon\right)\right\}^{1/v} \qquad \text{if} \quad k > su.
\end{aligned}$$

Since by Tchebichev's inequality and the Marcinkiewicz-Zygmund inequality, for $n \ge 1$,

$$P\left(\overline{U}_n - 1 < -\varepsilon\right) \le (\varepsilon n)^{-q} E|D_n|^q = O(n^{-q/2}) \qquad \text{for} \quad q \ge 2, \tag{3.7}$$

we obtain $J_1(n) \le M n^{s-q/(2v)}$ for $n \ge k$. If $k > 3$, then we can choose $s > 3$, $q \ge 2$ and (u, v) such that $k > su$ and $s - q/(2v) \le 0$, so that $J_1(n) \le M$ for $n \ge k$. For $J_2(n)$, since $\{\overline{V}_n/\overline{U}_n > 1/\varepsilon,\ \overline{U}_n \ge 1-\varepsilon\} \subset \{\overline{V}_n - 1 > \delta\}$ where $\delta = (1-2\varepsilon)/\varepsilon > 0$, we have, from Lemma 1, for $u > 1$ with $u^{-1} + v^{-1} = 1$ and the above $s > 3$,

$$\begin{aligned}
J_2(n) &\le (1-\varepsilon)^{-s} n^s\, E\left[\left(\overline{V}_n\right)^s I_{\{\overline{V}_n/\overline{U}_n > 1/\varepsilon,\ \overline{U}_n \ge 1-\varepsilon\}}\right] \\
&\le M\, n^s \left\{E\left(\overline{V}_n\right)^{su}\right\}^{1/u} \left\{P\left(\overline{V}_n - 1 > \delta\right)\right\}^{1/v} \\
&\le M\, n^s \left\{P\left(\overline{V}_n - 1 > \delta\right)\right\}^{1/v}.
\end{aligned}$$

By (3.7), $P(\overline{V}_n - 1 > \delta) = O(n^{-q/2})$ for $q \ge 2$, so that $J_2(n) \le M n^{s-q/(2v)}$ for $n \ge k$. Choosing q such that $s - q/(2v) \le 0$, we have $J_2(n) \le M$ for $n \ge k$. Therefore, $\{[(Z_n - \frac{n}{\varepsilon})^+]^3,\ n \ge k\}$ is uniformly integrable, that is, (C2) holds. Finally, we shall show (C3). From (3.2), Tchebichev's inequality, the independency of $\overline{U}_n$ and $\overline{V}_n$ and the Marcinkiewicz-Zygmund inequality, we have, for $0 < \varepsilon < 1$,

$$\begin{aligned}
P\{\xi_n < -\varepsilon n\} &= P\left\{\left(\overline{U}_n - 1\right)\left(\overline{V}_n - 1\right) - \overline{V}_n \left(\overline{U}_n - 1\right)^2 \eta_n^{-3} > \varepsilon\right\} \\
&\le P\left\{\left(\overline{U}_n - 1\right)\left(\overline{V}_n - 1\right) > \varepsilon\right\} \\
&\le \varepsilon^{-3} E|\overline{U}_n - 1|^3 E|\overline{V}_n - 1|^3 = O(n^{-3}),
\end{aligned}$$

which implies $\sum_{n=1}^{\infty} n\, P\{\xi_n < -\varepsilon n\} < \infty$, so that (C3) holds. $\square$

Let

$$H_c = Z_N - n^* = N - n^* - D_N + Q_N + \xi_N. \tag{3.8}$$

It follows from Propositions 2 and 3 of Aras and Woodroofe (1993) that as $c \to 0$,

$$\frac{N}{n^*} \xrightarrow{a.s.} 1 \qquad \text{and} \qquad \left(\frac{\boldsymbol{S}_N}{\sqrt{N}},\ \xi_N,\ H_c\right) \xrightarrow{d} (\boldsymbol{W},\ \xi,\ H) \tag{3.9}$$

with $\xi = \zeta_1^2 - \zeta_1\zeta_2$, where '$\xrightarrow{a.s.}$' and '$\xrightarrow{d}$' stand for almost sure convergence and convergence in distribution, respectively and H is a certain random variable with $\rho = E(H)$ which is given in (3.11). From Proposition 7 of Aras and Woodroofe (1993),

$$\{|\xi_N - H_c|^2,\ 0 < c \le c_0\} \text{ is uniformly integrable.} \tag{3.10}$$

Now we are in a position to prove Theorems 1–3.

Proof of Theorem 1. Using the notation (3.6), $N = \inf\{n \ge k :\ n + \langle \boldsymbol{c},\, \boldsymbol{S}_n\rangle + \xi_n \ge n^*\}$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes inner product. Let

$$t = \inf\{n \ge 1 :\ n + \langle \boldsymbol{c},\, \boldsymbol{S}_n\rangle > 0\} \qquad \text{and} \qquad \rho = \frac{E\{(t + \langle \boldsymbol{c},\, \boldsymbol{S}_t\rangle)^2\}}{2\,E(t + \langle \boldsymbol{c},\, \boldsymbol{S}_t\rangle)}\,. \tag{3.11}$$

It follows from Theorem 1 and Proposition 3 of Aras and Woodroofe (1993), Corollary 2.2 of Woodroofe (1982) and Lemma 4 that if $k > 3$, then

$$E(N) = n^* + \rho - E(\xi) + o(1) = n^* + \rho - 1 + o(1) \qquad \text{as} \quad c \to 0.$$

From Corollary 2.7 of Woodroofe (1982), $\rho = \frac{3}{2} - \sum_{n=1}^{\infty} \frac{1}{n} E\{(n - D_n + Q_n)^-\}$, where $(\,\cdot\,)^-$ denotes negative part such that $x^- \equiv \max(-x,\, 0)$, and so $0 \le \rho \le \frac{3}{2}$. Thus, the first assertion holds. We shall prove (ii). Observe that

$$R\left(\hat{\theta}_N\right) - 4cn^* = \theta^2 E\left(\overline{U}_N/\overline{V}_N - 1\right)^2 + 2cE(N) - 4cn^*$$

and by Taylor's theorem,

$$\begin{aligned}\left(\overline{U}_N/\overline{V}_N - 1\right)^2 &= \left\{\overline{U}_N - 1 - \left(\overline{V}_N - 1\right)\right\}^2 \left(\overline{V}_N\right)^{-2} \\ &= \left\{\overline{U}_N - 1 - \left(\overline{V}_N - 1\right)\right\}^2 \left\{1 - 2\left(\overline{V}_N - 1\right) + 3\left(\overline{V}_N - 1\right)^2 \varphi^{-4}\right\},\end{aligned}$$

where φ is a random variable lying between 1 and $\overline{V}_N$. Hence,

$$\begin{aligned} R\left(\hat{\theta}_N\right) - 4cn^* &= \theta^2 E\left\{\overline{U}_N - 1 - \left(\overline{V}_N - 1\right)\right\}^2 + 2cE(N) - 4cn^* \\ &\quad - 2\theta^2 E\left[\left\{\overline{U}_N - 1 - \left(\overline{V}_N - 1\right)\right\}^2 \left(\overline{V}_N - 1\right)\right] \\ &\quad + 3\theta^2 E\left[\left\{\overline{U}_N - 1 - \left(\overline{V}_N - 1\right)\right\}^2 \left(\overline{V}_N - 1\right)^2 \varphi^{-4}\right] \\ &= \quad K_1 \quad + \quad K_2 \quad + \quad K_3, \qquad \text{say.}\end{aligned} \tag{3.12}$$

Since from (2.1), $K_1 = 2c[\,\frac{1}{2}(n^*)^2 E\{\overline{U}_N - 1 - (\overline{V}_N - 1)\}^2 + E(N) - 2n^*\,]$, we get from Corollary 1 of Theorem 2 of Aras and Woodroofe (1993) with $\boldsymbol{b} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and Lemma 4,

$$\begin{aligned} K_1/(2c) &= E\{\xi(\zeta_1 - \zeta_2)^2\} - 2E(\xi) + 2 + 4 - E\{U_1 - 1 - (V_1 - 1)\}^3 + o(1) \\ &= E\{\zeta_1(\zeta_1 - \zeta_2)^3\} + 4 + o(1) = 10 + o(1), \end{aligned}$$

which implies

$$K_1 = 20c + o(c) \qquad \text{as} \quad c \to 0. \tag{3.13}$$

Observe from (2.1) that $K_3 = 3c\,E[\,(n^*)^2\{(\overline{U}_N - 1) - (\overline{V}_N - 1)\}^2(\overline{V}_N - 1)^2\varphi^{-4}\,]$. We shall show the uniform integrability of $\{(n^*)^2\{(\overline{U}_N - 1) - (\overline{V}_N - 1)\}^2 \times (\overline{V}_N - 1)^2\varphi^{-4},\, c \le c_0\}$. Clearly,

$$\begin{aligned} &(n^*)^2 \left\{\left(\overline{U}_N - 1\right) - \left(\overline{V}_N - 1\right)\right\}^2 \left(\overline{V}_N - 1\right)^2 \varphi^{-4} \\ &= (n^*)^2 \left(\overline{U}_N - 1\right)^2 \left(\overline{V}_N - 1\right)^2 \varphi^{-4} \\ &\quad - 2(n^*)^2 \left(\overline{U}_N - 1\right) \left(\overline{V}_N - 1\right)^3 \varphi^{-4} + (n^*)^2 \left(\overline{V}_N - 1\right)^4 \varphi^{-4} \\ &= \quad J_{31} \quad - \quad 2J_{32} \quad + \quad J_{33}, \qquad \text{say.} \end{aligned}$$

From the Hölder inequality, for $a > 1$,

$$\begin{aligned} E|J_{31}|^a &= E\left|(n^*/N)^4\,\{(n^*)^{-1/2}D_N\}^2\{(n^*)^{-1/2}Q_N\}^2\varphi^{-4}\right|^a \\ &\le \{E(n^*/N)^{12a}\}^{1/3}\{E|(n^*)^{-1/2}D_N|^{12a}\}^{1/6} \\ &\quad \times \{E|(n^*)^{-1/2}Q_N|^{12a}\}^{1/6}\{E(\varphi^{-12a})\}^{1/3} \end{aligned}$$

and by the convexity, $E(\varphi^{-12a}) \le 1 + E(\overline{V}_N)^{-12a}$. Thus, from Lemmas 1–3, if $k > 12$, then $\{|J_{31}|,\, c \le c_0\}$ is uniformly integrable. Similarly, we can show the uniform integrabilities of $\{|J_{32}|,\, c \le c_0\}$ and $\{|J_{33}|,\, c \le c_0\}$ provided $k > 12$, so that we obtain the uniform integrability of $\{(n^*)^2\{(\overline{U}_N - 1) - (\overline{V}_N - 1)\}^2 \times (\overline{V}_N - 1)^2\varphi^{-4},\, c \le c_0\}$. From (3.9) and the fact that $\varphi \xrightarrow{a.s.} 1$ as $c \to 0$,

$$(n^*)^2 \left\{\left(\overline{U}_N - 1\right) - \left(\overline{V}_N - 1\right)\right\}^2 \left(\overline{V}_N - 1\right)^2 \varphi^{-4} \ \xrightarrow{\ d\ }\ (\zeta_1 - \zeta_2)^2\zeta_2^2 \qquad \text{as} \quad c \to 0,$$

which yields

$$K_3 = 3cE\left\{(\zeta_1 - \zeta_2)^2\zeta_2^2\right\} + o(c) = 12c + o(c). \tag{3.14}$$

Finally, we shall calculate K_2. From (2.1),

(3.15)
$$\begin{aligned} K_2 &= -2c\,E\left\{(n^*)^2N^{-3}(D_N - Q_N)^2Q_N\right\} \\ &= -2c\,E\left\{(n^*)^{-1}\left((n^*/N)^3 - 1\right)(D_N - Q_N)^2Q_N + (n^*)^{-1}(D_N - Q_N)^2Q_N\right\} \\ &= -2c\,E\{\ J_{21}\ +\ J_{22}\ \}, \qquad \text{say.} \end{aligned}$$

Observe from (3.8) that

$$\begin{aligned}
J_{21} &= (n^*)^{-1}\left((n^*/N)^3-1\right)(D_N-Q_N)^2Q_N \\
&= \frac{(n^*)^2+n^*N+N^2}{n^*N^3}(n^*-N)(D_N-Q_N)^2Q_N \\
&= -\frac{(n^*)^2+n^*N+N^2}{n^*N^3}(D_N-Q_N)^3Q_N \\
&\quad+\frac{(n^*)^2+n^*N+N^2}{n^*N^3}(D_N-Q_N)^2Q_N(\xi_N-H_c) \\
&= \quad J_{211} \quad + \quad J_{212}, \qquad \text{say.}
\end{aligned}$$

For $a>1$, by the Hölder inequality,

$$\begin{aligned}
E|J_{211}|^a &= E\left|\frac{(n^*)^3+(n^*)^2N+n^*N^2}{N^3}\,\frac{(D_N-Q_N)^3Q_N}{(n^*)^2}\right|^a \\
&\le \left\{E\left(\frac{(n^*)^3}{N^3}+\frac{(n^*)^2}{N^2}+\frac{n^*}{N}\right)^{7a/3}\right\}^{3/7}\left\{E\left|\frac{(D_N-Q_N)^3Q_N}{(n^*)^2}\right|^{7a/4}\right\}^{4/7},
\end{aligned}$$

so that from Lemmas 2 and 3, $\{|J_{211}|,\, 0<c\le c_0\}$ is uniformly integrable provided $k>7$. Similarly, for $a>1$, $s>1$, $s^{-1}+u^{-1}=1$ and $v>1$, $v^{-1}+w^{-1}=1$,

$$\begin{aligned}
E|J_{212}|^a &= E\left|\frac{(n^*)^2+n^*N+N^2}{N^2}\,\frac{(D_N-Q_N)^2}{n^*}\left(\overline{V}_N-1\right)(\xi_N-H_c)\right|^a \\
&\le \left\{E\left(\frac{(n^*)^2}{N^2}+\frac{n^*}{N}+1\right)^{2as}\right\}^{1/2s}\left\{E\left|\frac{(D_N-Q_N)^2}{n^*}\right|^{2as}\right\}^{1/2s} \\
&\quad\times\left\{E|\overline{V}_N-1|^{auv}\right\}^{1/uv}\left\{E|\xi_N-H_c|^{auw}\right\}^{1/uw},
\end{aligned}$$

whence, taking $(s,u)=(\frac{11}{5},\frac{11}{6})$ and $(v,w)=(23,\frac{23}{22})$, from Lemmas 1–3 and (3.10), $\{|J_{212}|, 0<c\le c_0\}$ is uniformly integrable provided $k>8$. Since from (3.9), $J_{21}\xrightarrow{d}-3(\zeta_1-\zeta_2)^3\zeta_2$ as $c\to0$, we obtain

$$E(J_{21}) = -3E\left\{(\zeta_1-\zeta_2)^3\zeta_2\right\}+o(1) = 18+o(1). \tag{3.16}$$

For J_{22},

$$\begin{aligned}
J_{22} &= (n^*)^{-1}D_N^2Q_N-2(n^*)^{-1}D_NQ_N^2+(n^*)^{-1}Q_N^3 \\
&= \quad J_{221} \quad -2\,J_{222} \quad +J_{223}, \qquad \text{say.}
\end{aligned} \tag{3.17}$$

It follows from Theorem 9 of Chow *et al.* (1965), Lemma 2 and (3.9) that

$$\begin{aligned}
E(J_{223}) &= (n^*)^{-1}\left\{2E(N)+3E(NQ_N)\right\} \\
&= 2+3E\left\{(N/n^*)Q_N\right\}+o(1) \qquad \text{as} \quad c\to0,
\end{aligned}$$

where by Wald's lemma and (3.8),

$$E\left\{(N/n^*)Q_N\right\} = E\left\{\left(\frac{N}{n^*}-1\right)Q_N\right\} = E\left\{\frac{D_N - Q_N - \xi_N + H_c}{n^*}Q_N\right\}.$$

From (3.10) and Lemmas 2 and 3, for $a > 1$, if $k > 3$, then

$$\begin{aligned} &E\left|\frac{D_N - Q_N - \xi_N + H_c}{n^*}Q_N\right|^a \\ &\le M\left[E\left|\frac{(D_N - Q_N)Q_N}{n^*}\right|^a + \left\{E|\xi_N - H_c|^{3a/2}\right\}^{2/3}\left\{E\left|(n^*)^{-1/2}Q_N\right|^{3a}\right\}^{1/3}\right] \\ &\le M, \end{aligned}$$

and from (3.9), $(n^*)^{-1}(D_N - Q_N - \xi_N + H_c)Q_N \xrightarrow{d} (\zeta_1 - \zeta_2)\zeta_2$ as $c \to 0$. Therefore,

$$E\left\{(N/n^*)Q_N\right\} = E\{(\zeta_1 - \zeta_2)\zeta_2\} + o(1) = -1 + o(1) \qquad \text{as} \quad c \to 0, \tag{3.18}$$

which yields

$$E(J_{223}) = 2 + 3\{-1 + o(1)\} + o(1) = -1 + o(1). \tag{3.19}$$

From (3.18), as $c \to 0$,

$$\begin{aligned} E(J_{221}) &= (n^*)^{-1}E\{(D_N^2 - N)Q_N\} + E\{(N/n^*)Q_N\} \\ &= (n^*)^{-1}E\{(D_N^2 - N)Q_N\} - 1 + o(1) \end{aligned} \tag{3.20}$$

and we have

$$E\{(D_N^2 - N)Q_N\} = \frac{1}{2}\left\{E(D_N^2 - N + Q_N)^2 - E(D_N^2 - N)^2 - E(Q_N^2)\right\}. \tag{3.21}$$

We shall give the following lemma which will be proved later on.

LEMMA 5. *For every* $c \in (0, c_0]$, $E\{(D_N^2 - N)Q_N\} = E\{(Q_N^2 - N)D_N\} = 0$.

It follows from (3.20) and Lemma 5, we obtain

$$E(J_{221}) = -1 + o(1) \qquad \text{as} \quad c \to 0. \tag{3.22}$$

By the same argument as (3.22), we have that $E(J_{222}) = 1 + o(1)$, which, together with (3.17), (3.19) and (3.22), yields $E(J_{22}) = -4 + o(1)$. Therefore, from (3.15) and (3.16),

$$K_2 = -2c(18 - 4) + o(c) = -28c + o(c) \qquad \text{as} \quad c \to 0,$$

from which, together with (3.12)–(3.14), we get $R(\hat{\theta}_N) - 4cn^* = 4c + o(c)$. Thus, the proof is complete. □

Proof of Lemma 5. For $\boldsymbol{X}_i = (U_i - 1,\, V_i - 1),\ i = 1, 2, \ldots,$ let $\mathcal{F}_n = \sigma(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ for $n \geq 1$ be the σ-algebra generated by $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ with $\mathcal{F}_0 = \{\phi,\, \Omega\}$, and let $x_i = 2D_{i-1}(U_i - 1) + (U_i - 1)^2 - 1$ for $i \geq 1$ with $D_0 = 0$. By the same argument as (2.14) of Chow and Martinsek (1982), it follows from Lemma 2 (ii) and $E(N^2) < \infty$ that for fixed $c \in (0,\, c_0]$,

$$\int_{\{N>n\}} |D_n^2 - n| dP = o(1) \qquad \text{as} \quad n \to \infty.$$

Therefore, from Lemmas 3 and 6 of Chow *et al.* (1965),

$$\begin{aligned}(3.23) \qquad E\left(D_N^2 - N\right)^2 &= E\left(\sum_{i=1}^{N} x_i^2\right) = E\left\{\sum_{i=1}^{N} E\left(x_i^2 \,|\mathcal{F}_{i-1}\right)\right\} \\ &= E\left\{\sum_{i=1}^{N}\left(4D_{i-1}^2 + 8D_{i-1} + 8\right)\right\}.\end{aligned}$$

At the notation of (20) of Chow *et al.* (1965), letting $u_{r,i} = E(U_i - 1)^r$ and $U_{r,n} = \sum_{i=1}^{n} u_{r,i}$, since $E(NU_{4,N}) = E(U_1 - 1)^4 \cdot E(N^2) < \infty$, we have from Lemma 8 of Chow *et al.* (1965), $E(D_N U_{3,N}) = E(\sum_{i=2}^{N} D_{i-1} u_{3,i})$, which, together with $U_{3,N} = 2N$, $u_{3,i} = 2$ and $D_0 = 0$, implies $E(ND_N) = E(\sum_{i=1}^{N} D_{i-1})$. Hence, from (3.23), we have

$$E\left(D_N^2 - N\right)^2 = 4E\left(\sum_{i=1}^{N} D_{i-1}^2\right) + 8E(ND_N) + 8E(N),$$

which is finite because from Theorems 2, 7 and Lemma 9 of Chow *et al.* (1965),

$$\begin{aligned}&E|ND_N| \leq \left\{E\left(N^2\right)\right\}^{1/2}\left\{E\left(D_N^2\right)\right\}^{1/2} < \infty \qquad \text{and} \\ &E\left(\sum_{i=1}^{N} D_{i-1}^2\right) \leq E\left(\sum_{i=1}^{N} D_i^2\right) \leq E\left(ND_N^2\right) \leq \left\{E\left(N^2\right)\right\}^{1/2}\left\{E\left(D_N^4\right)\right\}^{1/2} < \infty.\end{aligned}$$

Similarly, we get

$$\begin{aligned}&E\left(D_N^2 - N + Q_N\right)^2 \\ &= E\left\{\sum_{i=1}^{N}(x_i + V_i - 1)^2\right\} = E\left[\sum_{i=1}^{N} E\left\{(x_i + V_i - 1)^2 \,|\mathcal{F}_{i-1}\right\}\right] \\ &= E\left[\sum_{i=1}^{N} E\left\{x_i^2 + 2x_i(V_i - 1) + (V_i - 1)^2 \,|\mathcal{F}_{i-1}\right\}\right] \\ &= E\left(D_N^2 - N\right)^2 + E(N) < \infty,\end{aligned}$$

which, together with (3.21) and $E(Q_N^2) = E(N)$, yields $E\{(D_N^2 - N)Q_N\} = 0$. By the same argument as above, we obtain $E\{(Q_N^2 - N)D_N\} = 0$. Thus, the lemma holds. $\square$

Proof of Theorem 2. From (2.1) and Taylor's theorem,

(3.24)
$$\begin{aligned}\frac{E(\hat{\theta}_N)-\theta}{\sqrt{c}} &= n^* E\left\{\frac{\overline{U}_N - 1 - \left(\overline{V}_N - 1\right)}{\overline{V}_N}\right\}\\ &= n^* E\left\{\overline{U}_N - 1 - \left(\overline{V}_N - 1\right)\right\} - n^* E\left[\left\{\overline{U}_N - 1 - \left(\overline{V}_N - 1\right)\right\}\left(\overline{V}_N - 1\right)\varphi^{-2}\right]\\ &= J_1 - J_2, \quad \text{say,}\end{aligned}$$

where φ is a random variable lying between 1 and $\overline{V}_N$. By Wald's lemma, (3.8) and (3.9),

(3.25)
$$\begin{aligned}J_1 &= E\left\{\frac{n^*-N}{N}(D_N - Q_N)\right\} = E\left\{\frac{-(D_N-Q_N)+\xi_N-H_c}{N}(D_N-Q_N)\right\}\\ &= -E\left(\zeta_1-\zeta_2\right)^2 + o(1) = -2 + o(1) \qquad \text{as} \quad c\to 0\end{aligned}$$

because for $a>1$,

$$\begin{aligned}&E\left|\frac{-(D_N-Q_N)+\xi_N-H_c}{N}(D_N-Q_N)\right|^a\\ &\le M\left[E\left(\frac{(D_N-Q_N)^2}{N}\right)^a + E\left|\left(\xi_N-H_c\right)\left(\overline{U}_N-\overline{V}_N\right)\right|^a\right]\\ &\le M\left\{E\left(\frac{n^*}{N}\right)^{3a}\right\}^{1/3}\left\{E\left|\frac{D_N-Q_N}{(n^*)^{1/2}}\right|^{3a}\right\}^{2/3}\\ &\quad + M\left\{E|\xi_N-H_c|^{3a/2}\right\}^{2/3}\left\{E|\overline{U}_N-\overline{V}_N|^{3a}\right\}^{1/3},\end{aligned}$$

which is bounded, that is, $\{\frac{n^*-N}{N}(D_N-Q_N),\ 0<c\le c_0\}$ is uniformly integrable by Lemmas 1–3 and (3.10), provided $k>3$. Since for $a>1$,

$$\begin{aligned}&E\left|n^*\left\{\overline{U}_N-1-\left(\overline{V}_N-1\right)\right\}\left(\overline{V}_N-1\right)\varphi^{-2}\right|^a\\ &\le\left\{E\left(\frac{n^*}{N}\right)^{6a}\right\}^{1/3}\left\{E\left|\frac{(D_N-Q_N)Q_N}{n^*}\right|^{3a}\right\}^{1/3}\left\{E\left(\varphi^{-6a}\right)\right\}^{1/3}\end{aligned}$$

and $E(\varphi^{-6a})\le 1+E(\overline{V}_N)^{-6a}$, it follows from Lemmas 1–3 that if $k>6$, then $\{n^*\{\overline{U}_N-1-(\overline{V}_N-1)\}(\overline{V}_N-1)\varphi^{-2},\ 0<c\le c_0\}$ is uniformly integrable. Thus, from (3.9) and the fact that $\varphi\xrightarrow{a.s.}1$ as $c\to 0$,

$$J_2 = E\{(\zeta_1-\zeta_2)\zeta_2\} + o(1) = -1 + o(1),$$

which, together with (3.24) and (3.25), yields $E(\hat{\theta}_N)-\theta = -\sqrt{c}+o(\sqrt{c})$. The theorem holds. □

Proof of Theorem 3. It follows from Theorems 1 and 2 that as $c \to 0$,

$$\begin{aligned} R\left(\hat{\theta}_N^*\right) &= R\left(\hat{\theta}_N\right) + 2\sqrt{c}\,E\left(\hat{\theta}_N - \theta\right) + c = R\left(\hat{\theta}_N\right) + 2\sqrt{c}\left\{-\sqrt{c} + o\left(\sqrt{c}\right)\right\} + c \\ &= R\left(\hat{\theta}_N\right) - c + o(c) = 4cn^* + 3c + o(c), \end{aligned}$$

proving Theorem 3. □

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