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SEQUENTIAL NONPARAMETRIC AGE REPLACEMENT POLICIES[1]

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Under an age replacement policy, a stochastically failing unit is replaced at failure or after being in service for t units of time, whichever comes first. An important problem is the estimation of ϕ^*, the optimal replacement time when the form of the failure distribution is unknown. Here, ϕ^* is optimal in the sense that it is the replacement time that achieves the smallest long-run expected cost. It is shown that substantial cost savings can be effected by estimating ϕ^* sequentially. The sequential methodology employed here is stochastic approximation (SA). When suitably standardized, convergence in distribution of the SA estimator to ϕ^* is established. This gives precise information about the rate of convergence. A sequential methodology introduced by Bather (1977) has roughly the same aims as ours, but it is not of the SA type. Rates of convergence apparently have not been established for Bather's procedure.

1. Introduction and summary. Consider a functioning unit with specified life distribution F, and probability of survival to age x, $S(x) = 1 - F(x)$. Suppose F is absolutely continuous with probability density f. Let C_1 and C_2 be fixed, known costs with $C_1 > C_2 > 0$. If the unit fails prior to t units of time after installation, it is replaced at that failure time with cost C_1. Otherwise, the unit is replaced t units of time after its installation with cost C_2. It is assumed that replacement is immediate. Under the age replacement policy, the replacement unit is available from a sequence of such units that fail independently with the same distribution function F. The objective is to minimize the long-run accumulation of costs in some sense. The cost function here is the expected long-run average cost,

$$R(t) = \{C_1F(t) + C_2S(t)\} \Big/ \int_0^t S(u)\,du. \tag{1.1}$$

(cf., Barlow and Proschan, 1965, page 87).

Under some fairly general conditions, there is a unique and finite time, say ϕ^*, where $R(t)$ attains a global minimum. An example of such a condition is that the failure rate $f(x)/S(x)$ be strictly increasing to infinity with x. See Bergman

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(1979) for other sufficient conditions. Call ϕ^* the optimal replacement time. We wish to estimate the parameter ϕ^*.

Estimation of the optimal replacement time based on a fixed number of i.i.d. units with distribution F has been examined in detail. See Arunkumar (1972) for asymptotic results of an interesting nonparametric approach. Using Monte-Carlo simulation, Ingram and Scheaffer (1976) give finite sample results for several cases where the form of F is known up to one parameter. Bergman (1977) and Barlow (1978) discuss graphical methods for the estimation problem.

Because fixed sample procedures rely on i.i.d. observations, during experimentation units must be left in service until failure. Therefore, the experimenter is constrained from using an estimator to achieve cost savings as the estimation procedure continues. Bather (1977) introduced a procedure that can be used on an ongoing basis which constantly updates the estimators using current observations. Suppose $\{X_n\}$ is an i.i.d. sequence with distribution function F and $\{\phi_n^*\}$ is a sequence of random variables that estimates ϕ^*. Let $N(t)$ be the number of replacements by time t, i.e.,

$$N(t) = \sum_{i=1}^{\infty} \mathbf{I}\{\min(X_1, \phi_1^*) + \cdots + \min(X_i, \phi_i^*) < t\} \tag{1.2}$$

where $\mathbf{I}(\cdot)$ is the indicator function. Bather showed how to construct the estimators $\{\phi_n^*\}$ sequentially so that

$$\phi_n^* \to \phi^* \quad \text{a.s.} \tag{1.3}$$

and

$$t^{-1} \sum_{i=1}^{N(t)} \{C_1\mathbf{I}(X_i < \phi_i^*) + C_2\mathbf{I}(X_i \geq \phi_i^*)\} \to R(\phi^*) \quad \text{a.s.} \tag{1.4}$$

Thus, the estimators are strongly consistent, and further, the actual average cost achieved by the experimenter is the same asymptotically as if the optimal replacement time were known a priori.

This paper gives alternative methods for constructing the sequential estimators $\{\phi_n^*\}$. These estimators are easier to calculate and have more fully understood asymptotic properties than the estimators introduced by Bather. In Section 2 sufficient conditions are given for (1.3) to imply (1.4). In Section 3 a sequential procedure of the stochastic approximation (SA) type is given. Not only do the resulting estimators satisfy (1.3) (and hence (1.4)), but using well-known theorems on SA we are able to establish rates of convergence for the estimators. In Section 4 the choice of several parameters needed in the algorithm is studied using Monte-Carlo techniques. Results of this section show that the algorithm behaves satisfactorily from a cost standpoint even in small samples. Section 5 contains the proofs of Section 3 results.

2. A preliminary result. Let X_i be the lifetime of the ith unit. Suppose that $\{X_i\}$ is an i.i.d. sequence with distribution function F having mean μ and finite variance σ^2. Assume that the experimenter has available at each stage a replacement time ϕ_i^*, so that if $X_i < \phi_i^*$, then the cost is C_1. Otherwise, the cost is C_2. Use $\{Z_i\}$ for the truncated observations, that is, $Z_i = \min\{X_i, \phi_i^*\}$. Thus, the cost for the first n units is

$$R_n = \sum_{i=1}^{n} \{C_1\mathbf{I}(Z_i < \phi_i^*) + C_2\mathbf{I}(Z_i \geq \phi_i^*)\}. \tag{2.1}$$

Using ideas of Bather (1977, Theorem 3), the following result shows that if ϕ_n^* estimates ϕ^* consistently, the best asymptotic cost is achieved.

THEOREM 2.1. *Let $\{X_n\}$ and $\{Z_n\}$ be as above and let $\mathbf{G}_n = \sigma(Z_1, \cdots, Z_n)$ be the sigma-field generated by $Z_1, \cdots, Z_n$. Suppose there exists a sequence of random variables $\{\phi_n^*\}$ such that ϕ_n^* is $\mathbf{G}_{n-1}$-measurable for $n \geq 2$ and* (1.3) *holds. Then, with R_n, $N(t)$ and $R(t)$ defined in* (2.1), (1.2) *and* (1.1), *respectively, we have*

$$\lim_{t\to\infty} R_{N(t)}/t = R(\phi^*) \quad a.s.$$

i.e., that (1.4) *holds.*

The proof of Theorem 2.1 follows from the application of martingale convergence theorems and can be found in Frees (1983, Theorem 1.1). It is easy to see that we may drop the assumption that F be absolutely continuous for Theorem 2.1. We only need require that ϕ^* be a continuity point of F. Further, the assumption of a finite variance can easily be weakened. However, these stronger assumptions are needed for the results in Section 3.

3. The sequential procedure and asymptotic results.

We now introduce a recursive estimation procedure. As with other stochastic approximation algorithms, its simple form makes it amenable both to practical implementation and to large sample calculations. It should be noted, however, that for this particular application of SA some of the intermediary estimators are complicated (e.g., (3.2) below).

Define the function

$$M(t) = (C_1 - C_2)f(t) \int_0^t S(u)\, du - S(t)\{C_1F(t) + C_2S(t)\}. \tag{3.1}$$

Now $\partial/\partial t R(t) = K_t M(t)$, where K_t is a positive function of t. Instead of assuming that $R(t)$ is uniquely minimized at some finite point ϕ^*, we use a slightly stronger but analytically more tractable assumption that $M(t)(t - \phi^*) > 0$ for each $t \neq \phi^*$. Since the function R is assumed to be differentiable, the latter assumption is equivalent to the assumption that $R(t)$ has no points of local relative minima. Let $g(\cdot)$ be a known, strictly increasing smooth function such that $g: \mathbf{R} \to [0, \infty)$. Define ϕ by $\phi^* = g(\phi)$. Note that ϕ is the unique minimum of $R(g(x))$ and thus the unique, finite zero of $g'(x)M(g(x))$ (where x may vary over the entire real line). Since unconstrained recursive estimation is particularly simple, we have introduced g and will estimate ϕ rather than the strictly positive parameter ϕ^*. Later we briefly discuss the issue of how one should choose g. After defining an estimate ϕ_n of ϕ, we will use the transform function g to calculate the estimate $\phi_n^* = g(\phi_n)$ of ϕ^*.

Let $\{X_{i,n}\}$, $i = 1, 2$, be two sequences of i.i.d. random variables that are mutually independent, each having distribution function F. Let $\mathbf{E}$ and $\mathbf{P}$ denote expectation and probability with respect to F. Suppose ϕ_1 is a random variable such that $\mathbf{E}\phi_1^2 < \infty$, and $\{\phi_n\}$, $\{a_n\}$ and $\{c_n\}$ are sequences of random variables. For $i = 1, 2$, define the truncated observations $\{Z_{in}\}$ by $Z_{in} = \min\{X_{in}, g(\phi_n + c_n)\}$. Let $\mathbf{F}_n = \sigma(\phi_1, Z_{ij}, i = 1, 2, j = 1, \cdots, n-1)$ and require that a_n and c_n be $\mathbf{F}_n$-measurable. In practice, we take $\{a_n\}$ and $\{c_n\}$ to be sequences such that for fixed A, $C > 0$ and $\gamma \in (0, 1)$, we have $a_n n \to A$ and $c_n n^\gamma \to C$.

Let B_0 be the class of all Borel-measurable real-valued functions $k(\cdot)$ where $k(\cdot)$ is bounded and equals zero outside $[-1, 1]$. For some positive integer r define

$$B_1 = \left(k \in B_0 \colon \int_{-1}^{1} y^j k(y)\, dy = \begin{cases} 1 & j = 0 \\ 0 & j = 1, \cdots, r-1 \end{cases}\right).$$

A class of kernel functions similar to B_1 is used by Singh (1977). See, for example, Wertz (1978), for a broad review on using kernel functions to estimate a probability density function.

Define $H(t) = F(g(t))$ and let $H^{(j)}$ be the jth partial derivative of H. For $i = 1, 2$, let $F_{in}(t) = I\{Z_{in} < t\}$ and $S_{in}(t) = 1 - F_{in}(t)$. In Lemma 5.2 below, we show that $h_n(t) = k[(g^{-1}(Z_{1n}) - t)/c_n]/c_n$ has desirable properties as an estimator of $h(t) = H^{(1)}(t)$. The estimator of $g'(t)M(g(t))$ is $M_{g,n}(t)$, where

$$M_{g,n}(t) = (C_1 - C_2)h_n(t) \int_0^{g(t)} S_{2n}(u)\, du \\ - g'(t)S_{1n}(g(t))\{C_1 F_{2n}(g(t)) + C_2 S_{2n}(g(t))\}. \tag{3.2}$$

The estimator in (3.2) is constructed so that the conditional expectation given $\mathbf{F}_n$ of $M_{g,n}(t)$ is sufficiently close to $g'(t)M(g(t))$. This proximity is made precise in Section 5. The estimators ϕ_n are constructed by the recursive algorithm,

$$\phi_{n+1} = \phi_n - a_n M_{g,n}(\phi_n). \tag{3.3}$$

Note that in (3.3) the updating of the estimator ϕ_n relies only on ϕ_n and the current observations Z_{1n} and Z_{2n}. While this simple form is desirable for practical implementation, it does raise fears about possible inefficiencies of the method. These fears are placated by the properties of the estimators stated in the theorems below. For convenience, a list of the most important assumptions is collected below.

A1. The distribution function F of the i.i.d. observations is absolutely continuous with density f, has support on $[0, \infty)$, finite mean μ and variance σ^2.

A2. Let g be a known, strictly increasing function such that $g\colon \mathbf{R} \to [0, \infty)$ and the first $r + 1$ derivatives exist and are bounded over the entire real line.

A3. For each $x \in \mathbf{R}$, $(x - \phi)M(g(x)) > 0$, $\forall x \neq \phi$.

A4. $H^{(1)}(x)$ and $H^{(r+1)}(x)$ exist for each x, are bounded over the entire real line and are continuous in a neighborhood of ϕ,

A5. Let $\lim c_n = 0$, $\sum_1^\infty a_n = \infty$, $\sum_1^\infty a_n c_n^r < \infty$, and $\sum_1^\infty a_n^2/c_n < \infty$.

A6. Let $\gamma = 1/(2r + 1)$ and $p = 2 + 1/r$. Assume that $\int_0^\infty t^p \, dF(t) < \infty$. For some $A, C > 0$, $a_n n \to A$, $c_n n^\gamma \to C$ and $1 - \gamma < 2\Gamma$, where $\Gamma = A(g'(\phi))^2 M'(g(\phi))$.
A7. $g^{(i)}(\phi) = 0$, $i = 2, \cdots, r + 1$.

REMARKS. A5 is a weaker condition than A6. A typical SA assumption, in general stronger than A3, that,

$$\inf\{|M(g(x))|: \varepsilon < |x - \phi| < \varepsilon^{-1}\} > 0 \quad \text{for each} \quad \varepsilon > 0,$$

is, in fact, implied by the continuity of $M(\cdot)$ and $g(\cdot)$. Note that assumption of a unique minimum (A3) and the smoothness assumptions on F and g (A2 and A4) obviate the need to take observations in the tail of F. Under these assumptions one can minimize long-run costs by sampling close to the point of interest ϕ^*. Assumption A7 requires that the transform function $g(\cdot)$ behave approximately like a line at ϕ. Under A7, the asymptotic distribution of the estimator of ϕ^* has a simple form, but otherwise we do not use this assumption.

Let $T = H^{(r+1)}(\phi) \int_{-1}^1 y^r/r! k(y) \, dy$, which is a factor in the asymptotic bias. The asymptotic variance of the ϕ_n will be proportional to Σ, where

$$\Sigma = (C_1 - C_2)^2 H^{(1)}(\phi) \int_0^{g(\phi)} uS(u) \, du.$$

We now state some asymptotic properties of our procedure.

THEOREM 3.1. *Assume* A1–A5. *Then, for the procedure defined in* (3.3),

(3.4) $$\phi_n \to \phi \quad a.s. \quad \textit{and thus}$$

(3.5) $$\phi_n^* = g(\phi_n) \to \phi^* \quad a.s.$$

THEOREM 3.2. *Assume* A1–A4 *and* A6. *Then, for the procedure defined in* (3.3),

(3.6) $$n^{(1-\gamma)/2}(\phi_n - \phi) \to_D N(\mu_1, \sigma_1^2)$$

where

$$\mu_1 = AC^r(C_1 - C_2) \int_0^{g(\phi)} S(u) \, du \; T/(2\Gamma - 1 + \gamma)$$

$$\sigma_1^2 = A^2 C^{-1} \Sigma \int_{-1}^1 k^2(y) \, dy/(2\Gamma - 1 + \gamma).$$

COROLLARY 3.3. *Under the assumptions of Theorem* 3.2 *and with* ϕ_n^* *defined in* (3.5),

(3.7) $$n^{(1-\gamma)/2}(\phi_n^* - \phi^*) \to_D N(\mu_2, \sigma_2^2)$$

where $\mu_2 = g'(\phi)\mu_1$ *and* $\sigma_2^2 = (g'(\phi))^2 \sigma_1^2$. *Further, assuming* A7, *we have* (3.7) *with*

(3.8)
$$\mu_2 = AC^r(C_1 - C_2) \int_0^{\phi^*} S(u)\, du\, f^{(r)}(\phi^*)$$
$$(g'(\phi))^{r+2} \int_{-1}^{1} y^r/r!k(y)\, dy/(2\Gamma - 1 + \gamma)$$

(3.9)
$$\sigma_2^2 = A^2C^{-1}(C_1 - C_2)^2 \int_0^{\phi^*} uS(u)\, du\, f(\phi^*)$$
$$(g'(\phi))^3 \int_{-1}^{1} k^2(y)\, du/(2\Gamma - 1 + \gamma).$$

Theorem 3.1 tells us that we may use the estimators constructed in (3.3) to achieve the best long-run cost. With some additional mild assumptions, in Theorem 3.2 we can quantify the speed of the convergence of the estimators of ϕ. Using the well-known "δ-method," Corollary 3.3 gives rates of convergence of the estimators of the optimal replacement time, ϕ^*. The choice of the transform function g does not affect the rate of convergence of the estimator ϕ_n^* but will, of course, affect the finite sample behavior of ϕ_n^*. While the best choice of g is not always clear, one criterion is available when the experimenter believes that ϕ^* falls in a specified, finite interval. Here, g may be taken to be a straight line over the interval subject to assumption A2. In this situation, or under the less restrictive assumption A7, the asymptotic distribution depends only on the slope of the transformation function at ϕ. Under A7, the asymptotic distribution of ϕ_n^* is left unchanged if $g'(\phi)$ is rescaled by a factor $K > 0$ and, simultaneouly, A is rescaled by K^{-2} and C is rescaled by K^{-1}. Thus, under A7, the choice of g may be subsumed under the question of choosing A and C. The parameters A and C may be chosen to be any positive constants, subject only to the restriction in assumption A6. One criterion for selection of parameters suggested by Abdelhamid (1973) is to choose A and C to minimize the asymptotic mean square error. Unfortunately, the best choice depends on knowledge of F and ϕ which are generally unknown a priori.

For a large class of distribution functions, we have constructed an estimator of the optimal replacement time that has desirable asymptotic properties. The following section shows that the estimator does well in some finite sample situations. However, there are at least three possible drawbacks in using the SA estimator. First, our results apply to a somewhat smaller class of distribution functions than the results of Bather. In particular, his estimator is strongly consistent when the cost function has a unique, finite global minimum and local relative minima. This situation is not considered under our stronger assumption A3. Second, a common criticism of stochastic approximation schemes is that the assumptions are difficult to check. Third, it may be argued that instead of constructing an estimator that converges in the quickest possible fashion a more important criterion is to construct an estimator so that the sample cost (2.1) converges to the optimal cost quickly. This is a different criterion which will be addressed in a later paper.

4. Monte-Carlo results. A Monte-Carlo study was undertaken to demonstrate the usefulness of the procedure proposed in Section 3 even in small samples. Further, the performance of the estimators at finite stages is improved dramatically by parameters that do not appear in the asymptotic theory.

For the model of the problem, we took $C_1 = 5$ and $C_2 = 1$. From (1.1), it can be seen that determining ϕ^* depends only on the ratio of the costs. The Weibull distribution was used with density

$$f(t) = \alpha\lambda(\lambda t)^{\alpha-1}\exp\{-(\lambda t)^{\alpha}\}$$

where $\alpha = 2$ and $\lambda = 2.2$. This produces a mean of 1.7712 and standard deviation of .8499 for the lifetime of the units. It also ensures a unique, finite $\phi^* = .99505$ and optimal cost $R(\phi^*) = 1.904$.

For the algorithm, let $r = 2$ which gives $\gamma = .2$ and use $k(y) = 1/2$ for $y \in [-1, 1]$. Thus, we used a very simple histogram estimator for the density. (Slightly better kernel estimators for the density when $r = 2$ are available, see Epanechnikov (1969) and Rosenblatt (1971).) Instead of using the simpler $a_n = An^{-1}$ and $c_n = Cn^{-\gamma}$, we used $a_n = A(n + k_A)^{-1}$ and $c_n = C(n + k_C)^{-\gamma}$, where k_A and k_C are nonnegative constants. Taking k_A and k_C to be positive provided dramatic improvements in finite samples over the more traditional $k_A = k_C = 0$. This form was first suggested by Dvoretsky (1956, equation (8.9)), and it has been employed by Ruppert et al. (1984). We used the transform function $g(x) = \log\{1 + \exp(x)\}$. Some easy calculations show that the values of A and C that minimize the asymptotic mean square error $(= \mu_2^2 + \sigma_2^2$, given in (3.8) and (3.9)) are $A = 2.3$ and $C = 1.5$. In this study we took these values to be fixed. For more complete tables where A and C are allowed to vary, see Frees (1983). In that study, the performance of the estimator was not sensitive to mildly different values of A and C but was best near the theoretical optimal values. The performance was slightly more sensitive to the parameter A than C. Too large a value of A caused large oscillations in the early stages which calmed down in the later stages when the asymptotics took over. For too small values of A, the estimator performed noticeably worse. Recall, to achieve convergence in distribution, we required in A6 that $A > (1 - \gamma)(2(g'(\phi))^2M'(g(\phi))) = .5834$.

Table 1 describes the performance of the estimator ϕ_n for various values of k_A, k_C and ϕ_1 (the starting value of the procedure). Stages at $n = 10$, 50 and 250 were chosen to reflect small, moderate and large sample sizes, respectively. Denote X_{ijk} to be the ith sample ($i = 1, 2$) at the jth stage ($j = 1, \cdots, 250$) from the kth trial ($k = 1, \cdots, 1000$). Let $\phi_{j,k}$ be the resulting estimator, $Z_{ijk} = \min\{X_{ijk}, g(\phi_{j,k} + C(j + k_C)^{-\gamma})\}$ and $\delta_{ijk} = \mathbf{I}\{Z_{ijk} < g(\phi_{j,k} + C(j + k_C)^{-\gamma})\}$. For the bias at the nth stage, use $\text{BIAS}_n = (.001) \sum_{k=1}^{1000} \phi_{n,k} - \phi$. Similarly, for the mean square error, use $\text{MSE}_n = (.001) \sum_{k=1}^{1000} (\phi_{n,k} - \phi)^2$. For the kth trial, the actual sample cost per unit time at the nth stage is

$$\text{SC}_{n,k} = \sum_{j=1}^{n} \{C_1(\delta_{1jk} + \delta_{2jk}) + C_2(2 - \delta_{1jk} - \delta_{2jk})\}/\sum_{j=1}^{n} (Z_{1jk} + Z_{2jk}).$$

The mean sample cost per unit time at the nth stage is $\text{MSC}_n = (.001) \cdot \sum_{k=1}^{1000} \text{SC}_{n,k}$. The sample variance for the sample cost per unit time at the nth stage is $\text{VSC}_n = (.001) \sum_{k=1}^{1000} (\text{SC}_{n,k} - \text{MSC}_n)^2$. While the asymptotic theory

TABLE 1
Performance of estimators

	ϕ_1	k_A	k_C	Stage of Algorithm 10	50	250	∞
BIAS_n	1.0	50	50	.3544	.1284	.0113	0
MSE_n				.2016	.1091	.0377	0
ASMSE_n				5.335	4.344	3.617	.8161
MSC_n				2.268	2.159	2.053	1.904
VSC_n				.1851	.0408	.0096	0
BIAS_n	1.0	0	0	.4053	.7246	.9354	
MSE_n				14.20	13.33	12.74	
ASMSE_n				89.60	304.8	1056.	
MSC_n				2.929	3.132	3.141	
VSC_n				1.283	2.812	3.263	
BIAS_n	1.0	0	50	−1.109	−1.112	−1.069	
MSE_n				34.89	34.16	33.56	
ASMSE_n				220.1	781.1	2781.	
MSC_n				4.900	13.18	43.44	
VSC_n				9.200	200.1	3452.	
BIAS_n	1.0	50	0	.3896	.1659	.0247	
MSE_n				.2030	.1082	.0374	
ASMSE_n				5.371	4.307	3.584	
MSC_n				2.472	2.261	2.091	
VSC_n				.1710	.0395	.0095	
BIAS_n	2.5	50	50	1.582	.7228	.1177	
MSE_n				2.640	.7360	.0649	
ASMSE_n				69.84	29.30	6.218	
MSC_n				2.745	2.522	2.210	
VSC_n				.1416	.0465	.0156	
BIAS_n	−1.0	50	50	−1.434	−1.132	−.4743	
MSE_n				2.058	1.289	.2473	
ASMSE_n				54.44	51.32	23.71	
MSC_n				2.256	2.136	1.970	
VSC_n				.1942	.0415	.0091	
BIAS_n	−2.0	50	50	−2.484	−2.333	−1.928	
MSE_n				6.171	5.443	3.718	
ASMSE_n				163.2	216.7	356.4	
MSC_n				4.334	4.169	3.648	
VSC_n				.1481	.0318	.0110	

(Theorem 3.2) indicates that $n^{.8}\text{MSE}_n = O_p(1)$, we found that an adjusted standardized mean square error $\text{ASMSE}_n = (n + k_A)^{.8}\text{MSE}_n$ (also $O_p(1)$) was more stable. Heuristically, in replacing An^{-1} with $A(n + k_A)^{-1}$, the procedure believes it is at the $(n + k_A)$th stage when only n iterations have been performed.

The results of the study indicate that the performance of the algorithm was greatly enhanced by the introduction of the parameter k_A and only somewhat by k_C. By (3.3), it can be seen that ϕ_n could fluctuate wildly for small n as compared to larger n. The introduction of positive k_A inhibits the fluctuation in finite samples without altering the asymptotic properties.

A practical upper bound to the asymptotic cost is a failure replacement policy, i.e., where the unit is never replaced prior to failure. The cost of this policy is easily seen from (1.1) by setting $t = \infty$. For our example, $R(\infty) = C_1/\mu = 2.823$. In each trial we achieved a lower expected cost, even by the tenth stage! The reduction was substantial in view of the fact that the best one could hope for is $R(\phi^*) = 1.904$.

In this example, since $\phi^* = .99505$ and $g(x) = \log\{1 + \exp(x)\}$, simple calculations show $\phi = .53349$. With a standard deviation of .8499, $\phi_1 = 1$ is not an unreasonable starting value for the algorithm. As is usual in SA schemes, starting far away from the optimal value will affect the bias and mean square error even for large n (=250). One happy note is that this adverse effect does not seem too severe on the expected cost. In fact, we seem to do even better by starting with a low starting value ($\phi_1 = -1$), an important practical point (but note that $g(-1) = .3133$, not so far from $\phi^* = .99505$).

5. Appendix. In this section, we first prove Theorem 3.1 and then Theorem 3.2. All relationships between random variables are meant to hold almost surely unless stated otherwise. We will use positive constants $K_1, K_2, \cdots$ in the inequalities. All random variables are defined on a fixed probability space (Ω, $\mathbf{F}$, $\mathbf{P}$). We begin by stating a martingale convergence result due to Robbins and Siegmund.

THEOREM 5.1 (Robbins-Siegmund, 1971, Theorem 1). *Let* $\mathbf{G}_n$ *be a nondecreasing sequence of sub* σ*-fields of* $\mathbf{F}$. *Suppose that* X_n, β_n, η_n *and* ζ_n *are nonnegative* $\mathbf{G}_n$*-measurable random variables such that*

$$\mathbf{E}_{\mathbf{G}_n} X_{n+1} \leq X_n(1 + \beta_n) + \eta_n - \zeta_n, \quad \text{for} \quad n = 1, 2, \cdots.$$

Then $\lim_{n\to\infty} X_n$ *exists and is finite and*

$$\sum \zeta_n < \infty \quad \text{on} \quad \{\sum \beta_n < \infty, \sum \eta_n < \infty\}.$$

Some additional notation will also be useful. Define

$$(5.1) \qquad \Delta_n = \mathbf{E}_{\mathbf{F}_n}\{M_{g,n}(\phi_n) - g'(\phi_n)M(g(\phi_n))\}$$

$$= (C_1 - C_2) \int_0^{g(\phi_n)} S(u)\, du\, \{E_{\mathbf{F}_n} h_n(\phi_n) - H^{(1)}(\phi_n)\}$$

$$(5.2) \qquad V_n = (c_n)^{1/2}\{M_{g,n}(\phi_n) - g'(\phi_n)M(g(\phi_n)) - \Delta_n\}.$$

A useful lemma which we use repeatedly is

LEMMA 5.2. *Assume* A1, A2 *and* A4. *Then, for* $\mathbf{F}_n$*-measurable* $x \le g(\phi_n + c_n)$,

$$\mathbf{E}_{\mathbf{F}_n} h_n(x) = H^{(1)}(x) + c_n^r \int_{-1}^{1} y^r/r!k(y)H^{(r+1)}(\eta_n(y))\, dy \tag{5.3}$$

where $|\eta_n(y) - x| \le c_n$.

PROOF. By a change of variables,

$$\begin{aligned}\mathbf{E}_{\mathbf{F}_n} h_n(x) &= \int k\{(g^{-1}(s) - x)/c_n\}/c_n f(s)\, ds \\ &= \int_{-1}^{1} k(y)g'(x + c_n y)f(g(x + c_n y))\, dy \\ &= \int_{-1}^{1} k(y)H^{(1)}(x + c_n y)\, dy.\end{aligned}$$

The result follows from a Taylor-series expansion and since $k \in B_1$. □

PROOF OF THEOREM 3.1. Using (5.2) in (3.3) gives,

$$\phi_{n+1} = \phi_n - a_n[g'(\phi_n)M(g(\phi_n)) + c_n^{-1/2}V_n + \Delta_n]. \tag{5.4}$$

Subtracting ϕ, squarirg and taking conditional expectations with respect to $\mathbf{F}_n$ gives,

$$\begin{aligned}\mathbf{E}_{\mathbf{F}_n}(\phi_{n+1} - \phi)^2 = (\phi_n - \phi)^2 - 2a_n(\phi_n - \phi)[g'(\phi_n)M(g(\phi_n)) + \Delta_n] \\ + a_n^2[(g'(\phi_n)M(g(\phi_n)) + \Delta_n)^2 + c_n^{-1}\mathbf{E}_{\mathbf{F}_n} V_n^2].\end{aligned} \tag{5.5}$$

Let $b_{n,1} = 2a_n |\Delta_n|$ and $b_{n,2} = a_n^2 c_n^{-1}\mathbf{E}_{\mathbf{F}_n} V_n^2$. Suppose

$$\sum b_{n,1} < \infty \quad \text{a.s.} \tag{5.6}$$

$$\sum b_{n,2} < \infty \quad \text{a.s.} \tag{5.7}$$

From A1, $\int_0^t S(u)\, du \le \int_0^\infty S(u)\, du = \mu < \infty$. Thus, by A2 and A4, $|g'(t)M(g(t))|$ is bounded, say, by K_1. Using the inequality $x \le 1 + x^2$ and (5.5)–(5.7), we get

$$\begin{aligned}\mathbf{E}_{\mathbf{F}_n}(\phi_{n+1} - \phi)^2 \le (\phi_n - \phi)^2(1 + b_{n,1}) - 2a_n(\phi_n - \phi)g'(\phi_n)M(g(\phi_n)) \\ + b_{n,1} + \tfrac{1}{2}b_{n,1}^2 + 2K_1^2 a_n^2 + b_{n,2}.\end{aligned} \tag{5.8}$$

By Theorem 5.1 and A5, we get that $\lim_{n\to\infty}\phi_n - \phi = X$ a.s. and $\sum a_n(\phi_n - \phi)g'(\phi_n)M(g(\phi_n)) < \infty$ a.s. This and A3 give the result. We need only show (5.6) and (5.7). By (5.1), Lemma 5.2 and A4, we have

$$\Delta_n = O(c_n^r). \tag{5.9}$$

This and A5 prove (5.6). From (3.2), for some $K_2, K_3 \geq 0$,

(5.10) $$\mathbf{E}_{\mathbf{F}_n}(M_{g,n}(\phi_n))^2 \leq K_2 + K_3 \mathbf{E}_{\mathbf{F}_n}(h_n(\phi_n))^2.$$

As in Lemma 5.2, we can show $\mathbf{E}_{\mathbf{F}_n}(h_n(\phi_n))^2 = O(c_n^{-1})$. This, the boundedness of $g'(t)M(g(t))$ and (5.9) prove (5.7) and hence the result. □

To prove Theorem 3.2, we use a special case of a theorem due to Fabian.

THEOREM 5.3 (Fabian, 1968, Theorem 2.2). *Suppose $\mathbf{G}_n$ is a nondecreasing sequence of sub σ-fields of $\mathbf{F}$. Suppose U_n, V_n, T_n, Γ_n, and Φ_n are random variables such that Γ_n, Φ_{n-1}, V_{n-1} are $\mathbf{G}_n$-measurable. Let β, T', Σ, Γ, and Φ be real constants with $\Gamma > 0$ such that*

(5.11) $$\Gamma_n \to \Gamma, \quad \Phi_n \to \Phi, \quad T_n \to T' \quad \textit{or} \quad \mathbf{E}\,|\,T_n - T'\,| \to 0, \quad \mathbf{E}_{\mathbf{G}_n} V_n = 0$$

and

(5.12) $$C > |\,\mathbf{E}_{\mathbf{G}_n} V_n^2 - \Sigma\,| \to 0.$$

Suppose, with $\sigma_{j,r}^2 = \mathbf{E}\mathbf{I}[V_j^2 \geq rj\,]V_j^2$, that

(5.13) $$\lim n^{-1} \textstyle\sum_1^n \sigma_{j,r}^2 = 0 \quad \forall r.$$

Let $0 \leq \beta < 2\Gamma$ and

(5.14) $$U_{n+1} = U_n[1 - n^{-1}\Gamma_n] - n^{-(1+\beta)/2}\Phi_n V_n + n^{-1-\beta/2} T_n.$$

Then,

$$n^{\beta/2} U_n \to_D N(T'/(\Gamma - \beta/2), \Sigma\Phi^2/(2\Gamma - \beta)).$$

PROOF OF THEOREM 3.2. By a Taylor-series expansion,

$$g'(\phi_n)M(g(\phi_n)) = (\phi_n - \phi)\{g''(\eta_n)M(g(\eta_n)) + (g'(\eta_n))^2 M'(g(\eta_n))\}$$

for some η_n such that $|\,\eta_n - \phi\,| \leq |\,\phi_n - \phi\,|$. This and (5.4) give

(5.15) $$\phi_{n+1} - \phi = (\phi_n - \phi)(1 - n^{-1}\Gamma_n) + n^{-1+\gamma/2}\Phi_n V_n + n^{-3/2+\gamma/2} T_n$$

where

$$\Gamma_n = a_n n\{g''(\eta_n)M(g(\eta_n)) + (g'(\eta_n))^2 M'(g(\eta_n))\}$$

$$\Phi_n = a_n c_n^{-1/2} n^{1-\gamma/2}$$

$$T_n = a_n n^{3/2-\gamma/2}\Delta_n.$$

By Theorem 3.1 and A6, $\Gamma_n \to \Gamma$. By A6, $\Phi_n \to \Phi = AC^{-1/2}$. By Theorem 3.1, Lemma 5.2 and (5.1),

(5.16) $$T_n \to TAC^r(C_1 - C_2)\int_0^{g(\phi)} S(u)\,du = T'.$$

Since $\mathbf{E}_{\mathbf{F}_n} V_n = 0$ by the definition of Δ_n, we have (5.11).

To prove (5.12), we first recall the boundedness of $g'(t)M(g(t))$ and (5.9).

Thus, we need only show for some K_4,

$$K_4 > | c_n \mathbf{E}_{\mathbf{F}_n}\{M_{g,n}(\phi_n)\}^2 - \Sigma | \to 0. \tag{5.17}$$

From (3.2), A2 and (5.17), we need only show for some K_5,

$$K_5 > \left| (C_1 - C_2)^2 c_n \mathbf{E}_{\mathbf{F}_n}\left\{ h_n(\phi_n) \int_0^{g(\phi_n)} S_{2n}(u)\, du \right\}^2 - \Sigma \right| \to 0. \tag{5.18}$$

By construction, $h_n(\cdot)$ and $S_{2n}(\cdot)$ are conditionally independent given $\mathbf{F}_n$. From Theorem 3.1, it is easy to show that

$$\infty > \mathbf{E}_{\mathbf{F}_n}\left(\int_0^{g(\phi_n)} S_{2n}(u)\, du \right)^2 \to 2 \int_0^{g(\phi)} uS(u)\, du \quad \text{a.s.} \tag{5.19}$$

Further,

$$c_n \mathbf{E}_{\mathbf{F}_n}(h_n(\phi_n))^2 = \frac{1}{2} \int_{-1}^{1} k^2(y) H^{(1)}(\phi_n + c_n y)\, dy$$

is bounded. This gives the boundedness of $\mathbf{E}_{\mathbf{F}_n} V_n^2$ and with (5.19) proves (5.18) and hence (5.12).

To prove (5.13), and hence the result, we need only show that

$$\sigma_{n,r}^2 = \mathbf{E}[V_n^2 \mathbf{I}[V_n^2 \geq rn]] \to 0 \quad \text{for each } r. \tag{5.20}$$

Suppose that for the p in A6

$$\mathbf{E}(c_n V_n^2)^{p/2} = o(1) \quad \text{as} \quad n \to \infty. \tag{5.21}$$

Let q be defined by $2/p + 1/q = 1$. Then, by Holder's and Markov's inequalities, we get

$$\begin{aligned} \sigma_{n,r}^2 &\leq (\mathbf{P}\{V_n^2 \geq rn\})^{1/q} (\mathbf{E}\{(c_n V_n^2)^{p/2}\})^{2/p}/c_n \\ &\leq (\mathbf{E}\{c_n V_n^2\}^{p/2}/\{rnc_n\}^{p/2})^{1/q} o(1)/c_n \\ &= o(1)(rn)^{-p/(2q)}(c_n)^{-p/(2q)-1} \\ &= o(1) n^{-p/(2q)+\gamma(1+p/(2q))} = o(1). \end{aligned}$$

Thus, to show (5.20), we need only prove (5.21). Recall the algebraic inequality for nonnegative constants a, b, c and d $(a + b + c)^d \leq 3^d(a^d + b^d + c^d)$. From (5.2),

$$(c_n V_n^2)^{p/2} \leq 3^p c_n^p \{ |M_{g,n}(\phi_n)|^p + |g'(\phi_n) M(g(\phi_n))|^p + |\Delta_n|^p \}. \tag{5.22}$$

As before, both $g'(\phi_n)M(g(\phi_n))$ and Δ_n are bounded. From (3.2), so is $c_n | M_{g,n}(\phi_n) |$. Further, from (5.22),

$$\mathbf{E}_{\mathbf{F}_n}(c_n V_n^2)^{p/2} \leq \mathbf{E}_{\mathbf{F}_n} | c_n M_{g,n}(\phi_n) |^p + o(1). \tag{5.23}$$

Since $\mathbf{E}_{\mathbf{F}_n} | c_n h_n(\phi_n) |^p = o(1)$, from (5.23) and (3.2) we have

$$\mathbf{E}_{\mathbf{F}_n}(c_n V_n^2)^{p/2} = o(1). \tag{5.24}$$

(5.21) follows immediately from the Bounded Convergence Theorem. □

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