SEQUENTIAL TEST FOR THE MEAN OF A NORMAL DISTRIBUTION III (SMALL t)

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1. Summary. Asymptotic expansions are derived for the behavior of the optimal sequential test of whether the unknown drift μ of a Wiener-Lévy process is positive or negative for the case where the process has been observed for a short time. The test is optimal in the sense that it is the Bayes test for the problem where we have an *a priori* normal distribution of μ, the regret for coming to the wrong conclusion is proportional to $|\mu|$ and the cost of observation is one per unit time. The Bayes procedure is compared with the best sequential likelihood ratio test and with the procedure which calls for stopping when no fixed additional sampling time is better than stopping. The derivations allow for generalizing to variations of this problem with different cost structure.

2. Introduction. In the Fourth Berkeley Symposium on Probability and Statistics Chernoff [3] indicated that the problem of sequentially testing whether the mean drift of a Wiener-Lévy process is positive or negative, given a normal *a priori* probability distribution, was relevant to the problem of deriving an asymptotically (as sampling cost approaches zero) optimal sequential test of whether the mean of a normally distributed variable is positive or negative. (The relationship between the two problems is examined in detail in [4].) The former problem was reduced to the solution of a free boundary problem involving a diffusion equation.

Subsequently Moriguti and Robbins [8] and Bather [1] derived asymptotic expansions for the optimal procedure and Bayes risk for time t large, i.e., for the case where the process has been observed for a long time. Breakwell and Chernoff [2] also derived these expansions and proved that they were valid in the sense of yielding asymptotic approximations to the desired optimum boundary and the corresponding risk. In Section 3 of the latter paper there is a brief review of the results and notation of [3], which is urgently recommended to the reader of the present paper. The case where $t \to 0$ is studied in this paper. Although this corresponds to the beginning of sampling in the normalized problem, it is relevant

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In honor of my teacher, colleague, and friend *Professor Charles Loewner* on his 70th birthday.

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to large sample situations when the cost of sampling is small. In fact, suppose that in a realistic version of the problem the scales are such that each observation has variance 1, the cost of making the wrong decision is roughly $10^3|\mu|$, the *a priori* probability of μ has variance 1 and the cost per observation is 10^{-3}. Then, in the normalized problem the starting point is set at (μ_0, t_0) where $t_0 = 10^{-4}$ and the discrete problem involves time steps of length $\delta = 10^{-4}$. Then after a substantial number of observations one will still be in a case of t small.

In Section 3, a formal derivation will be presented of expansions for the free boundary and the Bayes risk for the optimal solution. This derivation involves a special solution of the heat equation. The motivation for the use of this solution is briefly indicated in subsequent sections but can be bypassed and is no longer required. In Section 4 alternative expansions are presented. These have the advantage of being easier to compute and somewhat more elegant.

The problem of proving that these expansions do represent asymptotic approximations to the solution for small t is then attacked. In Section 5 it is shown that the expansion for the boundary is an asymptotic approximation. In Section 6 the same is done for the Bayes risk. The latter proof involves a representation of solutions of the heat equation which appears in Goursat [7]. This representation is particularly well adapted to our free boundary problem for it involves the value of the solution and its partial derivatives on the boundary both of which are described by our boundary conditions. The proofs in these two sections have been complicated substantially by the requirement that they be adaptable to variations of the stated problem. Simpler proofs could be presented which apply the symmetry of the special stated problem.

Section 7 is devoted to methods of deriving bounds on the behavior of the boundary and risk. This section serves two purposes. First it indicates some of the motivation which led to the formal expansions and the arguments used in preceding sections. Second, we describe the boundary and risk for the procedure which consists of stopping at a point where there is no fixed sampling time which will do better than stopping. This procedure arises naturally in this section because it leads to stopping early, thereby providing useful bounds. At the same time it is relatively easy to evaluate.

In Section 8 we compare the optimal procedure with the best procedure of the Wald type when the *a priori* probability distribution of μ has large variance. In Section 9 we decompose the Bayes risk of the optimal procedure into a Sampling risk and risk due to Error. Finally in Section 10, we discuss briefly the important question of the operating characteristics of the optimal procedure. That is to say we comment on the expected sample size and error probability corresponding to a specified μ when the Bayes procedure is applied. These characteristics are fundamental to a proper evaluation of the procedure for cases where the *a priori* normal assumption is not necessarily valid.

To facilitate reference we terminate this introduction by supplementing the reference to [2], Section 3, by a brief outline of some of the notation we shall use. The statistician observes $X_t = \mu t + Z_t$ where Z_t is a Wiener-Lévy process of independent Gaussian increments with mean zero and variance one per unit time. The *a priori* probability distribution of μ and a starting point (x_0, t_0) are chosen so that given $X_t = x$, the *a posteriori* probability distribution of μ is normal with mean xt^{-1} and variance t^{-1}, i.e., $\mathfrak{N}(xt^{-1}, t^{-1})$. Then we represent the risk due to additional sampling and error probability by $B(x, t)$ and the risk due to stopping by $D(x, t)$. In our problem $D(x, t) = t^{-\frac{1}{2}}\psi(xt^{-\frac{1}{2}})$ where $\psi(\alpha)$ is symmetric and $\psi(\alpha) = \varphi(\alpha) - \alpha[1 - \Phi(\alpha)]$ for $\alpha > 0$, and φ and Φ are the normal density and cdf respectively. For a given procedure represented by a continuation region, B satisfies the equation $1 + B_t + xt^{-1}B_x + \frac{1}{2}B_{xx} = 0$ subject to the restriction $B = D$ on the boundary. For the optimal procedure $B_x = D_x$ on the boundary. We shall find it convenient to transform to $y = xt^{-1}$ and $s = t^{-1}$. In these variables the differential equation and boundary conditions for $u(y, s) = t + B(x, t)$ are

$$
\begin{aligned}
u_s &= \tfrac{1}{2}u_{yy} && \\
u &= s^{-1} + s^{\frac{1}{2}}\psi(ys^{-\frac{1}{2}}) && \text{on the boundary} \\
u_y &= -[1 - \Phi(ys^{-\frac{1}{2}})] && \text{on the boundary.}
\end{aligned} \tag{2.1}
$$

The variable $\alpha = ys^{-\frac{1}{2}} = xt^{-\frac{1}{2}}$ will be used throughout the paper.

3. A formal expansion. To derive our formal expansion we shall consider solutions of the heat equation of a special form and apply the two boundary conditions alternately to obtain successive terms in the expansions for the boundary and optimal risk. The proof of the validity of these expansions is given in Sections 5 and 6, and some remarks on the motivation appear in those sections and Section 7. The special form of solution of the heat equation is

$$
u = Ks^{-\frac{1}{2}}\varphi(\alpha) + 2\varphi(\alpha)\int_0^\infty e^{-b^2/2}[\cosh \alpha b - 1]f(s^{\frac{1}{2}}b)\, db, \tag{3.1}
$$

while the boundary will be represented by an expansion of the form

$$
\beta = \log s = (\alpha^2/3) + k_0 + k_1\alpha^{-2} + k_2\alpha^{-4} + \cdots = (\alpha^2/3) + k_0 + \eta. \tag{3.2}
$$

First we indicate that u is a solution of the heat equation. It is easy to see that $s^{-\frac{1}{2}}\varphi(s^{-\frac{1}{2}}y)$ is a solution for $s > 0$. Similarly $\int s^{-\frac{1}{2}}\varphi[s^{-\frac{1}{2}}(y - y')]\, dF(y')$, which is a heat potential with sources distributed along the y axis is a solution provided the integral converges. A mass of K for dF at $y' = 0$ provides the $Ks^{-\frac{1}{2}}\varphi(\alpha)$ term of the proposed solution. Elsewhere let dF/dy' be represented by the symmetric density $f(y')$ which is compensated for by the mass $-\int f(y')\, dy'$ at $y' = 0$. Setting $y' = s^{\frac{1}{2}}b$, this process yields

$$
\begin{aligned}
g(\alpha, s) &= \int_0^\infty [\varphi(\alpha - b) + \varphi(\alpha + b) - 2\varphi(\alpha)]f(s^{\frac{1}{2}}b)\, db \\
&= 2\varphi(\alpha)\int_0^\infty e^{-b^2/2}[\cosh \alpha b - 1]f(s^{\frac{1}{2}}b)\, db.
\end{aligned} \tag{3.3}
$$

To assure convergence to a solution some regularity conditions are required. For the above representation it suffices that (i) f be measurable while $f(y)\exp(-cy^2) \to 0$ as $y \to \infty$ for each $c > 0$ and (ii) $y^2f(y)$ be absolutely integrable in every finite interval. Thus, as $y \to 0$, $f(y)$ may approach infinity like $y^{-2} \log y$ without causing difficulty.

Now we note that along the boundary, the term $Ks^{-\frac{1}{2}}\varphi(\alpha)$ is relatively negligible. To be more specific, along the boundary represented by (3.2), $\alpha = s^{-\frac{1}{2}}y = [3 \log s + O(1)]^{\frac{1}{2}}$. Then $s^{-\frac{1}{2}}\varphi(\alpha) \sim s^{-2}$ and its partial derivative with respect to y, $-s^{-1}\alpha\varphi(\alpha) \sim s^{-\frac{5}{2}}(\log s)^{\frac{1}{2}}$. (We use the symbol $\sim$ so that $f \sim g$ means that f/g is bounded away from 0 and ∞. The symbol $\approx$ is used for formal expansions. It is also used in the asymptotically equivalent sense. That is $f \approx g_1 + \cdots + g_r$ means that $[f - (g_1 + \cdots + g_r)]/g_r \to 0$.) On the other hand $s^{-1} + s^{\frac{1}{2}}\psi(\alpha) \sim s^{-1}$ and $1 - \Phi(\alpha) \sim s^{-\frac{3}{2}}(\log s)^{-\frac{1}{2}}$ along the boundary. From this it will follow that at the boundary the main part of u and u_y is contributed by g.

The function g may be approximated for large α by use of a simple argument similar to that used in the method of steepest descent. For large α, we have the formal expansions

$$g(\alpha, s) \approx E^* \{f[s^{\frac{1}{2}}(\alpha + \epsilon)]\} = f(s^{\frac{1}{2}} \alpha) + \frac{s}{2!} f^{(2)}(s^{\frac{1}{2}} \alpha) + \frac{1 \cdot 3}{4!} s^2 f^{(4)}(s^{\frac{1}{2}} \alpha) + \frac{1 \cdot 3 \cdot 5}{6!} s^3 f^{(6)}(s^{\frac{1}{2}} \alpha) + \cdots \tag{3.4a}$$

and

$$g_\alpha(\alpha, s) \approx E^* \{\epsilon f[s^{\frac{1}{2}}(\alpha + \epsilon)]\} = s^{\frac{1}{2}} f^{(1)}(s^{\frac{1}{2}} \alpha) + \frac{1 \cdot 3}{3!} s^{\frac{3}{2}} f^{(3)}(s^{\frac{1}{2}} \alpha) + \frac{1 \cdot 3 \cdot 5}{5!} s^{\frac{5}{2}} f^{(5)}(s^{\frac{1}{2}} \alpha) + \cdots \tag{3.4b}$$

where ϵ is regarded as normally distributed with mean 0 and variance 1 and E^* is the expectation operator applied after the operand has been expanded in a power series in ϵ. These expansions are asymptotic approximations if the derivatives of f are sufficiently well behaved.

Neglecting the $Ks^{-\frac{1}{2}}\varphi(\alpha)$ term, the boundary conditions of (2.1) transform to

$$E^*\{f[s^{\frac{1}{2}}(\alpha + \epsilon)]\} = s^{-1} + s^{\frac{1}{2}}\psi(\alpha) \tag{3.5a}$$

$$E^*\{\epsilon f[s^{\frac{1}{2}}(\alpha + \epsilon)]\} = -s^{\frac{1}{2}}[1 - \Phi(\alpha)]. \tag{3.5b}$$

Substituting (3.2) in the right hand side of (3.5) and using the expansions ([6], p. 179)

$$1 - \Phi(\alpha) \approx \varphi(\alpha)\{\alpha^{-1} - \alpha^{-3} + (1 \cdot 3)\alpha^{-5} - 1 \cdot 3 \cdot 5\alpha^{-7} + \cdots\} \tag{3.6a}$$

$$\psi(\alpha) \approx \varphi(\alpha)\{\alpha^{-2} - 1 \cdot 3\alpha^{-4} + 1 \cdot 3 \cdot 5\alpha^{-6} - \cdots\}, \tag{3.6b}$$

we have the boundary conditions

$$E^*\{f[s^{\frac{1}{2}}(\alpha + \epsilon)]\} \approx s^{-1}\{1 + (2\pi)^{-\frac{1}{2}}e^{3k_0/2}e^{3\eta/2}[\alpha^{-2} - 3\alpha^{-4} + 15\alpha^{-6} - \cdots]\} \tag{3.7a}$$

$$E^*\{\epsilon f[s^{\frac{1}{2}}(\alpha + \epsilon)]\} \approx -s^{-1}(2\pi)^{-\frac{1}{2}}e^{3k_0/2}e^{3\eta/2}[\alpha^{-1} - \alpha^{-3} + 3\alpha^{-5} - 15\alpha^{-7} + \cdots]. \tag{3.7b}$$

Let us compute a few terms in the apparently natural expansions of f and β which satisfy (3.7) in order to see how the other terms are derivable. Starting with the approximation

$$f_0(x) = 6x^{-2} \log x = 3x^{-2} \log x^2, \tag{3.8a}$$

the s^{-1} terms match in equation (3.7a) and equation (3.7b) leads to

$$(2\pi)^{-\frac{1}{2}} \exp (3k_0/2) = 2$$

or

$$k_0 = (\log 8\pi)/3. \tag{3.8b}$$

We apply the resulting approximation

$$\beta \approx \beta_0 = (\alpha^2/3) + (\log 8\pi/3) \tag{3.9}$$

to (3.7a), substituting $s\alpha^2$ for x^2 in the argument of f and we obtain a discrepancy (right side of (3.7a) minus left side] which is

$$-(s\alpha^2)^{-1}\{3 \log \alpha^2 + \log 8\pi + 1 + O^*(\alpha^{-2})\} \tag{3.10}$$

where $O^*(\alpha^{-2r})$ is used to represent an expression which is bounded by some power of $\log \alpha^2$ divided by α^{2r} as $\alpha \to \infty$. To compensate for this discrepancy we apply a correction to f_0 making use of the fact that if $x^2 \approx s\alpha^2$, $\log [3 \log x^2] \approx \log \alpha^2$ since $\log s = \beta \approx \alpha^2/3$. This gives

$$f_1(x) = x^{-2}\{3 \log x^2 - 3 \log [\log x^2] - 3 \log 3 - \log 8\pi - 1\}. \tag{3.11}$$

This approximation combined with (3.9) yields a discrepancy in (3.7b) which is

$$2(s\alpha)^{-1}\{3\alpha^{-2} + O^*(\alpha^{-4})\} \tag{3.12}$$

and the main part of this can be compensated for by the approximation

$$\beta = (\alpha^2/3) + (\log 8\pi/3) + 2\alpha^{-2}. \tag{3.13}$$

We are now in a position to state and prove the following theorem about the existence of a formal expansion.

THEOREM 3.1. *There are expansions of the form*

$$\beta = \log s \approx (\alpha^2/3) + (\log 8\pi/3) + K_1\alpha^{-2} + K_2\alpha^{-4} + \cdots \tag{3.14a}$$
$$= (\alpha^2/3) + (\log 8\pi/3) + \eta$$

and

$$f(x) \approx 3x^{-2}(\log x^2)\{1 + P_1(\log x^2)^{-1} + P_2(\log x^2)^{-2} + \cdots\} \tag{3.14b}$$

where the K_r *are polynomials in* $\log \alpha^2$ *and the* P_r *are polynomials in* $\log [\log x^2]$ *with the following property. If* β_r *and* f_r *represent the above expressions with the sums terminated with* $K_r\alpha^{-2r}$ *and* $P_r(\log x^2)^{-r}$, *substitution in equations* (3.5a) *and* (3.5b) *leave discrepancies of* $s^{-1}O^*(\alpha^{-2-2r})$ *and* $s^{-1}O^*(\alpha^{-3-2r})$ *respectively.*

PROOF. First we note that substituting β_r and f_r in both sides of (3.7a) and (3.7b) leads to asymptotic expansions. Each side of (3.7a) can be represented by an expansion of the form $s^{-1}\{1 + Q_1\alpha^{-2} + Q_2\alpha^{-4} + \cdots\}$ where Q_i is a *generic* polynomial in $\log \alpha^2$. Each side of (3.7b) can be represented by an expansion of the form $-2(s\alpha)^{-1}\{1 + Q_1\alpha^{-2} + Q_2\alpha^{-4} + \cdots\}$. Of course the generic polynomials Q_i are not to be assumed to be equal to one another.

Let us assume that the theorem holds for a certain value of $r \geqq 0$. As we have already seen, it does hold for $r = 0$ and 1. We observe that adding $x^{-2}(3 \log x^2)^{-r} \cdot P\{\log (3 \log x^2)\}$ to f_r has the effect of increasing the left side of (3.7a) by $s^{-1}\alpha^{-2-2r}P\{\log \alpha^2\}\{1 + O^*(\alpha^{-2})\}$. Furthermore it has the effect of changing the left side of (3.7b) by $s^{-1}O^*(\alpha^{-3-2r})$. It does not affect the right hand sides of these equations. Since the theorem applies for r, the first r Q's match on both sides for the expansions of (3.7a) and (3.7b). By letting P be the discrepancy of the two Q_{r+1} for equation (3.7a) the resulting f which we call f_{r+1} reduces the discrepancy in that equation to $s^{-1}O^*(\alpha^{-4-2r})$. Equation (3.7b) still has a discrepancy of $s^{-1}O^*(\alpha^{-3-2r})$.

Now we observe that adding $K(\log \alpha^2)\alpha^{-2r-2}$ to β_r does not change the first $(r + 1)$ Q's of the left hand sides of equations (3.7). Basically this is so because the change in β is equal to $\beta_r\ O^*(\alpha^{-4-2r})$, the change in $\log s\alpha^2$ is equal to the change in β, and the change in $\log [\log s\alpha^2]$ is $O^*(\alpha^{-4-2r})$. On the other hand the right hand sides are changed by $3s^{-1}\alpha^{-4-2r}K(\log \alpha^2)[1 + O^*(\alpha^{-2})]$ and $-3s^{-1}\alpha^{-3-2r} \cdot K(\log \alpha^2)[1 + O^*(\alpha^{-2})]$ respectively. Thus by setting K equal to the difference of the Q_{r+1} on the two sides of equation (3.7b) we can match the first $(r + 1)$ Q's of both sides of this equation without affecting the corresponding Q's of (3.7a). Hence the result holds for $r + 1$, and the theorem follows.

It should be noted that f_r is not defined for all x. If we replace x^2 by $x^2 + 2$ in the expression (3.14b) then $\log [\log x^2]$ and hence the new f_r are defined for all x. All of the asymptotic expansions of this section are unaffected. Furthermore the integral (3.3) is now defined when f is replaced by the new f_r.

In Section 5 we shall study the corresponding g. We remark here merely that while g is large compared to $Ks^{-\frac{1}{2}}\varphi(\alpha)$ near the boundary, it is relatively small, $O(s^{-1} \log s)$, for bounded α.

4. An alternative formal expansion. An alternative method of obtaining a formal expansion is somewhat easier for computational purposes. Essential use is made of the function $f_0^*(x)$ which is defined so that $f_0^*(x) = s_1^{-1}$ when $x = [s_1(3 \log s_1 - \log 8\pi)]^{\frac{1}{2}}$. This function was selected because $s^{-1} + s^{\frac{1}{2}}\psi(\alpha) \approx s^{-1}$ along the boundary and, using the approximation β_0 of the previous section, i.e., $\log s \approx (\alpha^2/3) + \frac{1}{3} \log 8\pi$, we have

$$s^{\frac{1}{2}}\alpha \approx s^{\frac{1}{2}}[3 \log s - \log 8\pi]^{\frac{1}{2}}$$

and hence

$$E^*\{f_0^*[s^{\frac{1}{2}}(\alpha + \epsilon)]\} \approx f_0^*(s^{\frac{1}{2}}\alpha) \approx s^{-1}.$$

A tedious computation indicates that an analogue of Theorem 3.1 applies using expansions of the form

$$f(x) = f_0^*(x) + c_1x^{-2} + c_2x^{-4}[f_0^*(x)]^{-1} + c_3x^{-6}[f_0^*(x)]^{-2} + \cdots \tag{4.2a}$$

and

$$\alpha^2 = 3w[1 + k_1w^{-2} + k_2w^{-3} + \cdots] \tag{4.2b}$$

where the c_i and k_i are constants and

$$w = \log s - \tfrac{1}{3}\log 8\pi \tag{4.2c}$$

corresponds to the approximation for $\alpha^2/3$ based on β_0. The constants thus far evaluated are

$$\begin{aligned} &c_1 = -1, \quad c_2 = -3, \quad c_3 = \frac{21}{2}, \\ &c_4 = \frac{-3(193)}{2}, \quad c_5 = \frac{9(179)}{4}, \quad c_6 = \frac{-27(28{,}937)}{20} \end{aligned} \tag{4.3a}$$

and

$$\begin{aligned} &k_1 = \frac{-2}{3}, \quad k_2 = \frac{-5}{9}, \quad k_3 = \frac{-22}{9}, \\ &k_4 = \frac{-427}{54}, \quad k_5 = \frac{-16{,}184}{405}, \quad k_6 = \frac{-17{,}152}{81}. \end{aligned} \tag{4.3b}$$

The following consideration is basic to the convenient computation of the above constants. If

$$\begin{aligned} z &= -\log f_0^*(x) - \tfrac{1}{3}\log 8\pi, \\ z + \log z &= \log x^2/3 - \tfrac{1}{3}\log 8\pi, \\ dz/dx &= (1 + z^{-1})^{-1}(2/x), \end{aligned}$$

and the derivative of $[f_0^*(x)]^{-r_1}x^{-r_2}z^{-r_3}$ is $[f_0^*(x)]^{-r_1}x^{-(r_2+1)}$ multiplied by a series in descending powers of z.

5. Asymptotic approximation to the optimal boundary. The object of this section is to establish that the formal expansions of Sections 3 and 4 yield asymptotic approximations to the optimal boundary. The proof will be preceded by several lemmas that detail properties of the function g defined by

$$g(\alpha, s) = \int_0^\infty [\varphi(\alpha - b) + \varphi(\alpha + b) - 2\varphi(\alpha)]f(s^{\frac{1}{2}}b)\, db \tag{5.1}$$

where

$$\begin{aligned} f(x) = {} & 3(x^2 + 2)^{-1}\log(x^2 + 2) \\ & \cdot\{1 + [\log(x^2 + 2)]^{-1}P_1 + \cdots + [\log(x^2 + 2)]^{-r}P_r\} \end{aligned} \tag{5.2}$$

and the P_i are polynomials in log [log $(x^2 + 2)$]. Several related quantities which are studied are

$$(5.3) \qquad \Delta_K(\alpha, s) = g(\alpha, s) + Ks^{-\frac{1}{2}}\varphi(\alpha) - s^{-1} - s^{\frac{1}{2}}\psi(\alpha),$$

and $\hat{\alpha}_K(s)$, the positive value of α, if any, for which $\Delta_K(\alpha, s)$ is maximized. Let $\theta_K(s)$ be the maximum value. Finally let $\alpha_K{}^*(s)$ be the greatest value of α, if any, for which $0 \leqq \alpha \leqq \hat{\alpha}_K(s)$ and $\Delta_K(\alpha, s) = 0$.

LEMMA 5.1 (a) *As* $s \to \infty$,

$$(5.4) \qquad \sup_\alpha \left|\frac{\partial^i g(\alpha, s)}{\partial \alpha^i}\right| = O(s^{-1} \log s) \quad \textit{for each integer} \quad i = 0, 1, 2, \cdots .$$

(b) *For fixed* s *and large* α, *and each integer* $i = 0, 1, 2, \cdots$

$$(5.5) \qquad \frac{\partial^i g(\alpha, s)}{\partial \alpha_i} \approx (-1)^i\, 3(i + 1)!\, (s\alpha^{2+i})^{-1} \log (s\alpha^2).$$

(c) *Equation* (5.5) *also holds for large* s when α *is bounded away from zero.*

PROOF. First we establish (b). Fix s and let $\alpha \to \infty$. Then

$$\frac{\partial^i g}{\partial \alpha^i} = \int_0^\infty \frac{\partial^i \varphi(\alpha - b)}{\partial \alpha^i} f(s^{\frac{1}{2}}b)\, db + O[\alpha^i \varphi(\alpha)]$$

$$\approx E^*\{H_i{}^*(\epsilon) f[s^{\frac{1}{2}}(\alpha + \epsilon)]\}$$

where $H_i{}^*(\epsilon)$ is the ith order Hermite Polynomial in ϵ, which is orthogonal to (uncorrelated with) the first $i - 1$ powers of ϵ. Hence

$$\frac{\partial^i g}{\partial \alpha^i} \approx s^{i/2} \frac{\partial^i f}{\partial x^i} [s^{\frac{1}{2}}\alpha] \approx (-1)^i 3(i + 1)!\, (s\alpha^{2+i})^{-1} \log (s\alpha^2).$$

Now we proceed to (a). For convenience we take $s > e$.

$$(5.6) \qquad \frac{\partial^i g}{\partial \alpha^i} = s^{-1} \int_0^\infty b^{-2} \frac{\partial^i}{\partial \alpha^i} [\varphi(\alpha - b) + \varphi(\alpha + b) - 2\varphi(\alpha)] \cdot s b^2 f(s^{\frac{1}{2}}b)\, db.$$

For $0 \leqq b \leqq 1$, $|b^{-2}\partial^i[\varphi(\alpha - b) + \varphi(\alpha + b) - 2\varphi(\alpha)]/\partial\alpha^i| \leqq J_1(\alpha) \exp [\alpha - \frac{1}{2}\alpha^2]$ where $J_1(\alpha)$ is a polynomial in α^2. Also $|sb^2 f(s^{\frac{1}{2}}b)| \leqq J_2 \log s$ where J_2 is a constant. Hence the interval from 0 to 1 contributes to (5.6) an amount bounded by $J_1(\alpha) J_2 \exp (\alpha - \frac{1}{2}\alpha^2) s^{-1} \log s = O(s^{-1} \log s)$. For $b > 1$, $|f(s^{\frac{1}{2}}b)| \leqq J_3(sb^2)^{-1} \log sb^2 \leqq J_3 s^{-1} \log s$ where J_3 is a constant. The partial derivatives in the integrand are uniformly bounded and (a) follows.

For the sake of brevity we shall prove (c) only for the case $i = 0$. Take $s > e$, $\alpha > 2$ and for given ξ, $0 < \xi < 1$, we decompose the integral into the parts from 0 to 1 to $\alpha(1 - \xi)$ to $\alpha(1 + \xi)$ to ∞. The contribution from 0 to 1 is bounded as above by $J_1(\alpha) J_2 \exp [\alpha - (\alpha^2/2)] s^{-1} \log s$. The contribution from 1 to $\alpha(1 - \xi)$ is bounded by $J_4 s^{-1} \log s\, \Phi(-\alpha\xi)$ where J_4 is a constant. The same bound applies to the part from $\alpha(1 + \xi)$ to ∞. Finally for $\alpha(1 - \xi) < b < \alpha(1 + \xi)$, $|f(s^{\frac{1}{2}}b)/f(s^{\frac{1}{2}}\alpha) - 1| \leqq J_5(\xi)$ for $s > s(\xi)$ and $\alpha > \alpha(\xi)$ where $J_5(\xi) \to 0$ as $\xi \to 0$.

It follows that given $\zeta > 0$, there exists $\xi(\zeta)$ such that for $s > s(\xi)$ and $\alpha > \alpha(\xi)$,

$$\left|\int_{\alpha(1-\xi)}^{\alpha(1+\xi)} [\varphi(\alpha - b) + \varphi(\alpha + b) - 2\varphi(\alpha)]f(s^{\frac{1}{2}}b)\, db - f(s^{\frac{1}{2}}\alpha)\right| \leqq \zeta f(s^{\frac{1}{2}}\alpha).$$

Combining the various segments, we have

$$|g(\alpha, s) - f(s^{\frac{1}{2}}\alpha)| \leqq \zeta f(s^{\frac{1}{2}}\alpha) + J_6(\alpha)s^{-1} \log s \exp(-\alpha^2\xi^2/2),$$

which implies (c) for $i = 0$. For $i > 0$, f must be treated more carefully for the section $\alpha(1 - \xi)$ to $\alpha(1 + \xi)$.

LEMMA 5.2. (a) *For large s, (in this lemma "large s" may depend on K as well as the particular form of f),*

$$\hat{\alpha}_K = \{3 \log s - \log 8\pi + o(1)\}^{\frac{1}{2}} \tag{5.7}$$

$$\frac{\partial \Delta_K}{\partial \alpha} > 0 \quad \textit{and} \quad \frac{\partial^2 \Delta_K}{\partial \alpha^2} < 0 \quad \textit{for} \quad \alpha < \hat{\alpha}_K \quad \textit{while} \quad \frac{\partial \Delta_K}{\partial \alpha} < 0 \quad \textit{for} \quad \alpha > \hat{\alpha}_K .$$

(b) *For* $\alpha^2 = 3 \log s - \log 8\pi + c + o(1)$, $\dfrac{\partial^2 \Delta_K}{\partial \alpha^2} \approx -2s^{-1}e^{-c/2}$.

(c) *Decreasing K increases* $\hat{\alpha}_K$ *and decreases* $\theta_K(s)$ *monotonically and continuously.*

(d) *For large s,* $\alpha_K^* \approx (3 \log s)^{\frac{1}{2}}$ *exists if* $\theta_K(s) \geqq 0$ *and is nonexistent if* $\theta_K(s) < 0$. *In the former case, increasing K decreases* $\alpha_K^*(s)$ *monotonically.*

(e) *For large s,* $\Delta_K < 0$ *for* $0 \leqq \alpha < \alpha_K^*(s)$ *when* $\alpha_K^*(s)$ *exists positive.*

PROOF. The first and second derivatives of $Ks^{-\frac{1}{2}}\varphi(\alpha)$ with respect to α are $-Ks^{-\frac{1}{2}}\alpha\varphi(\alpha)$ and $Ks^{-\frac{1}{2}}(\alpha^2 - 1)\varphi(\alpha)$ respectively. The first and second derivatives of $s^{-1} + s^{\frac{1}{2}}\psi(\alpha)$ with respect to α are $-s^{\frac{1}{2}}[1 - \Phi(\alpha)] \approx -s^{\frac{1}{2}}\alpha^{-1}\varphi(\alpha)$ and $s^{\frac{1}{2}}\varphi(\alpha)$ respectively. Applying Lemma (5.1a) for $\alpha^2 < \log s$ and Lemma (5.1c) for $\alpha^2 > \log s$, parts (a) and (b) follow. Parts (c), (d), and (e) are trivial.

Consider the function f in (5.2) where the P_i are obtained from the expansion of Theorem 3.1. with x^2 replaced by $x^2 + 2$. Adding (subtracting) the constant one to (from) P_r, we designate the resulting function $f^+(f^-)$. We label the corresponding g, Δ, $\hat{\alpha}$, α^*, θ etc. with the superscript $+(-)$ and the subscripts Kr. Let β_r represent the sum of the terms thru $K_r\alpha^{-2r}$ in the expansion (3.14a) and let $\alpha_r(s)$ represent the inversion of β_r. Our object is to prove that the optimal boundary $\tilde{\alpha}$ is approximated by $\alpha_r(s)$.

LEMMA 5.3. *For s large,*

$$\theta_{Kr}^+(s) \approx s^{-1}(\log s)^{-r}, \qquad \theta_{Kr}^-(s) \approx -s^{-1}(\log s)^{-r}$$

$$\hat{\alpha}_{Kr}^+(s) = \alpha_r + o(\log s)^{-r}, \qquad \hat{\alpha}_{Kr}^-(s) = \alpha_r + o(\log s)^{-r}$$

$$\alpha_{Kr}^{*+}(s) \approx \alpha_r - (\log s)^{-r/2}.$$

Furthermore both $\theta_{Kr}^+(s)$ *and* $\theta_{Kr}^-(s)$ *are monotone in s for s sufficiently large.*

PROOF. According to Theorem 3.1 and its derivation, the use of the terms thru $(\log x^2)^{-r}P_r$ and $K_r\alpha^{-2r}$ in the expansions for f and β yield a discrepancy $\Delta_{Kr}(\alpha, s) \approx s^{-1}O^*(\alpha^{-2-2r})$ and $\frac{\partial \Delta_{Kr}}{\partial \alpha} \approx s^{-1}O^*(\alpha^{-3-2r})$. Hence $\Delta_{Kr}^+(\alpha_r, s) \approx s^{-1}(\log s)^{-r}$ and $\frac{\partial \Delta_{Kr}^+(\alpha_r, s)}{\partial \alpha} \approx s^{-1}O^*(\log s)^{-r-\frac{1}{2}} = o[s^{-1}(\log s)^{-r}]$. Applying Lemma 5.2b, $\hat{\alpha}_{Kr}^+ - \alpha_r = o[\log s]^{-r}$ and $\Delta_{Kr}^+(\alpha_{Kr}^+, s) = \theta_{Kr}^+ \approx s^{-1}(\log s)^{-r}$. The same arguments apply to θ_{Kr}^- and α_{Kr}^-. Moreover α_{Kr}^{*+} exists and $\alpha_{Kr}^{*+} - \hat{\alpha}_{Kr}^+ \approx (\log s)^{-r/2}$, completing the first half of Lemma 5.3.

For large s,

$$\frac{d\theta_K}{ds} = \frac{\partial}{\partial s}\Delta_K(\hat{\alpha}_K, s) + \frac{\partial}{\partial \alpha}\Delta_K(\hat{\alpha}_K, s)\frac{\partial \hat{\alpha}_K}{\partial s} = \frac{\partial}{\partial s}\Delta_K(\hat{\alpha}_K, s).$$

Since $g + Ks^{-\frac{1}{2}}\varphi(\alpha)$ represents a solution of the heat equation, and $s^{-1} + s^{\frac{1}{2}}\psi(\alpha)$ corresponds to the solution of a modified version of it, it follows that $\frac{d\theta_K}{ds} = \frac{s^{-1}}{2}\frac{\partial^2 \Delta_K(\hat{\alpha}_K, s)}{\partial \alpha^2} + s^{-2} = O(s^{-2})$. The asymptotic expansion for g and $s^{\frac{1}{2}}\psi$ used in the proof of Theorem 3.1 can be differentiated termwise to give asymptotic expansions for the partial derivatives with respect to α. Substituting in the above equation and canceling we get $\frac{d\theta_K}{ds} \approx s^{-2}Q_r\hat{\alpha}_K^{-2r'-2}$ for some r' unless there is complete cancellation. In both the cases of θ_{Kr}^+ and θ_{Kr}^-, cancellation must fail for some $r' \leqq r$. In each case, for s sufficiently large $\frac{d\theta_K}{ds}$ will have only one sign determined by the polynomial Q_r. Clearly θ_{Kr}^+ is decreasing since it is positive and approaches zero. Similarly θ_{Kr}^- is increasing.

Lemma 5.4. *For large s, $\tilde{\alpha} \geqq \alpha_r - O^*[\log s]^{-r/2}$.*

Proof. For s sufficiently large, say $s > s_0$, $\theta_{0r}^+ > 0$. Increasing K from zero, increases θ_{Kr}^+, decreases $\hat{\alpha}_{Kr}^+$ and decreases α_{Kr}^{+*} wherever it exists positive. By increasing K sufficiently, we obtain $\Delta_{Kr}^+(\alpha, s_0) > 0$ for $0 \leqq \alpha < \hat{\alpha}_{0r}^+(s_0)$. Hence α_{Kr}^{*+} represents the boundary of a region which corresponds to a truncated sequential procedure. In this region $g_r^+ + Ks^{-\frac{1}{2}}\varphi(\alpha)$ satisfies the heat equation. At ordinary points of the boundary $g_r^+ + Ks^{-\frac{1}{2}}\varphi = s^{-1} + s^{\frac{1}{2}}\psi(\alpha)$. If, however, α_{Kr}^{*+} has discontinuities, this equality may fail. But, the fact that $\Delta_{Kr}^+ > 0$ between α_{Kr}^{*+} and $\hat{\alpha}_{Kr}^+$ implies that $g_r^+ + Ks^{-\frac{1}{2}}\varphi \geqq s^{-1} + s^{\frac{1}{2}}\psi(\alpha)$ on the boundary. Hence $g_r^+ + Ks^{-\frac{1}{2}}\varphi$ is the risk ($B + t$ value) corresponding to the truncated sequential procedure with a possibly increased stopping risk. Then the optimal Bayes risk u satisfies $u(\alpha s^{\frac{1}{2}}, s) \leqq g_r^+ + Ks^{-\frac{1}{2}}\varphi$ inside the region. Now for s sufficiently large, i.e., $s > s(K)$, $g_r^+ + Ks^{-\frac{1}{2}}\varphi < s^{-1} + s^{\frac{1}{2}}\psi(\alpha)$ for $0 < \alpha < \alpha_{Kr}^{*+}$ (Lemma 5.2e) and hence $u < s^{-1} + s^{\frac{1}{2}}\psi(\alpha)$ there. But this inequality characterizes continuation points for the optimal procedure. Thus $\tilde{\alpha} \geqq \alpha_{Kr}^{*+}$ for $s > s(K)$. Applying Lemma 5.3 yields the desired result.

Lemma 5.5. *For large s, $\tilde{\alpha} \leqq \alpha_r + o(\log s)^{-r}$.*

PROOF. Select s_0 so that $\theta_{0r}^-(s) < 0$ for $s > s_0$. Decreasing K reduces θ_{Kr}^- and increases $\hat{\alpha}_{Kr}^-$. For K sufficiently negative,

$$S = \sup_\alpha [g_r^-(\alpha, s_0) + Ks_0^{-\frac{1}{2}}\varphi(\alpha) - u(\alpha s_0^{\frac{1}{2}}, s_0)] < 0.$$

Since $\theta_{Kr}^-(s)$ is monotonic increasing (to zero) for s sufficiently large, there is an s_1 such that $\theta_{Kr}^-(s_1) > S$ and $\theta_{Kr}^-(s) < \theta_{Kr}^-(s_2) < 0$ whenever $s_2 > s > s_0$ and $s_2 \geqq s_1$. Now suppose that $\tilde{\alpha} > \hat{\alpha}_{Kr}^-$ for some $s_2 \geqq s_1$. Then

$$h = g_r^- + Ks^{-\frac{1}{2}}\varphi(\alpha) - \theta_{Kr}^-(s_2) - u(\alpha s^{\frac{1}{2}}, s)$$

satisfies the heat equation in the region bounded by $s = s_0$ and $(\pm\tilde{\alpha}(s), s)$ for $s > s_0$. Furthermore it is negative on that part of the boundary of the region corresponding to $s < s_2$. Thus $h(\hat{\alpha}_{Kr}^-(s_2), s_2) < 0$, and hence

$$u(\hat{\alpha}_{Kr}^-(s_2)s_2^{\frac{1}{2}}, s_2) - [s_2^{-1} + s_2^{\frac{1}{2}}\psi(\hat{\alpha}_{Kr}^-(s_2))] > 0$$

which is impossible. Hence $\tilde{\alpha} \leqq \hat{\alpha}_{Kr}^- = \alpha_r + o[\log s]^{-r}$ for large s which is the desired result.

THEOREM 5.1. $\tilde{\alpha} - \alpha_r = O^*[\log s]^{-r-\frac{3}{2}}$.

PROOF. Applying Lemmas 5.4 and 5.5 $\tilde{\alpha} - \alpha_{r'} = o[\log s]^{-r'/2}$. However

$$\alpha_{r+1} - \alpha_r = O^*(\log s)^{-r-\frac{3}{2}}$$

and hence

$$\alpha_{r'} - \alpha_r = O^*(\log s)^{-r-\frac{3}{2}} \qquad \text{for } r' > r.$$

Let $r' = 2r + 3$, and it follows that

$$\tilde{\alpha} - \alpha_r = O^*(\log s)^{-r-\frac{3}{2}}.$$

Theorem 5.1 is the desired result that the expansion (3.14a) of Theorem 3.1 furnishes an asymptotic approximation to $\tilde{\alpha}$. Essentially the same method could have been used with the alternative expansion of Section 4.

6. Asymptotic approximation to the Bayes risk. A considerable portion of this section is concerned with technicalities presented in the form of lemmas. To help the reader we outline this section briefly. Goursat ([7], p. 311) presents a representation of the solution of the heat equation which is equivalent to

$$u(y_1, s_1) = \int_B (s_1 - s)^{-\frac{1}{2}}\varphi\left(\frac{y_1 - y}{(s_1 - s)^{\frac{1}{2}}}\right)\left[u(y, s)\,dy + \tfrac{1}{2}u_y\,ds - \frac{u}{2}\left[\frac{y_1 - y}{s_1 - s}\right]ds\right] \tag{6.1}$$

where the integral is taken along a (y, s) path B which starts at $s = s_1$ with a value of y less than y_1, passes through points with $s < s_1$ and terminates at $s = s_1$ with a value of y greater than y_1. This representation is particularly well adapted for our problem with B consisting of a vertical section at $s_0 < s_1$ and the upper and lower optimal boundaries from s_0 to s_1. Along the upper and lower optimal boundaries, u and u_y are determined by the boundary conditions and coincide with $s^{-1} + s^{\frac{1}{2}}\psi(ys^{-\frac{1}{2}})$ and $1 - \Phi(ys^{-\frac{1}{2}})$ respectively.

A minor variation of (6.1) is applied to the difference between u and $g_r + Ks^{-\frac{1}{2}}\varphi(\alpha)$ where g_r is the rth order expression in the formal expansion and $s_0 = s_1^{\frac{1}{2}}$. It is required to show that the difference is small. To do so one must establish suitable bounds on various line integrals along the optimal boundary. Deriving these bounds and carrying out a number of other steps would be considerably easier if we used the symmetry of the Bayes risk and the monotonicity of the optimal boundary for our specific problem. To allow for the applicability to possibly asymmetric variations of our problem, more elaborate proofs were applied.

Lemmas 6.1 and 6.2 are used to bound integrals along the optimal boundary by similar integrals along monotone approximations to the boundary. Lemmas 6.3 and 6.4 express integrals of u and yu from the lower to upper boundary in terms of u and u_y along the boundaries and yield the variation of the Goursat representation in Lemma 6.5. Lemmas 6.6 to 6.9 consist of evaluating bounds on the various integrals in the above representation applied to $g_r + Ks^{-\frac{1}{2}}\varphi(\alpha) - u$. Finally we state and prove Theorem 6.1 which shows that the optimal solution is approximated by $C_0 s^{-\frac{1}{2}}\varphi(\alpha) + C_1 s^{-1}\alpha\varphi(\alpha) + g_r(\alpha, s)$. The C_1 term is zero for our symmetric problem. This is not the case for problems with asymmetric stopping risks.

LEMMA 6.1. *If*

(1) $y_1(x)$ *and* $y_2(x)$ *are monotonic increasing functions such that*

$$y_1(x) < y_0(x) < y_2(x) \quad \text{for} \quad x_1 < x < x_2,$$

(2) $y_i(x_1) = a_1$, $y_i(x_2) = a_2$ *for* $i = 0, 1, 2$, *and*

(3) f *is continuous and* $\frac{\partial f}{\partial x} > 0$ *in the region bounded by*

$$x_1 \leqq x \leqq x_2, \qquad y_1(x) \leqq y \leqq y_2(x),$$

then

$$(6.2) \quad \int_{x_1}^{x_2} f[x, y_2(x)]\, dy_2(x) < \int_{x_1}^{x_2} f[x, y_0(x)]\, dy_0(x) < \int_{x_1}^{x_2} f[x, y_1(x)]\, dy_1(x).$$

PROOF. Approximate $y_0(x)$ by a polygon which yields an approximation to the path integral. Suppose that the polygon decreases from b_2 to b_1 along one subinterval of (x_1, x_2) and increases from b_1 to b_2 along a "later" subinterval. Condition (3) implies that these two intervals contribute a positive amount to the integral. Replacing "later" by "earlier" in the first sentence leads to replacing "positive" by "negative" in the second. The repeated use of these facts lead to the desired result.

Incidentally the result applies when $y_1(x)$ and $y_2(x)$ have vertical sections. In our applications $y_2(x)$ will have a vertical section at x_1 and $y_1(x)$ may have one at x_2.

LEMMA 6.2. *If*

(1) $y_1(x)$ *and* $y_2(x)$ *are monotonic increasing functions such that*

$$y_1(x) < y_0(x) < y_2(x) \quad \text{for} \quad x_1 < x < x_2,$$

(2) $y_i(x_1) = a_1$, $y_i(x_2) = a_2$ *for* $i = 0, 1, 2$, *and*

(3) $\left|\frac{\partial f(x, y)}{\partial x}\right| < K(y)$ *in the region bounded by* $x_1 \leqq x \leqq x_2$, $y_1(x) \leqq y \leqq y_2(x)$,

then

$$(6.3) \quad \left|\int_{x_1}^{x_2} f[x, y_i(x)]\, dy_i(x) - \int_{x_1}^{x_2} f[x, y_0(x)]\, dy_0(x)\right| \leqq \int_{a_1}^{a_2} \rho(y)K(y)\, dy$$

where

$$(6.4) \quad \rho(y) = \sup\{|x_3 - x_4| : y_1(x_3) = y_2(x_4) = y\}.$$

PROOF. We simply elaborate on Lemma 1 using the fact that

$$|f(x_3, y) - f(x_4, y)| \leqq |x_3 - x_4| K(y).$$

Thus the interval $(y, y + dy)$ contributes less, in absolute value, then $\rho(y)K(y)\,dy$ to the difference $\int f[x, y_1(x)]\, dy_1 - \int f[x, y_0(x)]\, dy_0$ or to $\int f[x, y_0(x)]\, dy_0 - \int f[x, y_2(x)]\, dy_2$.

LEMMA 6.3. *For a solution u of the heat equation,*

$$(6.5) \quad \frac{d}{ds}\int_{y_1(s)}^{y_2(s)} u(y, s)\, dy = \left[\tfrac{1}{2}u_y(y_2, s) + u(y_2, s)\frac{dy_2}{ds}\right] - \left[\tfrac{1}{2}u_y(y_1, s) + u(y_1, s)\frac{dy_1}{ds}\right]$$

$$(6.6) \quad \frac{d}{ds}\int_{y_1(s)}^{y_2(s)} yu(y, s)\, dy = \frac{1}{2}\left\{u(y_2, s)\left[\frac{dy_2^2}{ds} - 1\right] + y_2 u_y(y_2, s)\right\} - \frac{1}{2}\left\{u(y_1, s)\left[\frac{dy_1^2}{ds} - 1\right] + y_1 u_y(y_1, s)\right\}.$$

PROOF.

$$\frac{d}{ds}\int_{y_1}^{y_2} u\, dy = \int_{y_1}^{y_2} u_s\, dy + u(y_2, s)\frac{dy_2}{ds} - u(y_1, s)\frac{dy_1}{ds}$$

$$\int_{y_1}^{y_2} u_s\, dy = \int_{y_1}^{y_2} \tfrac{1}{2}u_{yy}\, dy = \tfrac{1}{2}u_y(y_2, s) - \tfrac{1}{2}u_y(y_1, s)$$

and equation (6.5) follows. Similarly

$$\frac{d}{ds}\int_{y_1}^{y_2} yu\, dy = \int_{y_1}^{y_2} yu_s\, dy + y_2 u(y_2, s)\frac{dy_2}{ds} - y_1 u(y_1, s)\frac{dy_1}{ds}$$

$$\int_{y_1}^{y_2} yu_s\, dy = \int_{y_1}^{y_2} \tfrac{1}{2}yu_{yy}\, dy = \tfrac{1}{2}yu_y\Big|_{y_1}^{y_2} - \tfrac{1}{2}\int_{y_1}^{y_2} u_y\, dy = \tfrac{1}{2}[yu_y - u]_{y_1}^{y_2}$$

from which equation (6.6) follows.

LEMMA 6.4. *There is a constant C_0 such that the optimal risk satisfies*

(6.7a) $\int_{-\tilde{y}(s_0)}^{\tilde{y}(s_0)} u(y, s_0)\, dy = C_0 - \int_{s_0}^{\infty} [u_y(\tilde{y}, s)\, ds + 2u(\tilde{y}, s)\, d\tilde{y}(s)]$

(6.7b) $= C_0 - O[s_0^{-\frac{1}{2}}(\log s_0)^{\frac{1}{2}}].$

PROOF. Equation (6.7a) follows from equation (6.5) and symmetry provided the integral on the right converges. Along the optimal boundary, $u(\tilde{y}, s) = s^{-1} + s^{\frac{1}{2}}\psi(ys^{-\frac{1}{2}})$ whose partial derivative with respect to y is $-[1 - \Phi(ys^{-\frac{1}{2}})] = O[s^{-\frac{1}{2}}(\log s)^{-\frac{1}{2}}]$, and whose partial derivative with respect to s is $-s^{-2} + \frac{1}{2}s^{-\frac{1}{2}}\varphi(ys^{-\frac{1}{2}})$. Using the first few terms of the asymptotic expansion

$$\tilde{\alpha}^2 = \tilde{y}^2 s^{-1} \approx 3 \log s - \log 8\pi - 2(\log s)^{-1} + \cdots$$

it follows that the above partial derivative with respect to s is positive for s sufficiently large when $\tilde{\alpha}$ is between the lower and upper approximations obtained by changing the coefficient of $(\log s)^{-1}$ above to -3 and -1 respectively. Now we may apply Lemma 6.1 to

$$\int_{s_0}^{\infty} u(\tilde{y}, s)\, d\tilde{y}(s) = \int_{s_0}^{\infty} [s^{-1} + s^{\frac{1}{2}}\psi(\tilde{y}s^{-\frac{1}{2}})]\, d\tilde{y}\,(s)$$

using these monotonic lower and upper approximations to the boundary. Then

$$\int_{s_0}^{\infty} u(\tilde{y}, s)\, d\tilde{y}(s) = O[s_0^{-\frac{1}{2}}(\log s_0)^{\frac{1}{2}}]$$

and

$$\int_{s_0}^{\infty} u_y(\tilde{y}, s)\, ds = -\int_{s_0}^{\infty} [1 - \Phi(\tilde{y}s^{-\frac{1}{2}})]\, ds = O[s_0^{-\frac{1}{2}}(\log s_0)^{-\frac{1}{2}}]$$

and hence the integral in equation (6.7a) converges and equation (6.7b) follows.

Lemma 6.4 and its method of proof can be pursued further with profit. For example an analysis of the optimal procedure at $s = 0$ shows that the integral on the left of (6.7a) approaches zero as $s \to 0$. Hence

(6.8) $$C_0 = \int_{B_0} \tfrac{1}{2}u_y(\tilde{y}, s)\, ds + u(\tilde{y}, s)\, d\tilde{y}(s)$$

where B_0 is the path described by the optimal boundary going from the lower boundary and $s = \infty$ to $s = 0$ and then along the upper boundary to $s = \infty$. This result is also valid in the asymmetric case. Finally, Lemma 6.4 depends mainly on the convergence of the integral on the right of (6.7a). Thus equations (6.7a) and (6.7b) are valid if u is replaced by $g_r + K_0 s^{-\frac{1}{2}}\varphi(\alpha) + K_1 s^{-1}\alpha\varphi(\alpha) - u$ and C_0 is replaced by $C_{0r} + K_0 - C_0$.

In the asymmetric case an analogue of Lemma 6.4 concerning $\int yu\, dy$ would become relevant. This analogue would involve

(6.9) $$C_1 = \int_{B_0} \tfrac{1}{2}\{u(\tilde{y}, s)[d\tilde{y}^2(s) - ds] + \tilde{y}u_y(\tilde{y}, s)\, ds.$$

This analogue applied to $g_r - u$ appears in the proof of Lemma 6.6 (see Eq. 6.19).

We now apply Lemma 6.4 to the Goursat representation (6.1) to derive the following minor variation of the Goursat representation using sections of the optimal boundary and a vertical section for the path B.

LEMMA 6.5. *For* (y_1, s_1) *within the optimal continuation region, the optimal risk satisfies*

$$u(y_1, s_1) = C_0 s_1^{-\frac{1}{2}}\varphi(y_1 s_1^{-\frac{1}{2}}) + \int_{-\tilde{y}(s_0)}^{\tilde{y}(s_0)} \left\{\frac{1}{(s_1 - s_0)^{\frac{1}{2}}}\varphi\left[\frac{y_1 - y}{(s_1 - s_0)^{\frac{1}{2}}}\right] - \frac{1}{(s_1)^{\frac{1}{2}}}\varphi\left(\frac{y_1}{(s_1)^{\frac{1}{2}}}\right)\right\} u(y, s_0)\, dy$$

$$+ \int_{s_0}^{s_1} \left\{\frac{1}{(s_1 - s)^{\frac{1}{2}}}\left(\varphi\left(\frac{y_1 - \tilde{y}}{(s_1 - s)^{\frac{1}{2}}}\right) + \varphi\left(\frac{y_1 + \tilde{y}}{(s_1 - s)^{\frac{1}{2}}}\right)\right) - \frac{2}{(s_1)^{\frac{1}{2}}}\varphi\left(\frac{y_1}{(s_1)^{\frac{1}{2}}}\right)\right\} \{u(\tilde{y}, s)\, d\tilde{y} + \tfrac{1}{2}u_y(\tilde{y}, s)\, ds\}$$

$$(6.10) \qquad + \int_{s_0}^{s_1} \frac{1}{2(s_1 - s)^{\frac{3}{2}}}\left\{(y_1 + \tilde{y})\varphi\left(\frac{y_1 + \tilde{y}}{(s_1 - s)^{\frac{1}{2}}}\right) - (y_1 - \tilde{y})\varphi\left(\frac{y_1 - \tilde{y}}{(s_1 - s)^{\frac{1}{2}}}\right)\right\} u(\tilde{y}, s)\, ds$$

$$- \frac{1}{(s_1)^{\frac{1}{2}}}\varphi\left(\frac{y_1}{(s_1)^{\frac{1}{2}}}\right)\int_{s_1}^{\infty} \{2u(\tilde{y}, s)\, d\tilde{y} + u_y(\tilde{y}, s)\, ds\}, \qquad s_0 < s_1.$$

The same equation applies to $u_r(y, s) = g_r(ys^{-\frac{1}{2}}, s)$ *with* C_0 *replaced by* C_{0r}.

By applying equation (6.10) to $u_r - u$, we shall show that

$$(6.11) \qquad u(y, s) = C_0 s_1^{-\frac{1}{2}}\varphi(\alpha) + g_r(\alpha, s) + s^{-1}O^*[(\log s)^{-r-1}]$$

To accomplish this we require approximations and bounds for the various integrals appearing in (6.10). In order to avoid arguments which make essential use of the symmetry in our particular problem, some of the ensuing discussion will be considerably longer than necessary.

LEMMA 6.6. *Let* $s_0 = s_1^{\frac{1}{2}}$. *Then for* $r > 1$

$$(6.12) \qquad I_1 = \int_{-\tilde{y}(s_0)}^{\tilde{y}(s_0)} \left\{\frac{1}{(s_1 - s_0)^{\frac{1}{2}}}\varphi\left[\frac{y_1 - y}{(s_1 - s_0)^{\frac{1}{2}}}\right] - \frac{1}{(s_1)^{\frac{1}{2}}}\varphi\left[\frac{y_1}{(s_1)^{\frac{1}{2}}}\right]\right\} [u_r(y, s_0) - u(y, s_0)]\, dy$$

$$= C_{1r} s_1^{-1}\alpha_1 \varphi(\alpha_1) + O^*[s_1^{-1}(\log s_1)^{-r+1}]$$

where $C_{1r} = 0$ *for a symmetric stopping risk.*

PROOF.

$$(6.13) \qquad A_1 = \frac{1}{(s_1 - s_0)^{\frac{1}{2}}}\varphi\left(\frac{y_1 - y}{(s_1 - s_0)^{\frac{1}{2}}}\right) - \frac{1}{(s_1)^{\frac{1}{2}}}\varphi\left(\frac{y_1 - y}{(s_1)^{\frac{1}{2}}}\right) = O(s_0 s_1^{-\frac{3}{2}}).$$

Since $u_r - u$ is bounded

$$\text{(6.14)} \qquad \int_{-\tilde{y}(s_0)}^{\tilde{y}(s_0)} A_1[u_r(y, s_0) - u(y,s_0)]\, dy = O[s_0^{\frac{3}{4}}(\log s_0)^{\frac{1}{2}} s_1^{-\frac{3}{2}}] = O[s_1^{-9/8}(\log s_1)^{\frac{1}{2}}].$$

$$\text{(6.15)} \qquad A_2 = \frac{1}{(s_1)^{\frac{1}{2}}}\left[\varphi\left(\frac{y_1 - y}{(s_1)^{\frac{1}{2}}}\right) - \varphi\left(\frac{y_1}{(s_1)^{\frac{1}{2}}}\right)\right] = y s_1^{-1} \alpha_1 \varphi(\alpha_1) + y^2 O(s_1^{-\frac{3}{2}}).$$

First we note that

$$\text{(6.16)} \qquad \int_{-\tilde{y}(s_0)}^{\tilde{y}(s_0)} y^2[u_r(y, s_0) - u(y, s_0)]\, dy = O[s_0^{\frac{3}{4}}(\log s_0)^{\frac{3}{2}}] = O[s_1^{3/8}(\log s_1)^{\frac{3}{2}}].$$

This together with (6.14) and the fact that $\int_{-\tilde{y}(s_0)}^{\tilde{y}(s_0)} y(u_r - u)\, dy = 0$ in our symmetric problem establishes the lemma with $C_{1r} = 0$. However, for asymmetric versions of the problem, the latter integral does not necessarily vanish. For a proof which applies to such versions, we apply Equation (6.6) to transform the integral $\int y(u_r - u)\, dy$ to one with respect to s. Let $y_1(s)$ be the lower optimal boundary $-\tilde{y}(s)$ and $y_2(s)$ be the upper optimal boundary $\tilde{y}(s)$. Then along these optimal boundaries

$$\text{(6.17a)} \qquad u_r - u = [g_r + K s^{-\frac{1}{2}} \varphi(\alpha)] - [s^{-1} + s^{\frac{1}{2}} \psi(\alpha)] = \Delta(y s^{-\frac{1}{2}}, s)$$

and

$$\text{(6.17b)} \qquad (u_r - u)_y = s^{-\frac{1}{2}} \frac{\partial \Delta}{\partial \alpha}.$$

Thus we are interested in

$$\text{(6.18)} \qquad \int_{s_0}^{\infty} \tfrac{1}{2}\Delta \cdot [d\tilde{y}^2 - ds] + \tilde{y} s^{-\frac{1}{2}} \frac{\partial \Delta}{\partial \alpha}\, ds.$$

Using the results of Section 5 it is clear that there exist approximations $y_{3r}(s)$ and $y_{4r}(s)$, $r > 1$, such that

(i) $y_{3r}(s) < \tilde{y}(s) < y_{4r}(s)$

(ii) $y_{3r}(s)$ and $y_{4r}(s)$ are monotonic for s large

(iii) $\Delta(\alpha, s) = s^{-1} O^*[(\log s)^{-r-1}]$ along $\tilde{y}(s)$ and these approximations,

(iv) $\dfrac{\partial \Delta}{\partial \alpha} = s^{-1} O^*[(\log s)^{-r-\frac{3}{2}}]$,

(v) $\dfrac{\partial \Delta(y s^{-\frac{1}{2}}, s)}{\partial s} = (2s)^{-1} \dfrac{\partial^2 \Delta}{\partial \alpha^2} + s^{-2} \mp o(s^{-2})$

(vi) $y_{4r}^2 - y_{3r}^2 = O^*[s(\log s)^{-r}]$.

Then we may apply Lemma 6.2 to $\int_{s_0}^{\infty} \Delta\, d\tilde{y}^2(s)$ where the y_0, y_1, y_2 of Lemma 6.2 are replaced by y_{3r}^2, $\tilde{y}^2$ and y_{4r}^2 except that y_{3r}^2 and $\tilde{y}^2$ are modified with short vertical sections at s_0 so that the three functions coincide at s_0. The role of the function $\rho(y)$ of Lemma 6.2 is taken by

$$\rho[y^2] = O^*[s(\log s)^{-r-1}] = O^*[y^2(\log y^2)^{-r-2}]$$

and that of the function $K(y)$ of Lemma 2, is taken by

$$K(y^2) = o(s^{-2}) = o[(y^2)^{-2}(\log y^2)^2].$$

Then the integral

$$\int_{\tilde{y}^2(s_0)}^{\infty} \rho(y^2)K(y^2)\, dy^2 = o[\log y_{4r}^2(s_0)]^{-r+1} = o[\log s_1]^{-r+1}.$$

Hence

$$\begin{aligned} \left|\int_{s_0}^{\infty} \Delta\, d\tilde{y}^2(s)\right| &\leqq \int_{s_0}^{\infty} |\Delta|\, dy_{4r}^2(d) + o(\log s_1)^{-r+1} \\ &= \int_{s_0}^{\infty} s^{-1}O^*(\log s)^{-r-1}[\log s]\, ds + o(\log s_1)^{-r+1} \\ &= O^*[(\log s)^{-r+1}]. \end{aligned}$$

Furthermore

$$\int_{s_0}^{\infty} \Delta\, ds = O^*[(\log s_1)^{-r}]$$

and

$$\int_{s_0}^{\infty} \tilde{y}s^{-\frac{1}{2}} \frac{d\Delta}{d\alpha}\, ds = O^*[(\log s_1)^{-r}].$$

Thus the expression in (6.18) is $O^*[(\log s)^{-r+1}]$ and the application of equation (6.6) has yielded

$$\int_{-\tilde{y}(s_0)}^{\tilde{y}(s_0)} y[u_r(y, s_0) - u(y, s_0)]\, dy = C_{1r} + O^*[(\log s_1)^{-r+1}]. \tag{6.19}$$

Combining (6.14)–(6.17) with (6.19) we have

$$I_1 = O[s_1^{-9/8}(\log s_1)^{\frac{1}{2}}] + O[s_1^{-\frac{3}{2}+3/8}(\log s_1)^{3/8}] + \alpha_1\varphi(\alpha_1)s_1^{-1}\{C_{1r} + O^*[(\log s_1)^{-r+1}]\}$$

which gives the result claimed in Lemma (6.6). For the symmetric case $C_{1r} = 0$. For asymmetric stopping risks

$$C_{1r} = \frac{1}{2}\int_{B_0} \Delta[d\tilde{y}^2 - ds] + \tilde{y}s^{-\frac{1}{2}}\frac{\partial\Delta}{\partial\alpha}\, ds. \tag{6.20}$$

LEMMA 6.7. *Let* $s_0 = s_1^{\frac{1}{4}}$, $|y_1| \leqq (2.5\, s_1 \log s_1)^{\frac{1}{2}}$ *and* $r > 1$. *Then*

$$\begin{aligned} I_2 = \int_{s_0}^{s_1} &\left\{\frac{1}{(s_1 - s)^{\frac{1}{2}}}\left[\varphi\left(\frac{y_1 - \tilde{y}}{(s_1 - s)^{\frac{1}{2}}}\right) + \varphi\left(\frac{y_1 + \tilde{y}}{(s_1 - s)^{\frac{1}{2}}}\right)\right]\right. \\ &\left. - \frac{2}{(s_1)^{\frac{1}{2}}}\varphi\left(\frac{y_1}{(s_1)^{\frac{1}{2}}}\right)\right\}\{[u_r(\tilde{y}, s) - u(\tilde{y}, s)]\, d\tilde{y}(s) + \tfrac{1}{2}[u_{ry}(\tilde{y}, s) - u_y(\tilde{y}, s)]\, ds\} \\ = s_1^{-1}&O^*[(\log s_1)^{-r+1}]. \end{aligned}$$

PROOF. Once more we avoid the use of symmetry. Applying the condition on y_1 to $s_0 < s < s_1$ with $y = y(s) \approx (3s \log s)^{\frac{1}{2}}$, we obtain

$$\begin{aligned} \frac{1}{(s_1 - s)^{\frac{1}{2}}}\varphi\left(\frac{y_1 \pm y}{s_1 - s)^{\frac{1}{2}}}\right) - \frac{1}{(s_1)^{\frac{1}{2}}}\varphi\left(\frac{y_1}{(s_1)^{\frac{1}{2}}}\right) &= O[s_1^{-\frac{3}{2}}s] + O[ys_1^{-1}] \\ &= O[s_1^{-1}s^{\frac{1}{2}}(\log s_1)^{\frac{1}{2}}]. \end{aligned} \tag{6.21}$$

Along the optimal boundary $u_r - u$ and $u_{ry} - u_y$ coincide with Δ and $s^{-\frac{1}{2}}\partial\Delta/\partial\alpha$.

$$u_r(\tilde{y}, s) - u(\tilde{y}, s) = \Delta = s^{-1}O^*[(\log s_1)^{-r-1}] \tag{6.22}$$

and

$$u_{ry}(\tilde{y}, s) - u_y(\tilde{y}, s) = s^{-\frac{1}{2}}\frac{\partial\Delta}{\partial\alpha} = s^{-\frac{3}{2}}O^*[(\log s_1)^{-r-\frac{1}{2}}]. \tag{6.23}$$

That part of I_2 contributed by $u_{ry} - u_y$ is easily seen to be

$$s_1^{-1}O^*[(\log s_1)^{-r}].$$

On the other hand the remaining part of I_2 requires the application of Lemma 6.2. The y_0, y_1, and y_2 of Lemma 6.2 are $\tilde{y}$, y_{3r} and y_{4r} respectively. The distance function corresponding to ρ is now

$$\rho(y) = O^*[s(\log s)^{-r-1}] \tag{6.24}$$

where $s \sim y^2/\log y^2$.

Now we compute a bound for $K(y)$, the derivative with respect to s of the integrand which we identify with (6.21) multiplied by $\Delta(ys^{-\frac{1}{2}}, s)$. The derivative of (6.21) with respect to s (keeping y fixed) is

$$\tfrac{1}{2}(s_1 - s)^{-\frac{3}{2}}\left\{1 - \frac{(y_1 \pm y)^2}{s_1 - s}\right\}\varphi\left(\frac{y_1 \pm y}{(s_1 - s)^{\frac{1}{2}}}\right) = O[s_1^{-\frac{3}{2}}].$$

Since $\Delta + s^{-1}$ is a solution of the heat equation, the derivative of Δ with respect to s, (keeping y fixed) is $(2s)^{-1}[\partial^2\Delta/\partial\alpha^2] + s^{-2}$. Applying Lemma 5.2b, the required derivative is $O(s^{-2})$. Hence

$$K(y) = O^*[s_1^{-\frac{3}{2}}s^{-1}(\log s_1)^{-r-1}] + O[s_1^{-1}s^{-\frac{3}{2}}(\log s_1)^{\frac{1}{2}}]$$

and

$$\int_{y_{3r}(s_0)}^{y_{4r}(s_1)} \rho(y)K(y)\,dy = s_1^{-1}O^*(\log s_1)^{-r+1}.$$

Then Lemma 6.2 yields $s_1^{-1}O^*(\log s_1)^{-r+1}$ for that part of I_2 contributed by $(u_r - u)$ terms, and Lemma 6.7 follows.

LEMMA 6.8. *Let* $s_0 = s_1^{\frac{1}{4}}$ *and* $|y_1| \leqq (2.5\, s_1 \log s_1)^{\frac{1}{2}}$. *Then*

$$I_3 = \int_{s_0}^{s_1} \tfrac{1}{2}(s_1 - s)^{-\frac{3}{2}}\left\{(y_1 \pm \tilde{y})\varphi\left(\frac{y_1 \pm \tilde{y}}{(s_1 - s)^{\frac{1}{2}}}\right)\right\}[u_r(\tilde{y}, s) - u(\tilde{y}, s)]\,ds$$
$$= O\{s_1^{-1}(\log s_1)^{-r}\}.$$

PROOF. The result follows from the fact that the integrand is bounded by

$$s_1^{-1}s^{-1}O^*[(\log s_1)^{-r-1}].$$

LEMMA 6.9.

$$I_4 = \int_{s_1}^{\infty} \{2[u_r(\tilde{y}, s) - u(\tilde{y}, s)]\,d\tilde{y}(s) + [u_{ry}(\tilde{y}, s) - u_y(\tilde{y}, s)]\,ds\}$$
$$= s_1^{-\frac{1}{2}}O^*[(\log s_1)^{-r-\frac{1}{2}}].$$

PROOF. The contribution of $u_{ry} - u_y$ is $s_1^{-\frac{1}{2}}O^*[(\log s_1)^{-r-\frac{3}{2}}]$. The other part may be bounded using some of the computations of Lemma 6.8 in the application of Lemma 6.2. Using the same $\tilde{y}$, y_{3r}, y_{4r} we have the same $\rho(y)$, (see Eq. 6.24). On the other hand the new $K(y)$ is $O(s^{-2})$ and $\int K\rho\, dy = O[s_1^{-\frac{3}{2}}(\log s_1)^{-r-\frac{1}{2}}]$. The approximation using y_{3r} gives an amount equal to $s^{-\frac{3}{2}}O^*(\log s_1)^{-r-\frac{1}{2}}$. Combining these results we have our lemma.

THEOREM 6.1. *The formal expansion of Theorem 3.1 provides an asymptotic approximation to the optimal risk $u(y, s)$ in the following sense. For (y, s) within the optimal continuation region and $r > 1$,*

$$u(y, s) = C_0 s^{-\frac{1}{2}}\varphi(\alpha) + C_1 s^{-1}\alpha\varphi(\alpha) + g_r(\alpha, s) + s^{-1}O^*[(\log s)^{-r-1}] \tag{6.25}$$

where

$$C_0 = \int_B \tfrac{1}{2}u_y(\tilde{y}, s)\, ds + u(\tilde{y}, s)\, d\tilde{y}(s) \tag{6.26}$$

is the line integral along the optimal boundary and $C_1 = 0$ in our symmetric problem.

PROOF. Assembling Lemmas 6.6–6.9, we have for $|y_1| \leqq (2.5\, s_1 \log s_1)^{\frac{1}{2}}$

$$u(y_1, s_1) = C_0 s_1^{-\frac{1}{2}}\varphi(\alpha) + g_r(\alpha_1, s_1) + s_1^{-1}O^*[(\log s_1)^{-r+1}]. \tag{6.27}$$

The proofs of Lemmas 6.7 and 6.8 were not quite delicate enough to obtain the same result for $|y_1| \geqq (2.5\, s_1 \log s_1)^{\frac{1}{2}}$. However in this range we know from the proof of Lemma 5.4 that for some K, $u \leqq g_r^+ + Ks^{-\frac{1}{2}}\varphi(\alpha) = O(s^{-1})$ and $u_r = g_r = O(s^{-1})$. Hence

$$v(y, s) = g_r(\alpha, s) + C_0 s^{-\frac{1}{2}}\varphi(\alpha) - u(y, s) = O(s^{-1})$$

satisfies the heat equation in the region bounded by $s_1/12 \leqq s \leqq s_1$ and the optimal boundaries. Hence [5], $v(y, s) = E\{v(Y, S)\}$ where (Y, S) is the random point where a Wiener process going backwards in the s scale from the point (y_1, s_1) first intersects the boundary of the above region. But

$$v = s_1^{-1}O^*[(\log s_1)^{-r-1}]$$

along the optimal boundaries. Hence these sections contribute $s_1^{-1}O^*[(\log s_1)^{-r-1}]$ to v. Along the section where $s = s_1/12$, $v = O(s^{-1})$. The probability that the Wiener process from (y_1, s_1) with $y_1 \geqq (2.5\, s_1 \log s_1)^{\frac{1}{2}}$ intersects this vertical section is bounded by

$$1 - \Phi\left\{\frac{(2.5 s_1 \log s_1)^{\frac{1}{2}} - (3(s_1/12)\log(s_1/12))^{\frac{1}{2}}}{(11 s_1/12)^{\frac{1}{2}}}\right\} = o(s_1^{-.54}).$$

Hence the vertical section of the boundary contributes $o(s^{-1.04})$ to v and $v = s_1^{-1}O^*[(\log s_1)^{-r-1}]$ for $|y_1| \geqq [2.5\, s_1 \log s_1]^{\frac{1}{2}}$. Combining this with (6.27) we have the desired result except that the exponent $-r - 1$ of $\log s_1$ is replaced by $-r + 1$. However, since $g_{r+2} - g_r = s_1^{-1}O^*[(\log s_1)^{-r-1}]$, the exponent can be improved to $-r - 1$ by first applying the weaker result to $r^* = r + 2$.

7. Bounds and motivation. The original motivation for the expansions of

Section 3 came from the application of the Goursat formula (6.1). Note that in the second integral of (6.10) if we set $y_1/(s_1)^{\frac{1}{2}} = \alpha_1$ and $\tilde{y}/(s_1)^{\frac{1}{2}} = b$, part of the integrand is approximately $\varphi(\alpha + b) + \varphi(\alpha - b) - 2\varphi(\alpha)$. However the original application of the Goursat formula depended heavily on knowing that $u \approx s^{-1}$ on the boundary. To ascertain this it was necessary to obtain bounds for the boundary to show that

$$\tilde{y} \approx \{s[3 \log s + O(1)]\}^{\frac{1}{2}}.$$

While the bounds are no longer required, some of the methods used to obtain bounds seem to be of interest and will be sketched very briefly here.

(a) *Upper bound for optimal boundary.* We use the monotonicity of the upper boundary (in our special problem) and suppose that it passes through (y_0, s_0). We obtain a bound on the Bayes risk $b(y_1, t_1)$ where $y_1 < y_0$ and $s_1 = t_1^{-1} > s_0 = t_0^{-1}$.

$$d(y_1, t_1) = s_1^{\frac{1}{2}}\psi(y_1/(s_1)^{\frac{1}{2}}) \geqq b(y_1, t_1) \geqq (t_0 - t_1)P$$

where P is the probability that a Wiener process Y_s through (y_1, s_1) going backwards in s does not cross the boundary between s_0 and s_1. But

$$1 - P \leqq P\{\sup_{s_0<s<s_1} Y_s > y_0\}.$$

By the reflection principle

$$1 - P \leqq 2\Phi[(y_1 - y_0)/(s_1 - s_0)^{\frac{1}{2}}].$$

Let

$$y_1 - y_0 = -(2s_0/3 \log s_0)^{\frac{1}{2}} \quad \text{and} \quad s_1 - s_0 = 2s_0/(3 \log s_0).$$

Then

$$\psi\left[\frac{y_0 - (2s_0/3 \log s_0)^{\frac{1}{2}}}{\{s_0[1 + (2/3 \log s_0)]\}^{\frac{1}{2}}}\right] \geqq \frac{2s_0^{-\frac{3}{2}}}{3 \log s_0}\left[1 + \frac{2}{3 \log s_0}\right]^{-\frac{1}{2}} [1 - 2\Phi(-1)]$$

from which it follows that

$$y_0^2 \leqq s_0[3 \log s_0 + O(1)]. \tag{7.1}$$

The main point of the derivation is that if the boundary is too high at time t_0 there is a point (y_1, t_1) with $t_1 < t_0$ and $y_1 < y_0$ for which the probability of sampling until time t_0 is substantial because of the monotonicity of the optimal boundary. Then the expected cost of sampling outweighs the risk of stopping at (y_1, t_1) which leads to a contradiction. The derivation can be refined to get specific constants to replace the $O(1)$ term but I do not believe that $-\log 8\pi$ can be attained this way. In fact it is remarkable how effective this rather naive approach is.

(b) *A weak lower bound for the optimal boundary.* In this subsection we study the procedure which consists of terminating when there is no fixed sample time procedure which does as well as stopping. Obviously the boundary for this procedure is below the optimal boundary. The bound obtained by this argument is not strong enough to show that $\tilde{y} \geqq [s(3 \log s + O(1))]^{\frac{1}{2}}$ but the argument presents a simple use of a technique which can be extended to obtain that result. This technique consists of using the identity.

$$d(y_0, t_0) - \int_{-\infty}^{\infty} d(y_0 + \epsilon\gamma, t_1)\varphi(\epsilon)\, d\epsilon = \gamma\psi(y_0/\gamma) \qquad \text{for } t_0 < t_1 \tag{7.2}$$

where

$$\gamma^2 = t_0^{-1} - t_1^{-1} \tag{7.3}$$

and d represents the stopping risk (without the sampling cost) i.e.,

$$d(y, t) = t^{-\frac{1}{2}}\psi(yt^{\frac{1}{2}}).$$

The identity (7.2) is easily derived using the relations

$$d^{+}(y_0, t_0) = \int_{-\infty}^{\infty} d^{+}(y_0 + \epsilon\gamma, t_1)\varphi(\epsilon)\, d\epsilon, \tag{7.4}$$

$$d^{+}(y, t) - d^{-}(y, t) = -y, \tag{7.5}$$

and

$$\begin{aligned} d(y, t) &= d^{+}(y, t) \quad \text{for} \quad y > 0 \\ &= d^{-}(y, t) \quad \text{for} \quad y < 0 \end{aligned} \tag{7.6}$$

where d^{+} and d^{-} are the stopping costs associated with deciding that the mean is positive and negative respectively.

For the procedure which consists of stopping when there is no fixed sample time which does as well as stopping, the boundary is given by $y^*(t)$ where

$$d[y^*(t_0), t_0] = \inf_{t_1 > t_0} [(t_1 - t_0) + \int_{-\infty}^{\infty} d[y^*(t_0) + \epsilon\gamma, t_1]\varphi(\epsilon)\, d\epsilon] \tag{7.7}$$

$$d(y^*(t_0), t_0) = \inf_{t_1 > t_0} \{[t_1 - t_0] + d[y^*(t_0), t_0] - \gamma\psi[y^*(t_0)/\gamma]\} \tag{7.8}$$

$$\inf_{t_1 > t_0} \{(t_1 - t_0) - \gamma\psi[y^*(t_0)/\gamma]\} = 0, \qquad \gamma^2 = t_0^{-1} - t_1^{-1}. \tag{7.9}$$

Setting the derivative of $t_1 - t_0 - \gamma\psi[y^*(t_0)/\gamma]$ with respect to t_1 equal to zero we have two determining relations.

$$\varphi[y^*(t_0)/\gamma] = 2\gamma t_1^2 \tag{7.10a}$$

$$\psi[y^*(t_0)/\gamma] = (t_1 - t_0)/\gamma. \tag{7.10b}$$

Take $t_1/t_0 = u$ large, and substituting in the above relations, we derive

$$\begin{aligned} 2u \approx [y^*(t_0)/\gamma]^2 &= -3 \log t_0 - 4 \log[-\log t_0] - \log(81\pi/2) + o(1), \\ y^*(t_0)^2 &= t_0^{-1}\{-3 \log t_0 - 4 \log[-\log t_0] - \log(81\pi e^2/2)\} + o(1). \end{aligned} \tag{7.11}$$

Note how $t_0 y^{*2}(t_0)$ compares with $\tilde{\alpha}^2(t_0) = -3 \log t_0 - \log 8\pi + o(1)$ and that at the y^* boundary,

$$d[y^*(t_0), t_0] \approx (3e/2)t_0(-\log t_0)$$

Incidentally, if y^* is replaced by y, the right hand side of (7.8) represents the risk for the best fixed sample size procedure. For $y = 0$, a simple computation yields a risk of approximately $(2t_0/\pi)^{\frac{1}{2}}$ for t_0 small and $1/(2\pi t_0)^{\frac{1}{2}}$ for t_0 large.

8. The Wald procedure. For comparison purposes we derive an asymptotic approximation to the boundary and risk for the best Wald procedure when t is small. By this we mean, the following! A Wald procedure consists of stopping when $|X| \geqq a$. If μ has an *a priori* normal distribution with mean 0 and variance t^{-1}, the best a will be denoted by $a_0 = \alpha_0 t^{-\frac{1}{2}}$ and the corresponding Bayes risk by $R_0(0, t)$. In [2], the following expressions were derived and applied.

$$a_0^3 = \int_0^\infty v^2 \operatorname{sech}^2 v \exp(-v^2/2\alpha_0^2)\, dv \Big/ \int_0^\infty [\operatorname{sech}^2 v + v^{-1} \tanh v] \exp(-v^2/2\alpha_0^2)\, dv \tag{8.1}$$

$$R_0(0, t) = (2\pi\alpha_0^2 a_0^2)^{-\frac{1}{2}} \int_0^\infty \{v[1 - \tanh v] + 2a_0^3(\tanh v/v)\} \exp(-v^2/2\alpha_0^2)\, dv. \tag{8.2}$$

We shall use them to derive

THEOREM 8.1. *As* $t \to 0$,

$$\alpha_0 = t^{-\frac{1}{6}}(-\log t)^{-\frac{1}{2}}(2c_2)^{\frac{1}{3}}[1 + o(1)] \tag{8.3}$$

and

$$R_0(0, t) = (2\pi)^{-\frac{1}{2}} t^{\frac{1}{3}}(-\log t)^{2/3}(2c_2)^{-2/3}(c_3 + 2c_2)[1 + o(1)], \tag{8.4}$$

where

$$c_2 = \int_0^\infty v^2 \operatorname{sech}^2 v\, dv \approx .75 \tag{8.5}$$

and

$$c_3 = \int_0^\infty v[1 - \tanh v]\, dv \approx .4. \tag{8.6}$$

PROOF. The proof we present can be extended to yield more refined approximations involving

$$c_1 = \int_0^\infty \operatorname{sech}^2 v\, dv = 1 \tag{8.7}$$

and

$$c_4 = \int_0^\infty v^{-1}[\tanh v + e^{-v^2/2} - 1]\, dv \approx .95. \tag{8.8}$$

Expanding $\exp(-v^2/2\alpha_0^2)$ in a Taylor expansion and noting that

$$[e^{-x} - 1 + x - \tfrac{1}{2}x^2 - \cdots + (1/n!)(-1)^{n-1}x^n]x^{-(n+1)}$$

is bounded for $0 < x < \infty$, one immediately obtains

$$\int_0^\infty \operatorname{sech}^2 v \exp(-v^2/2\alpha_0^2)\, dv = c_1 + O(\alpha_0^{-2}), \tag{8.9}$$

$$\int_0^\infty v^2 \operatorname{sech}^2 v \exp(-v^2/2\alpha_0^2)\, dv = c_2 + O(\alpha_0^{-2}), \tag{8.10}$$

and

(8.11) $$\int_0^\infty v[1 - \tanh v] \exp(-v^2/2\,\alpha_0^2)\,dv = c_3 + O(\alpha_0^{-2}).$$

We now proceed to establish

(8.12) $$I = \int_0^\infty v^{-1} \tanh v \exp(-v^2/2\,\alpha_0^2)\,dv = \log \alpha_0 + c_4 + O(\alpha_0^{-2}).$$

First we note that $I = I_1 + I_2 + c_4$ where

$$I_1 = \int_0^\infty v^{-1}[\tanh v - 1][\exp(-v^2/2\,\alpha_0^2) - 1]\,dv = O(\alpha_0^{-2})$$

and

$$\begin{aligned} I_2 &= \int_0^\infty v^{-1}[\exp(-v^2/2\,\alpha_0^2) - \exp(-v^2/2)]\,dv, \\ &= \int_0^1 v^{-1}[\exp(-v^2/2\,\alpha_0^2) - 1]\,dv - \int_0^1 v^{-1}[\exp(-v^2/2) - 1]\,dv \\ &\qquad + \int_1^\infty v^{-1} \exp(-v^2/2\,\alpha_0^2)\,dv - \int_1^\infty v^{-1} \exp(-v^2/2)\,dv, \\ &= \int_0^{\alpha_0} v^{-1}[\exp(-v^2/2) - 1]\,dv - \int_0^1 v^{-1}[\exp(-v^2/2) - 1] \\ &\qquad + \int_{\alpha_0}^\infty v^{-1} \exp(-v^2/2)\,dv - \int_1^\infty v^{-1} \exp(-v^2/2)\,dv, \\ &= \int_1^{\alpha_0} v^{-1}[\exp(-v^2/2) - 1]\,dv - \int_1^{\alpha_0} v^{-1} \exp(-v^2/2)\,dv = \log \alpha_0, \end{aligned}$$

and (8.12) follows. Substituting (8.9)–(8.12) into (8.1) and (8.2) we have

(8.13) $$a_0^3 = c_2[\log \alpha_0 + (c_1 + c_4)]^{-1}[1 + O(\alpha_0^{-2})]$$

(8.14) $$R_0(0, t) = (2\pi)^{-\frac{1}{2}} t^{\frac{1}{2}}[\log \alpha_0 + (c_1 + c_4)]^{2/3} c_2^{-2/3}[c_3 + 2c_2][1 + o(1)].$$

Combining (8.13) with $\alpha_0 = a_0 t^{-\frac{1}{2}}$ we have

$$3 \log \alpha_0 = -\tfrac{3}{2} \log t - \log[-\log t] + \log 2c_2 + o(1)$$

from which (8.3) and (8.4) follow.

As is to be expected both α_0 and $R_0(0, t)$ are larger than $\tilde{\alpha}$ and $B(0, t)$. What is not so expected is that they would be larger by an order of magnitude. However this order of magnitude is determined by $(-\log t)^{2/3}$ and for practical considerations it is worth remarking that this term increases very slowly as $t \to 0$.

9. Sampling cost and error cost. In Sections 3 and 4 asymptotic expansions were presented for the Bayes risk and the optimal boundary as $t \to 0$. It is of some interest to decompose the Bayes risk into two parts; one corresponding to the sampling cost or expected time of sampling and the other to the cost due to the possibility of coming to the wrong conclusion.

Let u_1 represent the sampling cost (including the time necessary to go from 0 to $t = s^{-1}$), and let u_2 represent the cost due to error. Then both u_1 and u_2 satisfy the heat equation subject to the boundary conditions $u_1 = s^{-1}$ and $u_2 = s^{-\frac{1}{2}}\psi(\alpha)$ respectively. Since $u_1 + u_2 = u$ the derivation of an asymptotic expansion for u_1 will suffice to yield one for u_2. To derive an expansion for u_1 we may substitute in the formal expansions of Sections 3 or 4 and apply the expansion for the optimal boundary.

Thus one easily derives an expansion of the form

$$u_1 \approx K_1 s^{-\frac{1}{4}}\varphi(\alpha) + \int_0^b [\varphi(\alpha + b) + \varphi(\alpha - b) - 2\varphi(\alpha)]f_1(s^{\frac{1}{2}}b)\, db$$

where f_1 is an expansion determined by the expansion for α on the optimal boundary and

$$E^*\{f_1[s^{\frac{1}{2}}(\alpha + \epsilon)]\} = s^{-1}.$$

Substitution in the expansion of Section 4 yields

$$f_1(x) \approx f_0^*(x) + c_{11}x^{-2} + c_{21}x^{-4}[f_0^*(x)]^{-1} + c_{31}x^{-6}[f_0^*(x)]^{-2} + \cdots .$$

As in Section 6, the constant K_1 may be expressed in terms of the integral

$$K_1 = \int_{B_0} [\tfrac{1}{2}u_{1y}(\tilde{y}, s)\, ds + u_1(\tilde{y}, s)\, d\tilde{y}].$$

When t is small (s large), $u_1(\tilde{y}, s) = s^{-1}$ and $u_{1y}(\tilde{y}, s) \approx s^{-\frac{1}{2}}E^*\{\epsilon f_1[s^{\frac{1}{2}}(\alpha + \epsilon)]\}$. To evaluate K_1 seems to require the numerical solution of the heat equation for the region determined by the optimal boundary and the boundary condition $u_1 = s^{-1}$.

Applying the methods of Section 4, we compute the coefficients

$$c_{11} = -3, \quad c_{21} = 3, \quad c_{31} = -27/2, \quad c_{41} = -351/2,$$
$$c_{51} = -6705/4, \quad c_{61} = -592119/20.$$

Consequently the coefficients corresponding to

$$f_2 = c_{12}x^{-2} + c_{22}x^{-4}[f_0^*(x)]^{-1} + c_{32}x^{-6}[f_0^*(x)]^{-2} + \cdots$$

which generates the expansion for u_2 are

$$c_{12} = 2, \quad c_{22} = -6, \quad c_{32} = 24, \quad c_{42} = -114, \quad c_{52} = 2079, \quad c_{62} = -9459.$$

At the same time the computations used in deriving the results of Section 4 yield

$$u_{1y}(\tilde{y}, s) \approx s^{-\frac{3}{2}}(3w)^{-\frac{1}{3}}\{-2 - 2w^{-2} - (25/9)w^{-3} + \cdots\}$$

and

$$u_{2y}(\tilde{y}, s) \approx s^{-\frac{4}{3}}(3w)^{-\frac{1}{3}}\{-(4/3)w^{-1} + (4/3)w^{-2} - (52/9)w^{-3} + \cdots\}.$$

In the case where the two costs are combined, the evaluation of the main term $Ks^{-\frac{1}{4}}\varphi(\alpha)$ requires knowledge only of the optimal boundary because the boundary conditions express u and u_y in terms of the stopping risk along the optimal boundary. Suppose however that the problem were modified. For example suppose that that for $t > 2$, a prescribed non-optimal boundary must be used. Then all of our expansions for $t \to 0$ would remain unaltered except that the evaluation of the coefficient of $s^{-\frac{1}{4}}\varphi(\alpha)$ would depend either on u_{1y} along the boundary for $t > 2$ or alternatively on $\int u\, dy$ where the integral corresponds to that part of the line $t = 2$ which is in the continuation region (for $t = 2-$). Thus we see that in many of these modified problems there is no essential difference in the problem of finding the coefficient of $s^{-\frac{1}{2}}\varphi(\alpha)$ for the combined risk and for the separate risks.

10. Operating characteristics. In cases where the *a priori* distribution is not necessarily normal or where the statistician is not willing to accept an *a priori* probability distribution at all, it becomes important to derive the operating characteristics of the procedure studied. Each starting point (x, t) characterizes a procedure. For each starting point and each value of the mean μ, the functions $T(\mu; x, t)$, $\beta(\mu; x, t)$, and $R(\mu; x, t) = T + |\mu|\beta$ represent the expected sampling cost, the probability of error, and the risk respectively. Expressing these functions in terms of $v = x - \mu t$ and $t^* = t$, they satisfy the diffusion equations

$$1 + \tfrac{1}{2}T_{vv} + T_t = 0$$

$$\tfrac{1}{2}\beta_{vv} + \beta_t = 0$$

and

$$1 + \tfrac{1}{2}R_{vv} + R_t = 0$$

with the obvious boundary conditions. While the techniques of this paper should suffice to yield asymptotic expansions for $t \to 0$, there may be some difficulty in establishing that these expansions give approximations which hold uniformly in x or α.

It seems reasonable to expect that as $t \to 0$ for α bounded, T, β, and R converge to functions of μ which are independent of α. When μ has the normal *a priori* distribution $\mathfrak{N}(\alpha_0 t_0^{-\frac{1}{2}}, t_0^{-1})$, t_0 small, the density of μ is approximately $t_0^{\frac{1}{2}}\varphi(\alpha_0)$ over a large range of μ. Then the asymptotic behavior $B(x, t) \approx C_0 t^{\frac{1}{2}}\varphi(\alpha)$ is consistent with the above mentioned expectation and the further one that $\int_{-\infty}^{\infty} R(\mu; x, t)\, d\mu \approx C_0 = \int_{-\infty}^{\infty} R(\mu; 0, 0)\, d\mu$.

This conjecture may be applied to the non-normalized problem as follows. Suppose that the cost per unit time of observation (in the t^* scale) is c^* (small) the cost of a wrong decision is $k^*|\mu^*|$, μ^* is the mean drift per unit time of the observed Wiener process which has variance σ^{*2} per unit time. Let the not necessarily normal *a priori* distribution of μ^* be G^* which has a density approximately $g^*(0)$ in the neighborhood of $\mu^* = 0$. Then according to the normalizing transformations of [2], Section 3,

$$\mathcal{B}^* = c^{*\frac{1}{3}} k^{*2/3} \sigma^{*2/3} \mathcal{B}$$

where $\mathcal{B}$ is the Bayes risk for the normalized problem involving

$$\mu = c^{*-\frac{1}{3}} k^{*\frac{1}{3}} \sigma^{*2/3} \mu^*$$

whose *a priori* distribution G has a density which is approximately $g^*(0)c^{*\frac{1}{3}}k^{*-\frac{1}{3}}\sigma^{*2/3}$ over a large range of μ. Then one would expect

$$\mathcal{B} \approx \int R(\mu; 0, 0)\, dG(\mu) \approx C_0 g^*(0) c^{*\frac{1}{3}} k^{*-\frac{1}{3}} \sigma^{*2/3}$$

and

(10.1) $$\mathcal{B}^* \approx C_0 g^*(0) c^{*2/3} k^{*\frac{1}{3}} \sigma^{*4/3}.$$

To put some rigor into a derivation of (10.1) it would be necessary to develop some bounds for $R(\mu; 0, t)$ or $R(\mu; 0, 0)$ as μ becomes large. We terminate this section with a brief sketch of the derivation of such bounds for $R(\mu; 0, 0)$. A tool would be the distribution of the time at which a Wiener process Z_t with mean 0 and variance one per unit time first crosses a straight line $a + mt$. Implicit in Wald ([9], pp. 191–193) is the result that the distribution has density

(10.2) $$f_{a,m}(t) = (2\pi)^{-\frac{1}{2}} a t^{-\frac{3}{2}} \exp\{-\tfrac{1}{2}[at^{-\frac{1}{2}} + mt^{\frac{1}{2}}]^2\}, \qquad t > 0, a > 0,$$

and moment generating function M (2) $= \exp\{a[m - (m^2 - 2\lambda)^{\frac{1}{2}}]\}$. Note that when $m > 0$, the probability of crossing the line is less than one but the above formulae are still correct and yield a total probability of crossing the line equal to $\exp(-2am)$.

The optimal boundary may be approximated from above by $\tilde{x}^+$ and from below by $\tilde{x}^-$ where both of these approximations are concave for t sufficiently small. Consider a large specified value of μ. The process X_t will intersect the boundary at least as soon as it will intersect a line which is tangent to $\tilde{x}^+$ for some small t. Take the tangent line at the point where $\mu t = \tilde{x}^+(t)$. The process $X_t - \mu t$ is a Wiener process with zero drift and the values of m and a corresponding to the above tangent line are $\frac{d\tilde{x}^+}{dt} - \mu < 0$ and $\tilde{x}^+ - t\frac{d\tilde{x}^+}{dt}$. From the characteristic function the mean time till X_t intersects the tangent line is a/m. Substituting $\tilde{x}^+ = [t(-3\log t - \log 8\pi + 2(\log t)^{-1} - \cdots)]^{\frac{1}{2}}$ we have $a/m \approx t \approx 3\log\mu^2/\mu^2$. Hence for large μ,

(10.3) $$T(\mu; 0, 0) \leqq 3\mu^{-2}(\log\mu^2)[1 + o(1)].$$

Because the above bound is approximately the time required for the "mean drift line" to intersect the boundary, it isn't difficult to show that it is a good approximation to T.

The probability of error is dominated by the probability that the process crosses $-\tilde{x}^-(t)$. The probability that it first crosses this boundary between t and $t + dt$ (t small) is less than the probability that it first crosses the tangent line to $-\tilde{x}(-t)$ in this time interval. Hence small values of t contribute at most

$$\int (2\pi)^{-\frac{1}{2}} a(t) t^{-\frac{3}{2}} \exp\{-\tfrac{1}{2}[a(t)t^{-\frac{1}{2}} + m(t)t^{-\frac{1}{2}}]^2\}\, dt$$

to the error. Substituting $a(t) = \tilde{x}^- - t\frac{d\tilde{x}^-}{dt}$ and $m(t) = \frac{d\tilde{x}^-}{dt} + \mu > 0$, observing where $at^{-\frac{1}{2}} + mt^{\frac{1}{2}}$ is minimized, and applying the steepest descent argument, the above integral is approximately $O(\mu^{-3}(\log\mu^2)^{-1})$. The section where t is not small can be disposed of easily. Thus we have

(10.4) $$\beta(\mu; 0, 0) = O[\mu^{-3}(\log\mu^2)^{-1}].$$

Thus, the contribution of error to risk is smaller than that of sampling by a factor of $(\log \mu^2)^2$ when μ is large.

To summarize this section, we have sketched derivations of bounds on the operating characteristics $\beta(\mu; 0, 0)$ and $T(\mu; 0, 0)$. These bounds together with the conjecture $C_0 = \int R(\mu; 0, 0)\, d\mu$ suggest that using the optimal boundary (starting from (0, 0)) leads to a Bayes risk of approximately $C_0 g^*(0) c^{*2/3} k^{*\frac{1}{3}} \sigma^{*4/3}$, if the *a priori* probability distribution has density $g^*(0)$ at $\mu^* = 0$ and $c^* \to 0$. The same result is also conjectured for the optimal procedure with the starting point at (y, t), t small and $\alpha = y/t^{\frac{1}{2}}$ bounded.

REFERENCES

[1] BATHER, J. A. (1962). Bayes procedures for deciding the sign on a normal mean. *Proc. Cambridge Philos. Soc.* **58** 599–620.

[2] BREAKWELL, J. and CHERNOFF, H. (1964). Sequential tests for the mean of a normal distribution II (large *t*). *Ann. Math. Statist.* **35** 162–173.

[3] CHERNOFF, HERMAN (1961). Sequential tests for the mean of a normal distribution. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **1** 79–91. Univ. of California Press.

[4] CHERNOFF, HERMAN (1965). Sequential tests for the mean of a normal distribution IV (discrete case). *Ann. Math. Statist.* **36** 55–68.

[5] DOOB, J. L. (1955). A probability approach to the heat equation. *Trans. Amer. Math. Soc.* **80** 216–280.

[6] FELLER, WILLIAM (1957). *An Introduction to Probability Theory and its Applications.* **1** (2nd ed.). Wiley, New York.

[7] GOURSAT, E. (1942). *Cours D'Analyse Mathématique.* **3** (5th ed.). Gauthier-Villars, Paris.

[8] MORIGUTI, S. and ROBBINS, H. (1961). A Bayes test of '$p \leqq \frac{1}{2}$' versus '$p > \frac{1}{2}$'. *Rep. Statist. Appl. Res., Un. Japan. Sci. Engrs.* **9** 39–60.

[9] WALD, ABRAHAM (1947). *Sequential Analysis.* Wiley, New York.