SEQUENTIAL TESTS FOR THE MEAN OF A NORMAL DISTRIBUTION IV (DISCRETE CASE)

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1. Summary. The problem of sequentially testing whether the mean of a normal distribution is positive has been approximated by the continuous analogue where one must decide whether the mean drift of a Wiener-Lévy process is positive or negative [3]. The asymptotic behavior of the solution of the latter problem has been studied as $t \to \infty$ and as $t \to 0$ [1], [2], [4], [5]. The original (discrete) problem, can be regarded as a variation of the continuous problem where one is permitted to stop observation only at the discrete time points t_0, $t_0 + \delta$, $t_0 + 2\delta, \cdots$.

Especially since the numerical computation of the solution of the continuous version can be carried out by solving the discrete version for small δ, it is important to study the relationship between the solutions of the discrete and continuous problems. These solutions are represented by symmetric continuation regions whose upper boundaries are $\tilde{x}_\delta(t)$ and $\tilde{x}(t)$ respectively. The main result of this paper is that

$$\tilde{x}_\delta(t) = \tilde{x}(t) + \hat{z}\sqrt{\delta} + o(\sqrt{\delta}). \tag{1.1}$$

This result involves relating the original problem to an associated problem and studying the limiting behavior of the solution of the associated problem. This solution corresponds to the solution of a Wiener-Hopf equation. Results of Spitzer [6], [7] can be used to characterize the solution of the Wiener-Hopf equation and yield $\hat{z}$ as an integral, which, as Gordon Latta pointed out to the author, is equal to $\zeta(\frac{1}{2})/(2\pi)^{\frac{1}{2}} = -.5824$.

The associated problem referred to above is the following. A Wiener-Lévy process Z_t starting at a point (z, t), $t < 0$ is observed at a cost of one per unit time. If the observation is stopped before $t = 0$, there is no payoff. If $t = 0$ is reached, the payoff is Z_0^2 if $Z_0 < 0$ and 0 if $Z_0 \geqq 0$. Stopping is permitted at times $t = -1, -2, \cdots$.

2. Introduction. The reader is referred to [3], and particularly to Section 3 of [2] for a description of the problem and notation. Briefly, assuming that the mean drift μ has a normal *a-priori* probability distribution $\mathfrak{N}(\mu_0, \sigma_0^2)$, one begins observing, starting at the point $(x_0, t_0) = (\mu_0/\sigma_0^2, 1/\sigma_0^2)$, the Wiener process X_t

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with independent normal increments with mean μ per unit time and variance 1 per unit time, at a cost of one per unit time. If the process reaches $X_t = x$ at time t the *a-posteriori* probability distribution of μ is $\mathfrak{N}(x/t, 1/t)$. In the normalized version of the problem the cost of reaching the wrong decision is $|\mu|$ and the *a-posteriori* risk of stopping may be computed to be

$$D(x, t) = t^{-\frac{1}{2}}\psi(xt^{-\frac{1}{2}}) \tag{2.1}$$

where

$$\begin{aligned} \psi(y) &= \varphi(y) - y[1 - \Phi(y)] && \text{for } y \geqq 0 \\ &= \varphi(y) + y\Phi(y) && \text{for } y < 0, \end{aligned} \tag{2.2}$$

φ is the standard normal density and Φ the corresponding c.d.f. For any sequential stopping rule which may be represented by a continuation set in the (x, t) space the *a-posteriori* risk given $X_t = x$, (not counting the sampling cost required to reach (x, t)) is given by $B(x, t)$ where

$$1 + B_t + (x/t)B_x + \tfrac{1}{2}B_{xx} = 0 \tag{2.3}$$

in the interior of the continuation set and $B = D$ on the stopping set. It was seen that the optimal procedure is represented by a continuation set such that $B < D$ on the continuation set and the additional boundary condition $B_x = D_x$, is satisfied. Incidentally it was also seen that $B_{xx} - D_{xx} = 2$ on the boundary.

Outside the continuation set $B - D$ vanishes. Inside but near the optimal boundary $\tilde{x}$, $B - D$ behaves like $-(x - \tilde{x})^2$.

Consider the statistician who is permitted to use the optimal continuous procedure for $t \geqq t_1$ but for $t < t_1$ may stop only at time points $t_1 - \delta$, $t_1 - 2\delta$, $\cdots$, etc. If he measures $B - D$ near the boundary of the continuation region his problem resembles the associated problem described in the summary. This remark is the key to the applicability of the associated problem to our result.

In Section 3, the solution of the associated problem is studied with the help of Spitzer's results. In Section 4, we examine variations of the associated problem relevant to our application and to the stability of an iterative solution of the Wiener-Hopf equation. In Section 5, these results are applied to obtain the desired result concerning the relation of the boundary $\tilde{x}_\delta$ of the discrete problem to $\tilde{x}$.

3. The associated problem. The associated problem is defined in the last paragraph of Section 1. In this section we show that the solution consists of stopping if $Z_{-n} \geqq \hat{z}(-n)$ where the $\hat{z}(-n)$ are negative, decrease as $n \to \infty$ and converge to $\hat{z} = -.5824$. The conditional risk for the optimal procedure, given $Z_t = z$ is studied. Bounds and limiting properties are derived using Spitzer's results on the solution of certain Wiener-Hopf equations. On occasion, the proofs have been designed to avoid using the analytic and special properties of the normal distribution so that some results can be generalized.

Let $\hat{v}(z, t)$ represent the infimum, among all procedures, of the conditional risks (not counting the cost of reaching time $t < 0$) given that $Z_t = z$. The procedure, which consists of stopping when $\hat{v}(z, -n) = 0$ and continuing if $\hat{v}(z, -n) < 0$, is optimal and does not involve past history and therefore corresponds to a continuation set in the half plane $t < 0$. For any procedure represented by a continuation set, the conditional risk $v(z, t)$ is easily seen to satisfy

$$1 + v_t + \tfrac{1}{2}v_{zz} = 0 \tag{3.1}$$

subject to the boundary conditions $v = 0$ for $t < 0$ and

$$\begin{aligned} v(z, 0) &= -z^2 \quad \text{for } z \leqq 0 \\ &= 0 \quad\quad \text{for } z > 0. \end{aligned} \tag{3.2}$$

We shall find it convenient to define the following operators.

$$H_\delta u(z) = \delta + \int_{-\infty}^{\infty} u(z + \epsilon\sqrt{\delta})\varphi(\epsilon)\, d\epsilon = \delta + S_\delta u(z), \qquad \delta \geqq 0 \tag{3.3}$$

$$Tu(z) = \int_{-\infty}^{0} u(y)\varphi(y - z)\, dy = \int_{-\infty}^{-z} u(z + \epsilon)\varphi(\epsilon)\, d\epsilon. \tag{3.4}$$

These operators are defined and analytic for complex z provided u is measurable and does not grow too rapidly as $z \to -\infty$. The function $v = S_\delta u(z)$, regarded as a function of z and δ, satisfies the heat equation $v_\delta = v_{zz}/2$. From the probabilistic interpretation of S_δ as an expectation the relation $H_{\delta_1}[H_{\delta_2}u] = H_{\delta_1+\delta_2}u$ follows immediately.

As an analytic function we see that

$$\frac{dTu(z)}{dz} = \int_{-\infty}^{0} (y - z)u(y)\varphi(y - z)\, dy.$$

For our applications we shall have functions u which are identically zero for z greater than some negative number a. Furthermore the derivatives will be bounded by a polynomial for $z < a$. Then

$$\frac{dTu(z)}{dz} = -\varphi(a - z)u(a-) + Tu'(z). \tag{3.5}$$

For any procedure with risk $v(z, t)$ we define

$$h(z, -n - 1) = 1 + \int_{-\infty}^{\infty} v(z + \epsilon, -n)\varphi(\epsilon)\, d\epsilon \tag{3.6}$$

which represents the risk given that $Z_{-n-1} = z$ and that we do not stop at time $-n - 1$. When referring to the optimal procedure we shall use a circumflex. Hence the equations

$$\hat{h}(z, -n - 1) = 1 + \int_{-\infty}^{\infty} \hat{v}(z + \epsilon, -n)\varphi(\epsilon)\, d\epsilon \tag{3.7}$$

$$\hat{v}(z, -n) = \min\, [\hat{h}(z, -n), 0] \tag{3.8}$$

define the backward induction which gives the optimal procedure and the corresponding risk for the associated problem.

It is convenient to define

$$I_n(z) = Tz^n = \int_{-\infty}^{0} y^n \varphi(y - z)\, dy = \int_{-\infty}^{-z} (z + \epsilon)^n \varphi(\epsilon)\, d\epsilon \tag{3.9}$$

$$J_n(z) = \int_0^{\infty} y^n \varphi(y - z)\, dy = \int_{-z}^{\infty} (z + \epsilon)^n \varphi(\epsilon)\, d\epsilon. \tag{3.10}$$

Note that $(-1)^n I_n(z) = J_n(-z) > 0$ and $I_n(z) + J_n(z) = E[(z + \epsilon)^n]$ where ϵ is normally distributed with mean 0 and variance 1. Furthermore $I_n'(z) = nI_{n-1}(z)$ and $J_n'(z) = nJ_{n-1}(z)$. In particular

$$\begin{aligned} J_0(z) &= \Phi(z), & I_0(z) + J_0(z) &= 1 \\ J_1(z) &= z\Phi(z) + \varphi(z), & I_1(z) + J_1(z) &= z \\ J_2(z) &= (z^2 + 1)\Phi(z) + z\varphi(z), & I_2(z) + J_2(z) &= z^2 + 1 \\ J_3(z) &= (z^3 + 3z)\Phi(z) + (z^2 + 2)\varphi(z), & I_3(z) + J_3(z) &= z^3 + 3z. \end{aligned} \tag{3.11}$$

Finally $J_n(z) \to 0$ as $z \to -\infty$ and hence is bounded for real $z \leqq 0$, as is $I_n(z)$ for real $z \geqq 0$.

LEMMA 3.1.

(a) *The function* $\hat{h}(z, -1)$ *is an entire function. As a function of real* z, $\hat{h}'(z, -1) > 0$, $\hat{h}''(z, -1) < 0$, $\hat{h}(z, -1) \to 1$ *as* $z \to \infty$, $\hat{h}(z, -1) + z^2 \to 0$ *as* $z \to -\infty$.

(b) $\hat{h}(z, -1)$ *vanishes for a unique value of* $z = \hat{z}(-1) < 0$ *and*

$$\begin{aligned} \hat{v}(z, -1) &= \hat{h}(z, -1) > v(z, 0) && \text{for } z < \hat{z}(-1) \\ \hat{v}(z, -1) &= 0 \geqq v(z, 0) && \text{for } z \geqq \hat{z}(-1). \end{aligned}$$

PROOF. We compute $\hat{h}$ and apply (3.5) to represent its derivatives.

$$\begin{aligned} \hat{h}(z, -1) &= 1 - Tz^2 = -z^2 + J_2(z) \\ \hat{h}'(z, -1) &= -2Tz = -2I_1(z) \\ \hat{h}''(z, -1) &= -2T1 = -2I_0(z) = -2\Phi(-z). \end{aligned}$$

Each property in the lemma is easy to establish now.

LEMMA 3.2.

$$\begin{aligned} \hat{v}(z, -n - 1) &\geqq \hat{v}(z, -n) && \text{for} \quad n \geqq 0 \\ \hat{h}(z, -n - 1) &\geqq \hat{h}(z, -n) && \text{for} \quad n \geqq 1 \end{aligned} \tag{3.12}$$

and

$$\hat{v}(z, -n) = 0 \qquad \text{for} \quad z \geqq 0.$$

PROOF. If the discrete problem is modified by replacing $v(z, 0)$ by $\hat{v}(z, -1)$ the corresponding optimal solution would replace $\hat{v}(z, -n)$ by $\hat{v}(z, -n - 1)$. But since the termination risk is increased by the modification, the optimal risk is also increased. The lemma follows.

LEMMA 3.3. *The statement of Lemma* 3.1 *is valid with* $\hat{h}(z, -n)$, $\hat{v}(z, -n)$ *and*

$\hat{z}(-n)$ *replacing* $\hat{h}(z, -1)$, $\hat{v}(z, -1)$ *and* $\hat{z}(-1)$. *Furthermore* $\hat{z}(-n) \leqq \hat{z}(-n+1)$.

PROOF. The lemma holds for $n = 1$. Assume it for a given value of n. Then

$$\hat{h}(z, -n-1) = 1 + T[\hat{v}(z, -n)]$$

$$\hat{h}'(z, -n-1) = T\hat{v}'(z, -n) > 0$$

$$\hat{h}''(z, -n-1) = -\varphi[\hat{z}(-n) - z]\hat{h}'[\hat{z}(-n), -n)] + T\hat{v}''(z, -n) < 0.$$

Clearly $T\hat{v}(z, -n) \to 0$ as $z \to \infty$ and hence $\hat{h}(z, -n-1) \to 1$. For $z < \hat{z}(-n)$, $\hat{v}(z, -n) = \hat{h}(z, -n)$ and $\hat{h}(z, -n) + z^2 \to 0$ as $z \to -\infty$. Therefore, as $z \to -\infty$, $\hat{v}(z, -n) = -z^2 + o(1)$. Invoking Lemma 3.2 and the Lebesgue convergence theorem it follows that $\hat{h}(z, -n-1) + z^2 \to 0$. The lemma follows.

Combining Lemmas 3.1 to 3.3 yields

THEOREM 3.1. *The optimal procedure for the associated problem consists of stopping if* $Z_{-n} \geqq \hat{z}(-n)$, *where* $\hat{z}(-n-1) \leqq \hat{z}(-n) < 0$.

To obtain the limiting behavior of $\hat{v}(z, -n)$ and $\hat{z}(-n)$ as $n \to \infty$, it is useful to derive an upper bound. One method for doing so involves considering the continuous time procedure which calls for stopping when $Z_t \geqq 0$. An alternative, which would generalize to problems without a continuous time analogue, involves the procedure which calls for stopping when $Z_{-n} \geqq 0$. The two procedures are applied in the following lemmas to obtain the desired bound.

LEMMA 3.4. *The continuous time procedure which consists of stopping when* $Z_t \geqq 0$ *has risk* $v(z, t) = v(z, 0)$.

PROOF. This result could be established directly by an argument which involves integrating payoff and sampling time with respect to the probability elements corresponding to paths which terminate before time zero and those which intersect the t axis for Z_0 between y and $y + dy$ but which never go above the z axis.

As an alternative we simply note that for $z \leqq 0$, $v(z, t) = -z^2$ is a solution of the associated partial differential equation (3.1) which satisfies the boundary condition for $z = 0$, $t \leqq 0$ and for $t = 0$, $z \leqq 0$.

LEMMA 3.5.

(a) $-z^2 \leqq \hat{v}(z, -n) \leqq -z^2 + 1$ *for* $z \leqq 0$, $n \geqq 0$,

(b) $-1 \leqq \hat{z}(-n) < 0$ *for* $n > 0$.

PROOF. The first inequality of (a) is included in Lemma 3.3. Let v^* be the risk associated with the procedure which consists of terminating at $t = -n$ if Z_t goes above zero for some t between $-n-1$ and $-n$. Relating this procedure to that of Lemma 3.4, $v^*(z, t) \leqq 1 + v(z, 0)$. On the other hand it is suboptimal and hence the second inequality of part (a) follows. Thus $\hat{v}(z, -n) < 0$ for $z < -1$ and $\hat{z}(-n) \geqq -1$.

LEMMA 3.6. *The procedure which consists of stopping when* $Z_{-n} \geqq 0$ *has risk* $v(z, -n) \leqq v(z, 0) + \frac{1}{2}$. *Then* $\hat{v}(z, -n) \leqq v(z, 0) + \frac{1}{2}$, *and* $\hat{z}(-n) \geqq -1/2^{\frac{1}{2}}$.

PROOF. Let $X_i = Z_{-i+1} - Z_{-i}$, $z < 0$, and $Z_0 = z + X_n + X_{n-1} + \cdots + X_1$.

Then

$$E(n - Z_0^2) = -z^2 = E_1(n - Z_0^2) + E_2(n - Z_0^2),$$
$$v(z, -n) = E_1(n - Z_0^2) + E_2(n - M)$$

where E_1 is the contribution to the expectation of paths which do not stop till time 0, E_2 is the contribution of the remaining paths and $-M$ is the stopping time. Now

$$E\{Z_0^2 \mid Z_{-M}, M\} = M + Z_{-M}^2, \qquad E_2(n - Z_0^2) = E_2(n - M) - E_2(Z_{-M}^2).$$

Applying the type of argument used by Wald in deriving bounds for the operating characteristics of the sequential probability ratio test

$$E_2(Z_{-M}^2) = E_2\{\textstyle\int_0^\infty y^2\varphi(y - Z_{-M-1})\,dy\} \leqq J_2(0) = \tfrac{1}{2}.$$

Combining these equations the first part of the lemma follows. The rest is an immediate consequence.

THEOREM 3.2.

(a) *As* $n \to \infty$, $\hat{v}(z, -n)$, $\hat{h}(z, -n)$ *and* $\hat{z}(-n)$ *converge monotonically to* $\hat{v}(z)$, $\hat{h}(z)$ *and* $\hat{z}$, *where* $\hat{z}$ *is between* 0 *and* $-1/2^{\frac{1}{2}}$. *Also* $\hat{h}(z, -n)$ *and* $\hat{h}'(z, -n)$ *converge uniformly to* $\hat{h}(z)$ *and* $\hat{h}'(z)$ *for* z *bounded from below.*

(b) *The limiting functions satisfy*

$$\hat{h}(z) = 1 + T\hat{v}(z) \tag{3.13}$$
$$\hat{v}(z) = \min [\hat{h}(z), 0]$$

and $\hat{h}(z)$ *is strictly increasing, concave, and approaches* 1 *as* $z \to \infty$ *while* $\hat{v}(z) = 0$ *for* $z \geqq \hat{z}$ *and is negative for* $z < \hat{z}$.

(c) *The function* $v_0(z) = \hat{v}(z + \hat{z})$ *satisfies*

$$(3.14)\qquad \begin{aligned} v_0(z) &= 1 + Tv_0(z) && \text{for } z \leqq 0,\\ v_0'(z) &= Tv_0'(z) > 0 && \text{for } z < 0,\\ -(z + \hat{z})^2 \leqq v_0(z) &\leqq -(z + \hat{z})^2 + \tfrac{1}{2} && \text{for } z < 0. \end{aligned}$$

PROOF. The monotone convergence is trivial. We restrict ourselves to sets where z is bounded from below. Then $\hat{h}(z, -n)$ is bounded. Since $\hat{h}'(z, -n)$ is positive and decreasing, $\hat{h}'(z, -n)$ is bounded. Then every subsequence of $\hat{h}'(z, -n)$ has a uniformly convergent subsequence to a non-negative and monotone limit which must be $\hat{h}'(z)$. It follows that $\hat{h}(z, -n)$ and $\hat{h}'(z, -n)$ converge uniformly to $\hat{h}(z)$ and $\hat{h}'(z)$. We may take limits of both sides of the equations (3.7, 3.8) defining $\hat{h}(z, -n)$ and $\hat{v}(z, -n)$, since $|\hat{v}(z, -n)| \leqq z^2$. Part (b) follows readily.

Because $\hat{v}(z, -n) \geqq -z^2$ for $z \leqq 0$ and $\hat{h}(z, -n)$ is concave the tangent line at a point on the $\hat{h}(z, -n)$ curve does not intersect the parabola. This implies that $\hat{h}'(z, -n) \leqq -2z + 2$ for $z < 0$. With this bound the Lebesgue convergence theorem yields $\hat{h}' = T\hat{v}'$. Part (c) follows immediately.

COROLLARY 1.

(a) $\int_{-\infty}^{0-} e^{\lambda z} v_0''(z)\, dz$

$$= 2^{\frac{1}{2}}[1 - \exp\{-(1/2\pi)\int_{-\infty}^{\infty}[\lambda/(\lambda^2 + \xi^2)] \log [1 - e^{-\xi^2/2}]\, d\xi\}].$$

(b) $\hat{z} = -(1/2\pi)\int_{-\infty}^{\infty} \lambda^{-2} \log \{\lambda^2/2[1 - e^{-\lambda^2/2}]\}\, d\lambda = -.5824.$

(c) $v_0'(0) = 2^{\frac{1}{2}}$.

PROOF. Spitzer [6], [7] has shown that there is a unique non-decreasing positive solution of the equation

$$F(z) = \int_0^{\infty} F(\epsilon)\varphi(z - \epsilon)\, d\epsilon \tag{3.15}$$

subject to $F(0) = 1$ and that this function satisfies the following properties

$$1 + \int_{0+}^{\infty} e^{-\lambda x}\, dF(x) = \exp\{-(1/2\pi)\int_{-\infty}^{\infty} [\lambda(\lambda^2 + \xi^2)] \log [1 - e^{-\xi^2/2}]\, d\xi\}$$

$$\lim_{x\to\infty} (F(x)/x) = 2^{\frac{1}{2}}.$$

$$\lim_{x\to\infty} [(F(x)/2^{\frac{1}{2}}) - x] = (1/2\pi)\int_{-\infty}^{\infty} \lambda^{-2} \log \tfrac{1}{2}\lambda^2[1 - \exp(-\lambda^2/2)]^{-1}\, d\lambda.$$

For $c > 0$, $F(x) = cv_0'(-x)$ satisfies (3.15). Hence for the appropriate c, $cv_0'(0) = 1$, and $\lim_{z\to\infty} cv_0'(-z)/z = 2^{\frac{1}{2}}$. The bounds on $\hat{v}(z)$ of Theorem 3.2 imply that, if the limits exist as $z \to -\infty$, $v_0'(z)/z \to -2$, and $v_0'(z) + 2z \to -2\hat{z}$. Thus $c = 2^{-\frac{1}{2}}$ and (a), (b), and (c) follow.

For $0 < \delta < 1$, $\hat{v}(z, -n - \delta) = H_\delta \hat{v}(z, -n)$ converges to $H_\delta \hat{v}(z)$. Thus the following corollary will be useful.

COROLLARY 2.

(a) *$H_\delta \hat{v}(z)$ is strictly increasing and concave in z.*

(b) $-z^2 \leqq H_\delta \hat{v}(z) \leqq -z^2 + (\delta + 1)/2$ *for* $z \leqq 0$.

(c) $H_{\delta_1+\delta_2}\hat{v}(z) \leqq H_{\delta_1}\hat{v}(z) + \delta_2$.

(d) $H_{\delta_1+\delta_2}\hat{v}(z) \geqq H_{\delta_1}\hat{v}(z) + O(\delta_2^{\frac{1}{2}})$,

where the term O is uniform, for z bounded from below and δ_1 bounded, as $\delta_2 \to 0$. For z unbounded the O term is bounded by a constant as $\delta_2 \to 0$.

PROOF.

$$H_\delta \hat{v}(z) = \delta + \int_{-\infty}^{\infty} \hat{v}(z + \epsilon\sqrt{\delta})\varphi(\epsilon)\, d\epsilon = \delta + \int_{-\infty}^{(\hat{z}-z)\delta^{-\frac{1}{2}}} \hat{v}(z + \epsilon\sqrt{\delta})\varphi(\epsilon)\, d\epsilon$$

$$\frac{d}{dz} H_\delta\, \hat{v}(z) = \int_{-\infty}^{(\hat{z}-z)\delta^{-\frac{1}{2}}} \hat{v}'(z + \epsilon\sqrt{\delta})\varphi(\epsilon)\, d\epsilon > 0.$$

Increasing z decreases the integrand and the range of integration and (a) follows. Since $\hat{v}(z) \geqq -z^2$ for $z \leqq 0$, $H_\delta \hat{v}(z) \geqq \delta[1 - I_2(z/\sqrt{\delta})] = \delta\hat{h}(z/\sqrt{\delta}, -1) \geqq -z^2$. Since $\hat{v}(z) \leqq v(z, 0) + \frac{1}{2}$, $H_\delta \hat{v}(z) \leqq \delta\hat{h}(z/\sqrt{\delta}, -1) + \frac{1}{2} \leqq \delta[(-z^2/\delta) + \frac{1}{2}] + \frac{1}{2}$ for $z \leqq 0$. Thus we have part (b).

The concavity of $H_{\delta_1}\hat{v}$ implies that

$$H_{\delta_1+\delta_2}\hat{v}(z) = \delta_2 + E\{H_{\delta_1}\hat{v}(z + \epsilon\sqrt{\delta_2})\} \leqq \delta_2 + H_{\delta_1}\hat{v}(z).$$

Finally, in the proof of the theorem, we noted that $\hat{h}'(z, -n) \leqq -2z + 2$ for

$z < 0$. A similar argument yields $(d/dz)H_{\delta_1}\hat{v}(z) \leqq -2z + K_1$ for some constant K_1 if δ_1 is bounded. For z bounded from below and δ_1 bounded, there is a K_2 such that

$$\begin{aligned} H_{\delta_1}\hat{v}(z + \epsilon\sqrt{\delta_2}) &\geqq H_{\delta_1}\hat{v}(z) && \text{for } \epsilon > 0 \\ H_{\delta_1}\hat{v}(z + \epsilon\sqrt{\delta_2}) &\geqq H_{\delta_1}\hat{v}(z) + K_2\epsilon\sqrt{\delta_2} && \text{for } 0 \geqq \epsilon \geqq -\delta_2^{-\frac{1}{2}} \\ H_{\delta_1}\hat{v}(z + \epsilon\sqrt{\delta_2}) &\geqq -(z + \epsilon\sqrt{\delta_2})^2 && \text{for } \epsilon \leqq -\delta_2^{-\frac{1}{2}}. \end{aligned}$$

Part (d) follows by integrating.

4. Modified versions of the associated problem. For our purposes it is necessary to consider a modification of the associated problem. The modification will involve changing $v(z, 0)$, the time points at which stopping is permitted, and the cost of observation.

Two other changes are of general interest and will also be dealt with even though they are not necessary for the main result. First, the results of Section 3 indicate an iterative technique of approximating a solution of the equation $1 + Tv = v$ or of the equation $Tv = v$. We shall observe that changing $v(z, 0)$ by a function which is bounded does not affect $\lim_{n\to\infty} \hat{v}(z, -n)$. This implies that the iterative technique is stable to the extent that the limiting effect of an error approaches zero. We shall also show that the procedure of stopping when $Z_t \geqq 0$ is optimal among procedures where the time of stopping is not restricted.

For notational uniformity we use the following conventions. If $v(z, 0)$ is replaced by $v_i(z, 0)$, $v_i(z, t)$ will represent the risk associated with a given procedure. A circumflex is used to indicate the optimal risk, a dagger is used to indicate that the set of possible time points for stopping prior to $t = 0$ is $\{t_n : n = 1, 2, \cdots\}$, where $t_n \not\equiv -n$, and a subscript c is used to indicate that the cost of sampling is given by a rate $c(t) \not\equiv 1$. Finally, the optimal procedure for the continuous case where one is permitted to stop at any time is denoted by a tilde.

LEMMA 4.1. *Let $v(z, 0)$ of the associated discrete problem be replaced by $v_1(z, 0) = v(z, 0) - K$, $K > 0$. The optimal procedure consists of stopping if $Z_{-n} \geqq z^*(-n)$ and continuing if $Z_{-n} < z^*(-n)$. The corresponding optimal risk $\hat{v}_1(z, -n)$ is monotone increasing and concave in z. It is monotone increasing in n, $z^*(-n)$ is monotone decreasing in n, and $z^*(-n)$ is finite for n sufficiently large.*

PROOF. Let $\hat{h}_1(z, -n - 1) = 1 + \int_{-\infty}^{\infty} \hat{v}_1(z + \epsilon, -n)\varphi(\epsilon)\, d\epsilon$. Following the development of the preceding section, we find that $\hat{h}_1(z, -1)$ is concave with positive derivative and $\hat{h}_1(z, -1) \geqq v_1(z, 0)$. Hence

$$\hat{v}_1(z, -1) = \min [\hat{h}_1(z, -1), 0] \geqq v_1(z, 0)$$

and, as before, this implies that $\hat{v}_1(z, -n)$ is monotone increasing in n. Besides $\hat{v}_1(z, -1)$ is concave with non-negative derivative. By induction we find the concavity and positive derivative of $\hat{h}_1(z, -n)$ for all n. Thus the optimal procedure consists of stopping when $Z_{-n} > z^*(-n)$, which is the root of $\hat{h}_1(z, -n) = 0$, and continuing otherwise. Note that if $K > 1$, there may be no root $z^*(-n)$. In that case we may say $z^*(-n) = \infty$. In any case there is at most one root. The monotonicity of $\hat{v}_1(z, -n)$ in n implies the monotonicity of $z^*(-n)$. Note that if $\hat{v}_1(z, -n) \to c < 0$ as $z \to \infty$, $\hat{h}_1(z, -n-1) \to c + 1$. Thus $z^*(-n)$ becomes finite in a finite number of steps.

THEOREM 4.1. *Let $v(z, 0)$ of the associated discrete problem be replaced by a measurable function $v_1(z, 0)$ such that $|v(z, 0) - v_1(z, 0)| \leqq K$. Then the optimal risk $\hat{v}_1(z, t) = \hat{v}(z, t) + o(1)$ where the $o(1)$ term approaches zero as $t \to -\infty$ uniformly for z bounded from below. As $n \to \infty$ the optimal procedure for the modified problem calls for stopping when $Z_{-n} \geqq \hat{z} + o(1)$ and continuing if $Z_{-n} \leqq \hat{z} + o(1)$.*

PROOF. For any specified procedure, the difference between the risks for the modified and original problems $|v_1 - v|$ is bounded by K times the probability of reaching time 0. The optimal stopping sets for both problems include that for the problem where $v^*(z, 0) = v(z, 0) - K$. From Lemma 4.1 it follows that the probability of a path emanating from (z, t) reaching time 0 approaches 0 as $t \to -\infty$ uniformly for z in any interval bounded from below, for each optimal procedure. The risk, in the modified problem, for using the procedure optimal for the original problem is no larger than $\hat{v}(z, t) + o(1)$. Hence $\hat{v}_1(z, t) \leqq \hat{v}(z, t) + o(1)$. Similarly $\hat{v}(z, t) \leqq \hat{v}_1(z, t) + o(1)$.

Combining this with the trivial bound $|\hat{v}_1(z, t) - \hat{v}(z, t)| \leqq K$, it follows that as $n \to \infty$

$$\hat{h}_1(z, -n-1) = 1 + \int_{-\infty}^{\infty} \hat{v}_1(y, -n)\varphi(y - z)\, dy$$

converges uniformly to $\hat{h}(z)$ for z bounded from below. Finally, for z sufficiently negative, $\hat{h}_1(z, -n-1) \leqq \hat{h}(z, -n-1) + K + 1 < 0$. Since $\hat{v}_1(z, -n) = \min[\hat{h}_1(z, -n), 0]$ all stopping points $(z, -n)$ for the optimal procedure for the modified problem are such that z is bounded from below. Since $\hat{h}_1(z, -n)$ converges uniformly to the strictly monotone function $\hat{h}(z)$ for z bounded from below, the optimal procedure is as described in the statement of the theorem.

LEMMA 4.2. *If $|t_n + n| \leqq \eta < \frac{1}{2}$, and $t_{n+1} < t \leqq t_n$, then $|\hat{v}^\dagger(z, t) - H_{t_n - t}\hat{v}(z)| \leqq o(1) + \eta^{\frac{1}{2}}O(1)$ where o and O are uniform for z bounded from below as $t \to -\infty$. For z unbounded, o and O are uniformly bounded by a quadratic in z.*

PROOF. The optimal procedures with stopping restricted to times $\{-n + \eta\}$ and $\{-n - \eta\}$ have risks $\hat{v}_1(z, t + 1 - \eta)$ for $t + 1 - \eta < 0$ and $\hat{v}_2(z, t + 1 + \eta)$ for $t + 1 + \eta < 0$ where $v_1(z, 0) = \min[H_{1-\eta}v(z, 0), 0]$ and $v_2(z, 0) = \min[H_{1+\eta}v(z, 0), 0]$. Both $v_1(z, 0) - v(z, 0)$ and $v_2(z, 0) - v(z, 0)$ are bounded and hence $\hat{v}_1(z, t) = v(z, t) + o(1)$ and $\hat{v}_2(z, t) = v(z, t) + o(1)$.

The difference between the risks for (i) the optimal procedure for the modified time problem and (ii) the procedure which stops at times $-n + \eta$ if the former stops at time t_n is less than 2η. Hence

$$\hat{v}^\dagger(z, t_n) \geqq \hat{v}_1(z, t_n + 1 - \eta) - 2\eta. \tag{4.1}$$

Similarly, we compare (i) the optimal procedure subject to stopping at times $-n + 1 - \eta, -n + 2 - \eta, \cdots, -1 - \eta$, with (ii) the procedure which stops at

t_r if the former stops at $-r - \eta$, $r \leqq n - 1$. Then

$$\hat{h}^\dagger(z, t_n) = H_{t_{n-1}-t_n}\hat{v}^\dagger(z, t_{n-1}) \leqq H_{1-n-t_n-\eta}\hat{v}_2(z, -n+2) + 2\eta. \tag{4.2}$$

Let $\delta_1 = \eta - t_n - n \leqq 2\eta$ and $\delta_2 = \eta + n + t_n \leqq 2\eta$. Then applying part (d) of Corollary 2

$$\hat{v}^\dagger(z, t_n) \geqq H_{\delta_1}[\hat{v}(z) + o(1)] - 2\eta \geqq \hat{v}(z) + o(1) + \eta^{\frac{1}{2}}O(1)$$
$$\hat{h}^\dagger(z, t_n) \leqq H_{1-\delta_2}[\hat{v}(z) + o(1)] + 2\eta \leqq \hat{h}(z) + o(1) + \eta^{\frac{1}{2}}O(1)$$
$$\hat{v}^\dagger(z, t_n) \leqq \hat{v}(z) + o(1) + \eta^{\frac{1}{2}}O(1).$$

The lemma follows easily.

Let $u_r(z) = -z^r$ for $z \leqq 0$ and 0 for $z \geqq 0$.

LEMMA 4.3.

$$S_\delta[u_1(z)] \leqq u_1(z) + \delta^{\frac{1}{2}}/(2\pi)^{\frac{1}{2}} \tag{4.3}$$

$$S_\delta[u_3(z)] \leqq u_3(z) + 3\delta u_1(z) + 2\delta^{\frac{3}{2}}/(2\pi)^{\frac{1}{2}}. \tag{4.4}$$

PROOF.

$$S_\delta[u_{2n+1}(z)] = -\delta^{n+\frac{1}{2}}I_{2n+1}(z\delta^{-\frac{1}{2}}) = -E[(z + \epsilon\delta^{\frac{1}{2}})^{2n+1}] + \delta^{n+\frac{1}{2}}J_{2n+1}(z\delta^{-\frac{1}{2}}).$$

Also

$$-I_{2n+1}(z) \leqq -I_{2n+1}(0) = J_{2n+1}(0) = 2^n n!/(2\pi)^{\frac{1}{2}} \qquad \text{for} \quad z \geqq 0$$
$$J_{2n+1}(z) \leqq J_{2n+1}(0) \qquad \text{for} \quad z \leqq 0.$$

Substituting in the first equality for S_δ for $z \geqq 0$ and the second for $z \leqq 0$, the result follows.

LEMMA 4.4. *If* $|t_n + n| \leqq \eta_1 < \frac{1}{2}$, *and* $v_2(z, 0) \leqq v_1(z, 0) + \eta_2 u_3(z)$, $\eta_2 > 0$, *then there is a constant K such that*

$$\hat{v}_2^\dagger(z, t) \leqq \hat{v}_1^\dagger(z, t) + \eta_2[u_3(z) + 3|t|u_1(z) + K(t^2 - t)]. \tag{4.5}$$

PROOF. Applying Lemma 4.3, the result holds for $t_1 \leqq t \leqq 0$. Suppose it holds for $t_n \leqq t \leqq 0$. Then

$$H_\delta\hat{v}_2^\dagger(z, t_n) \leqq H_\delta\hat{v}_1^\dagger(z, t_n) + \eta_2 S_\delta[u_3(z) + 3|t_n|u_1(z) + K(t_n^2 - t_n)$$
$$\leqq H_\delta\hat{v}_1^\dagger(z, t_n) + \eta_2[u_3 + 3|t_n| + \delta]u_1$$
$$+ K(t_n^2 - t_n) + [2\delta + 3|t_n|]\delta^{\frac{1}{2}}/(2\pi)^{\frac{1}{2}}].$$

Applying this inequality for $\delta < t_n - t_{n+1}$, gives the desired result for $t_{n+1} < t \leqq 0$. Applying the inequality for $\delta = t_n - t_{n+1}$ and using the fact that $\hat{v}_i^\dagger(z, t_{n+1}) = \min [H_{t_n - t_{n+1}}\hat{v}_i^\dagger(z, t_n), 0]$, the result follows for $t = t_{n+1}$, which completes the induction.

The following lemma is trivial and is stated without proof.

LEMMA 4.5. *If two modifications of the associated problem have the same stopping* *times t_n and initial risk $v_1(z, 0)$ but differ in their observation cost rates $c_1(t)$ and $c_2(t)$, then*

$$\hat{v}^\dagger_{1c_1}(z, t) \leqq \hat{v}^\dagger_{1c_2}(z, t) + \sup_{t<t'<0} \int_t^{t'} [c_1(x) - c_2(x)]\, dx.$$

THEOREM 4.2. *If the associated problem is modified so that*

(i) *the possible stopping times are t_n, where $|t_n + n| \leqq \eta_1 < \frac{1}{2}$,*

(ii) *$v(z, 0)$ is replaced by $v_1(z, 0)$ where $v_1(z, 0) = v(z, 0) = 0$ for $z \geqq 0$ and $|v_1(z, 0) - v(z, 0)| < -\eta_2 z^3$ for $z \leqq 0$, and*

(iii) *the observation cost rate is $c(t)$ where $|c(t) - 1| < \eta_3$, then as $t \to -\infty$*

$$(4.6)\qquad \begin{aligned} |\hat{v}^\dagger_{1c}(z, t) - H_{t_n - t}\hat{v}(z)| \leqq \eta_3|t| + \eta_2[u_3(z) + 3|t|u_1(z) + K(t^2 - t)] \\ + (\eta_1)^{\frac{1}{2}}O(1) + o(1) \qquad \text{if} \quad t_{n+1} < t \leqq t_n \end{aligned}$$

and the same bound applies to $\hat{h}^\dagger_{1c}(z, t_n) - \hat{h}(z)$. The o and O terms are uniform for z bounded from below and the η_i bounded. They are also bounded by a quadratic in z for all z.

PROOF. From Lemma 4.5 we have $|\hat{v}^\dagger_{1c}(z, t) - \hat{v}_1{}^\dagger(z, t)| \leqq \eta_3|t|$. Then applying Lemma 4.4 $|\hat{v}_1{}^\dagger(z, t) - \hat{v}^\dagger(z, t)| \leqq \eta_2[u_3(z) + 3|t|u_1(z) + K(t^2 - t)]$. Finally, from Lemma 4.2, $|\hat{v}^\dagger(z, t) - H_{t_n - t}\hat{v}(z)| \leqq o(1) + (\eta_1)^{\frac{1}{2}}O(1)$. Thus (4.6) follows. As $t \to t_{n+1}$ from above $\hat{v}^\dagger_{1c}(z, t) \to \hat{h}^\dagger_{1c}(z, t)$ and $H_{t_n - t}\hat{v}(z) \to H_{t_n - t_{n+1}}\hat{v}(z)$. Since $|t_n - t_{n+1} - 1| \leqq 2\eta_1$ and $H_1\hat{v} = \hat{h}$, $|H_{t_n - t_{n+1}}\hat{v}(z) - \hat{h}(z)| \leqq (\eta_1)^{\frac{1}{2}}O(1)$. Thus we have the desired result for $\hat{h}^\dagger_{1c} - \hat{h}$.

Inasmuch as a number of variations of the associated problem have been discussed, we digress briefly to study the continuous time version of the problem.

LEMMA 4.6. *The associated problem is invariant under the transformations $t^* = a^2 t$, $Z^*_{t^*} = aZ_t$, and $v^*(z^*, t^*) = a^2 v(z, t)$. [The stopping times are transformed accordingly.]*

The proof is trivial but the result will be used to relate the continuous and discrete versions of the problem.

THEOREM 4.3. *The optimal procedure for the continuous version of the associated problem [with the original initial risk $v(z, 0)$] consists of stopping when $Z_t \geqq 0$ and continuing otherwise. This procedure has risk $\bar{v}(z, t) = v(z, 0)$.*

PROOF. Applying Lemma 4.6, the discrete problem converts to one in which the stopping times are $t_n{}^* = -na^2$ and $v^*(z^*, 0) = a^2 v(z, 0) = v(z^*, 0)$. Hence the original solution with risk $v(z, t)$ converts to stopping if $Z^*_{-na^2} \geqq a\hat{z}(-n)$ which has risk $\hat{v}^*(z^*, t^*) = a^2\hat{v}(z^*/a, t^*/a^2)$. The transformed continuous problem goes into itself. But $|\hat{v}^*(z^*, t^*) - \bar{v}^*(z^*, t^*)| \leqq a^2$ and thus the transformed discrete solution and risk converge to the continuous solution and risk as $a \to 0$. Applying Lemma 3.5 with $n = -t^*/a^2$ where $a^2 \to 0$ so that n is integral, $\hat{v}^*(z^*, t^*) \to v(z^*, 0)$ and the theorem follows.

5. The comparison of the discrete and continuous solution. To relate the discrete and continuous solutions for the sequential analysis problem, we find it convenient to convert the latter to the $(y, t) = [(x/t), t]$ scale. Let $\tilde{y}$ be the upper boundary for the optimal continuous time procedure and $\tilde{y}_\delta$ be the upper boundary for the optimal discrete time procedure where the stopping times are $t_0 \pm n\delta$.

We introduce several new procedures. Let y_δ^* represent the optimal procedure subject to the restriction that for $t < t_0$, one can stop only at the discrete time points $t_0 - \delta, t_0 - 2\delta, \cdots$. For $t \geqq t_0$, y_δ^* coincides with $\tilde{y}$. Let y_δ^{**} coincide with y_δ^* for $t < t_0$ and with $\tilde{y}_\delta$ for $t \geqq t_0$. (The risks will use the same designation as the procedures. Thus $\tilde{b}(y, t) = B(x, t)$ and $\tilde{b}_\delta(y, t) = B_\delta(x, t)$.)

The four risk functions are closely related. It is clear that

$$\tilde{b} \leqq b_\delta^* \leqq \tilde{b}_\delta \leqq b_\delta^{**}.$$

Furthermore comparing $\tilde{b}_\delta$ with the risk of the non-optimal procedure which stops at $t_0 + r\delta$ if $\tilde{y}$ leads to stopping for $t_0 + (r - 1)\delta < t \leqq t_0 + r\delta$, we see that $\tilde{b}_\delta \leqq \tilde{b} + \delta$. For $t \geqq t_0$, $b_\delta^{**} = \tilde{b}_\delta$ and $b_\delta^* = b$ and hence $0 \leqq b_\delta^{**}(y, t) - b_\delta^*(y, t) \leqq \delta$ for $t \geqq t_0$. But y_δ^* and y_δ^{**} coincide for $t > t_0$ and thus the above inequality holds also for $t > t_0$. Thus

$$0 \leqq b_\delta^{**}(y, t) - \tilde{b}(y, t) \leqq 2\delta.$$

The outermost boundary, i.e., the procedure with the largest continuation set at the times $t_0 - n\delta$ and $t \geqq t_0$, is $\tilde{y}$. But $\tilde{b}(\tilde{y} - a, t) - d(\tilde{y} - a, t) \approx -a^2t^2$ for small a. This is due to the fact that $B_{xx} - D_{xx} = -2$ and $\tilde{b}_{yy} - d_{yy} = -2t^2$ at the optimal boundary. Hence the boundaries are all within $O(\delta^{\frac{1}{2}})$ of one another, where the O is uniform for t in any interval I bounded away from zero and infinity where b_{yyy} is uniformly bounded within some fixed distance of $\tilde{y}$.

Consider a process Y_t starting at $(y_0 + \delta^{\frac{1}{2}}z, t_0 - n\delta)$ where $y_0 = \tilde{y}(t_0)$. Let $n \to \infty$ in such a way that $n = o(\delta^{-1})$. For each of the four procedures, the probability of termination along the upper boundary before time t_0 approaches one as $n \to \infty$ since the increments of Y_t are normally distributed with means 0 and variances $(t_0 - n\delta)^{-1} - (t_0 - (n - 1)\delta)^{-1} \approx \delta t_0^{-2}$. Hence for each of these procedures, $b(y_0 + \delta^{\frac{1}{2}}z, t_0 - n\delta)$ can be expressed as an 'average' of values of d along the boundary and of $\tilde{b}(y, t_0)$ or $\tilde{b}_\delta(y, t_0)$ where the weight attached to $\tilde{b}(y, t_0)$ or $\tilde{b}_\delta(y, t_0)$ approaches zero as $n \to \infty$. Thus the direct effect (in this weighted average) of a discrepancy of $O(\delta)$ in $b\ (y, t_0)$ is $o(\delta)$ on $b(y_0 + \delta^{\frac{1}{2}}z, t_0 - n\delta)$.

Since y_δ^* and y_δ^{**} coincide for $t < t_0$

$$b_\delta^{**}(y_0 + \delta^{\frac{1}{2}}z, t_0 - n\delta) - b_\delta^*(y_0 + \delta^{\frac{1}{2}}z, t_0 - n\delta) = o(\delta). \tag{5.1}$$

As indicated above $b_\delta^* \leqq \tilde{b}_\delta \leqq b_\delta^{**}$. Thus it follows that

$$\tilde{b}_\delta(y_0 + \delta^{\frac{1}{2}}z, t_0 - n\delta) - b_\delta^*(y_0 + \delta^{\frac{1}{2}}z, t_0 - n\delta) = o(\delta). \tag{5.2}$$

Now we introduce a variation $\hat{y}_\delta$ of y_δ^* by not permitting the rejection of $H_1 : \mu > 0$ before time t_0. Suppose n is large but $n\delta$ is small and $y > a > 0$. The probability that a process starting at $(y, t_0 - n\delta)$ would lead to rejection of H_1 before time t_0 (using the y_δ^* procedure) is $o[\exp(-a^2/2n\delta)]$. We may adjust

$n \to \infty$, e.g., $n = o(\delta^{-\frac{1}{4}})$, so that this probability is $o[\delta^2]$. Then we have

$$(5.3) \quad \hat{b}_\delta(y_0 + \delta^{\frac{1}{2}}z, t_0 - n\delta) - b_\delta^*(y_0 + \delta^{\frac{1}{2}}z, t_0 - n\delta) = o(\delta) \qquad \text{for } y_0 + \delta^{\frac{1}{2}}z > a.$$

Finally we relate $\hat{y}_\delta$ to the associated problem. Let

$$(5.4) \quad \begin{aligned} t^* &= t_0^2\delta^{-1}[t_0^{-1} - t^{-1}], \qquad z = (y - y_0)t_0\delta^{-\frac{1}{2}}, \\ & \qquad\qquad v_1(z, 0) = [\tilde{b}(y, t_0) - d(y, t_0)]\delta^{-1}. \end{aligned}$$

Restricting ourselves to $t \varepsilon I$, we have $\tilde{b}_{yy} - d_{yy} = -2t^2$ at the optimal boundary, $\tilde{b} - d$ is bounded, and $\tilde{b}(y, t) - d(y, t) + 2t^2(y - \tilde{y})^2 \leqq K|y-\tilde{y}|^3$ for some K and for y in the continuation region. It follows that

$$(5.5) \quad \begin{aligned} v_1(z, 0) &= 0 && \text{for } z > 0 \\ v_1(z, 0) &= -z^2 + z^3O[\delta^{\frac{1}{2}}] && \text{for } z \leqq 0. \end{aligned}$$

With this transformation the $\hat{y}_\delta$ problem becomes a variation of the associated problem, where the initial risk is $v_1(z, 0)$ the stopping times are

$$t_n^* = t_0^2\delta^{-1}[t_0^{-1} - (t_0 - n\delta)^{-1}] = -n + O(n^2\delta),$$

and the rate of sampling cost is δ^{-1} per unit time in the t scale and hence $[\delta\, dt^*/dt]^{-1} = 1 + O(n\delta)$.

Now we let $n \to \infty$ so that $n^4\delta \to 0$ and apply Theorem 4.2. It follows that for z bounded

$$\hat{b}_\delta(y_0 + \delta^{\frac{1}{2}}zt_0^{-1}, t_0 - n\delta) - d(y_0 + \delta^{\frac{1}{2}}zt_0^{-1}, t_0 - n\delta) = \delta[\hat{v}(z) + O(n^2\delta) + o(1)].$$

Hence

$$(5.6) \quad \begin{aligned} &\tilde{b}_\delta(y_0 + \delta^{\frac{1}{2}}zt_0^{-1}, t_0 - n\delta) - d(y_0 + \delta^{\frac{1}{2}}zt_0^{-1}, t_0 - n\delta) \\ &\qquad\qquad = \delta[\hat{v}(z) + O(n^2\delta) + o(1)] \end{aligned}$$

and the $\tilde{y}_\delta$ procedure calls for stopping if $Y_{t_0-n\delta} > y_0 + \delta^{\frac{1}{2}}t_0^{-1}[\hat{z} + o(1)]$ and continuing if $Y_{t_0-n\delta} < y_0 + \delta^{\frac{1}{2}}t_0^{-1}[\hat{z} + o(1)]$. Finally three steps lead to the following theorem. First we apply Theorem 4.2 to time $t_0 - (n + \rho)\delta, 0 \leqq \rho < 1$. Then we note that (5.6) holds uniformly for t_0 in I, and t_0 may be shifted to $t_0 + n\delta$ if $t_0 + n\delta \varepsilon I$. Finally we transform to the (x, t) space. We have

THEOREM 5.1. *The relation between the optimal discrete and optimal continuous procedures are described by*

(a) $\tilde{x}_\delta(t) = \tilde{x}(t) + \delta^{\frac{1}{2}}[\hat{z} + o(1)]$

(b) $B_\delta(\tilde{x} + \delta^{\frac{1}{2}}z, t_0 - \rho\delta) - D(\tilde{x} + \delta^{\frac{1}{2}}z, t_0 - \rho\delta) = \delta H_\rho\hat{v}(z) + o(\delta), 0 \leqq \rho < 1$

where o applies uniformly for ρ and z bounded and t in an interval I bounded away from 0 and ∞ where B_{xxx} is uniformly bounded within some fixed distance from $\tilde{x}$.

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