Sequentially Rationalizable Choice

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A sequentially rationalizable choice function is a choice function that can be retrieved by applying sequentially to each choice problem the same fixed set of asymmetric binary relations (rationales) to remove inferior alternatives. These concepts translate into economic language some human choice heuristics studied in psychology and explain cyclical patterns of choice observed in experiments. We study some properties of sequential rationalizability and provide a full characterization of choice functions rationalizable by two and three rationales. (*JEL* D01).

Cyclical choice is persistently observed in experimental evidence. It typically occurs in simple decision problems (involving only binary comparisons and few alternatives) and in significant proportions, sometimes nearing or even exceeding 50 percent.[1] This is obviously incompatible with the classical model of rational choice, in which choice is constructed as the maximizer of a single preference relation (which we call a *rationale*), or of a utility function. If a decision maker exhibits cycles of choice over some set of alternatives, for any candidate "best" alternative there is always another one in the set that is judged better still: it is not possible to express a decision maker's preferences by a utility function, since it is not possible to find a maximizer for it. In this paper, we propose and study a family of boundedly rational choice procedures that can account for these observed anomalies.

In line with some prominent psychology and marketing studies (see below), in our model we assume that the decision maker uses sequentially two rationales to discriminate among the available alternatives. These rationales are applied in a fixed order, independently of the choice set, to remove inferior alternatives. This procedure "sequentially rationalizes" a choice function if, for any feasible set, the process identifies the unique alternative specified by the choice function. In this case, we say that a choice function is a *Rational Shortlist Method* (RSM). Intuitively, the first rationale identifies a shortlist of candidate alternatives from which the second rationale selects. The special case in

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[1] See, e.g., Amos Tversky (1969), Graham Loomes, Chris Starmer, and Robert Sugden (1991), and Peter H. M. P. Roelofsma and Daniel Read (2000). Roelofsma and Read (2000) find that the majority (52 percent) of choices exhibited binary cycles in a universal choice set of four alternatives. In the experiment carried out in Loomes, Starmer, and Sugden (1991), between 14 percent and 29 percent of choices made by all subjects were cyclical, and a staggering 64 percent of subjects exhibited at least one binary cycle in a universal choice set of just three alternatives. More recent results in this same line are in Pavlo Blavatskyy (2003), who finds that 55 percent of his experimental subjects violate transitivity of choice. Humans seem to fare better than nonhuman animals: for instance, in an experiment of choice behavior of gray jays, Thomas A. Waite (2001) finds that *all* the birds preferred choices a to b and b to c, but none preferred a over c, where all alternatives (n, l) consisted in going and getting n raisins at the end of a $l$cm long tube, with $a = (1 \text{ raisin, } 28 \text{ cm})$, $b = (2 \text{ raisins, } 42 \text{ cm})$, and $c = (3 \text{ raisins, } 56 \text{ cm})$. Thus, none of the birds exhibited transitive choice; moreover, 25 percent of them exhibited consistently intransitive choice.

which the first rationale always yields a unique maximal element corresponds to the standard model of rationality.

A notable aspect of these procedures is that they are testable based on a "revealed preference" type of analysis that, despite the highly nonstandard choices to be explained, is not more demanding than the standard one.[2] In other words, we ask the following question: when are observed choices compatible with the use of our boundedly rational choice procedure? The answer is: if and only if the choice data satisfy two testable conditions. Of these conditions, one is a standard Expansion axiom, and the other is a modification of Samuelson's Weak Axiom of Revealed Preference (WARP).[3] The simplicity of our tests stands in contrast to the indirect estimation algorithms normally used (notably in the marketing literature) to infer boundedly rational procedures.[4]

Typically, RSMs will lack standard menu-independence properties, so that it may be possible for an alternative to be revealed as preferable to another alternative in some choice set, but for that preference to be reversed in a different choice set (thus violating WARP). Because of this feature, RSMs can exhibit cyclical patterns of choice; however, they still rule out other types of irrational choice. In this sense, an RSM is a nonvacuous notion and this gives it empirical content: it can be tested by observable choice data.

For a simple example of how an RSM works, suppose that an arbitrator has to pick one from the available allocations a, b, or c. Suppose that c Pareto dominates a, while no other Pareto comparisons are possible. Assume further that the arbitrator deems a fairer than b and b fairer than c. The arbitrator decides first on the basis of the Pareto criterion, invoking the fairness criterion only when Pareto is not decisive. Then, the arbitrator's choice function γ would be such that $\gamma(\{a,b,c\}) = b$, since, first, a is eliminated by c using the Pareto criterion, and, second, c is eliminated by b using the fairness criterion. On the other hand, $\gamma(\{a,b\}) = a$, given that the Pareto criterion has no bite, and the arbitrator would select on the basis of fairness. Similarly, $\gamma(\{b,c\}) = b$, whereas $\gamma(\{a,c\}) = c$ by Pareto. This seems an entirely reasonable way for the arbitrator to come to a decision. In fact, this procedure has been proposed in a social choice setting by Koichi Tadenuma (2002). Yet it produces a violation of WARP and pairwise cyclical pattern of choice.

One can think of a wide array of other practical situations where RSMs may apply. A cautious investor comparing alternative portfolios first eliminates those that are too risky relative to others available, and then ranks the surviving ones on the basis of expected returns. A recruiting selector first excludes candidates with lower levels of some desired skills than other applicants he is considering, and then selects based on merit from the remaining ones. The notion of RSM is relevant also in other fields in the social sciences. For instance, psychologists have often insisted on sequential "noncompensatory"[5] heuristics, as opposed to one single rationale, to explain choices (though axiomatic characterizations of such boundedly rational procedures are lacking). Notable in this respect are the "Elimination by Aspects" procedure of Tversky (1972) and the idea of "fast and frugal heuristics" of Gerd Gigerenzer, Peter M. Todd, and the ABC Research Group (1999). Similarly, this type of model is widely used and documented in the management/marketing literature. Yee et al. (forthcoming) provide recent and compelling evidence of the use by consumers of "two-stage consideration and choice" decision-making procedures, and also refer to firms taking account of this fact in product development.

In summary, RSMs are simple boundedly rational procedures that are introspectively plausible and can explain empirically relevant "anomalies" of choice patterns. Above all, whether the choice pattern of a decision maker can be explained by an RSM is a testable hypothesis. Last but not least, RSMs provide rigorous formal underpinnings to the heuristics

[2] See Hal R. Varian (2005) for a recent survey on standard revealed preference theory.

[3] Recall that WARP, in its general form, states that if an alternative a is chosen from some menu of alternatives where some other alternative b is present (i.e., a is directly revealed preferred to b), then it can never be the case that alternative b is selected from any other menu including both a and b.

[4] For recent examples, see, e.g., Michael Yee et al. (forthcoming) and Rajeev Kohli (forthcoming).

[5] That is, in which the several "criteria" used for choice cannot be traded off against each other.

approach central to much psychology and marketing literature.

In addition to providing a characterization of RSMs, we consider a natural extension whereby the decision maker applies sequentially more than two rationales, much in the same way as they are used in the elimination procedure described before for RSMs. We call choice functions recoverable in this way *sequentially rationalizable*. Although a full characterization of sequentially rationalizable choice functions remains a nontrivial open problem, we are able to present some partial results, notably including a full characterization of rationalizability by *three* rationales. Interestingly, even when the number of rationales allowed is unboundedly large, not all choice functions are sequentially rationalizable.

The rest of the paper is organized as follows. In the next section we define and characterize RSMs. In Section II we extend RSMs to sequential rationalizability. Section III presents an application to choice over time. We conclude in Section IV. Some technical examples are in the Appendices.

I. Rational Shortlist Methods

A. *Basic Definitions*

Let X be a set of alternatives, with $|X| > 2$. Given $S \subseteq X$ and an asymmetric binary relation $P \subseteq X \times X$, denote the set of P-maximal elements of S by

$$\max(S;P) = \{x \in S | \nexists y \in S \text{ for which } (y,x) \in P\}.$$

Let $\mathcal{P}(X)$ denote the set of all nonempty subsets of X. A *choice function* on X selects one alternative from each possible element of $\mathcal{P}(X)$, so it is a function $\gamma : \mathcal{P}(X) \rightarrow X$ with $\gamma(S) \in S$ for all $S \in \mathcal{P}(X)$. We abuse notation by often suppressing set delimiters, e.g., writing $\gamma(xy)$ in place of $\gamma(\{x,y\})$.

The main result in this section (Theorem 1) goes through (as can be easily checked by an inspection of the proof) whether the choice sets S are finite or not. For simplicity of notation, however, we confine ourselves to the case where X is finite.

Since Paul A. Samuelson's (1938) paper, economists have sought to express choice as the outcome of maximizing behavior. Formally, a choice function γ is *rationalizable* if there exists an acyclic binary relation P, such that

$$\{\gamma(S)\} = \max(S;P) \text{ for all } S \in \mathcal{P}(X).$$

The main new concept we introduce is the following.

DEFINITION 1: *A choice function γ is an RSM whenever there exists an ordered pair (P_1, P_2) of asymmetric relations, with $P_i \subseteq X \times X$ for $i = 1, 2$, such that:*

$$\{\gamma(S)\} = \max(\max(S;P_1);P_2) \text{ for all } S \in \mathcal{P}(X).$$

In that case we say that (P_1, P_2) sequentially rationalize γ. We call each P_i a rationale.

So the choice from each S can be represented as if the decision maker went through two sequential rounds of elimination of alternatives. In the first round, he retains only the elements that are maximal according to rationale P_1. In the second round, he retains only the element that is maximal according to rationale P_2: that is his choice. Note that, crucially, the rationales and the sequence in which they are applied are invariant with respect to the choice set.

This choice procedure departs from (standard) rational choice only when the relation P_1 is incomplete. The relation P_2 may or may not be complete, though it needs to be decisive on the shortlist created after the first round of elimination, i.e., select from it a single element.

B. *An Example*

To glean some intuition on what RSMs can and cannot do, let us consider an example where two types of "pathologies of choice" are displayed. We show in the next section that the decomposition of pathologies illustrated in the example is very general; of these, only one can be accommodated by an RSM.

Suppose that the decision maker can conceivably choose among three alternative routes to go to work, A, B, and C. Because of periodic road closures, we can observe his choices also between subsets of the grand set $\{A, B, C\}$. Up to a relabelling of the alternatives, it is not difficult to check that there are only three possible configurations of choice behavior. Fix the route that is taken when all are available, say route *A*. Then, consider the situation when, at any one time, only two routes are available. Those that follow exhaust all possible choices:[6]

Case 1 (Dominance of the best route).—Route A (the choice from the grand set) is also taken whenever only one other route is available, regardless of the choice when A is not available.

Case 2 (Pairwise cycle of choice).—Route *A* is taken when *B* is the only other available route; route *B* is taken when *C* is the only other available route; route *C* is taken when *A* is the only other available route.

Case 3 (Default route).—Some route different from *A* is always taken when only one other route is available, regardless of the choice when *A* is available.

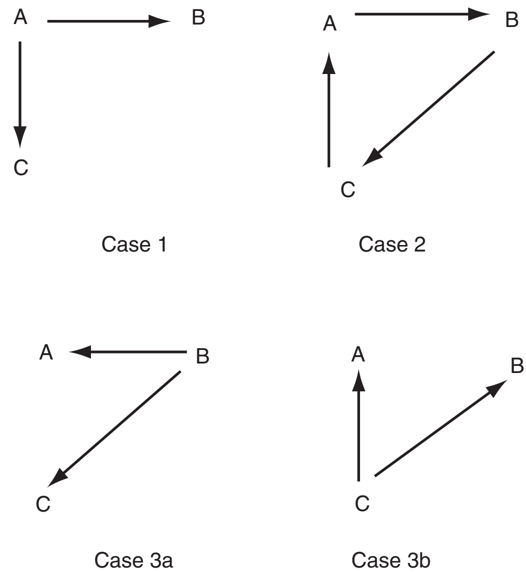

FIGURE 1

These cases are depicted in Figure 1, where arrows point away from the selected route to the rejected one in pairwise choice.

Case 1 can be rationalized in the standard way, with only one transitive preference relation such that *A* is preferred to both *B* and *C*.

Case 2 is pathological from the point of view of standard economic rationality. Nonetheless, it can be sequentially rationalized by two rationales—let us call them "traffic" and "length"—as follows. The decision maker prefers less traffic to more, and prefers shorter routes. Route *C* is shorter than Route *A,* and Route *A* is shorter than route *B*. Route *B* has less traffic than route *C*, but traffic comparisons are hard to make between other routes. The decision maker looks first at traffic to eliminate routes, and then at length. It is immediate to see that the criteria applied in the given sequence generate the choice behavior of Case 2.

In Case 3, a different pathology of choice is observed. There is one route, say *B* to fix ideas, that is revealed preferred in pairwise choices to all other routes, yet it is not chosen when all routes are available (as in Figure 1, Case 3a). This pattern of choice is not an RSM. To see this, suppose to the contrary that this were an RSM, again with rationales "traffic" and "length" applied in that order. If so, the fact that *B* is chosen in pairwise comparison over *A* means that if *B* and *A* are comparable by traffic, then *B* has less traffic than *A*. Otherwise, *B* must be shorter than *A*. Similarly, since *B* is chosen in pairwise comparison over *C*, either *B* has less traffic than *C*, or is shorter (or both). But then, when all three routes are available, *B* can never be eliminated by either the traffic or the length criterion. This contradicts the initial hypothesis that the choice was an RSM. We shall see later that this reasoning can be generalized to more complex cases, and in fact it would stand even if the number of possible criteria were not limited

[6] Let *XtY* denote "route *X* is taken when route *Y* is also available." Then, it is easy to see that, once we fix the route selected when all are available, there are eight possible combinations of routes chosen in each of the three possible pairwise comparisons between *A* and *B*, *A* and *C,* and *B* and *C*, namely: (1) *AtB*, *AtC*, and *BtC*; (2) *AtB*, *AtC,* and *CtB*; (3) *AtB*, *BtC,* and *CtA*; (4) *BtA*, *AtC,* and *CtB*; (5) *BtA*, *BtC,* and *AtC*; (6) *BtA*, *BtC*, and *CtA*; (7) *CtA*, *CtB*, and *AtB*; and (8) *CtA*, *CtB*, and *BtA*. Of these possibilities, (1) and (2) correspond to Case 1 in the text; (3) and (4) are the same, subject to relabelling by switching *B* and *C*, and correspond to Case 2 in the text; and, finally, both (5) and (7), and (6) and (8) are the same subject to swapping *B* for *C*, and correspond to Case 3 in the text.

to two. It is this type of pathological behavior that gives our theory empirical content.

C. *Characterization of Rational Shortlist Methods*

In general, suppose that we observed the choices of a decision maker. How could we test whether his behavior is consistent with the sequential maximization of two rationales? Surprisingly, it turns out that RSMs can be simply characterized through two familiar observable properties of choice.

Recall, first, the standard WARP pioneered by Samuelson (1938) for consumer theory.

WARP: *If an alternative x is chosen when y is available, then y is not chosen when x is available. Formally, for all S, T* $\in \mathcal{P}(X)$: $[x = \gamma(S), y \in S, x \in T] \Rightarrow [y \neq \gamma(T)]$.

It is well known that (in the present setting) WARP is a necessary and sufficient condition for choice to be rationalized by an ordering (i.e., a complete transitive binary relation).[7] WARP essentially asserts the absence of a certain type of "menu effects" in choice: if an alternative is revealed preferred to another within a certain "menu" of alternatives, changing the menu cannot reverse this judgement. The property we introduce allows menu effects, but requires some consistency in the way they operate. It is in the following spirit: if you are observed to choose steak over fish when they are the only items on the menu, and also when a *large* selection of pizzas is on the menu, then you do not choose fish over steak when a *small* selection of pizzas is on the menu. A pairwise preference for *x* over *y* does not exclude in principle that in larger menus some reason can be found to reject *x* and choose *y* instead. However, if a large menu does not contain any such reason, no smaller menu contains such a reason either. Although this property may look introspectively plausible, here we are not interested in issues of plausibility: we simply propose this property as an observable test for the RSM model.

[7] See, e.g., Hervé Moulin (1985) and Kotaro Suzumura (1983).

WEAK WARP: *If an alternative x is chosen both when only y is also available and when y and other alternatives* $\{z_1, \ldots, z_K\}$ *are available, then y is not chosen when x and a subset of* $\{z_1 \ldots, z_K\}$ *are available. Formally, for all S, T* $\in \mathcal{P}(X)$: $[\{x, y\} \subset S \subset T, x = \gamma(xy) = \gamma(T)] \Rightarrow [y \neq \gamma(S)]$.

The second property we use in our characterization is called Expansion, and it directly rules out pathologies of the type considered in Case 3 of the route example above.

EXPANSION: *An alternative chosen from each of two sets is also chosen from their union. Formally, for all S, T* $\in \mathcal{P}(X)$: $[x = \gamma(S) = \gamma(T)] \Rightarrow [x = \gamma(S \cup T)]$.

Our main result can now be stated as follows.

THEOREM 1: *Let X be any (not necessarily finite) set. A choice function* γ *on X is an RSM, if and only if it satisfies Expansion and Weak WARP.*

PROOF:

Necessity: Let γ be an RSM on X and let P_1 and P_2 be the rationales.

(a) Expansion. Let $x = \gamma(S) = \gamma(T)$ for $S, T \in \mathcal{P}(X)$. We show that for any $y \in S \cup T$, it cannot be $(y,x) \in P_1$, and for any $y \in \max(S \cup T; P_1)$, it cannot be $(y,x) \in P_2$. If $(y,x) \in P_1$, this would immediately contradict $x = \gamma(S)$ or $x = \gamma(T)$ and γ being rationalized. Suppose, now, that for some $y \in \max(S \cup T; P_1)$ we had $(y, x) \in P_2$. Since $\max(S \cup T; P_1) \subseteq \max(S; P_1) \cup \max(T; P_1)$, we have $y \in \max(S; P_1)$ or $y \in \max(T; P_1)$, contradicting $x \in \max(\max(S; P_1); P_2)$ or $x \in \max(\max(T; P_1); P_2)$.

Therefore, x survives both rounds of elimination and we can conclude that $x = \gamma(S \cup T)$.[8]

[8] Note that this argument cannot be iterated further in the case of more than two rationales. For any set $S \in \mathcal{P}(X)$, let $M_1(S) = \max(S; P_1)$ and $M_2(S) = \max(\max(S; P_1); P_2)$. Then, observe that it is not necessarily true that $M_2(S \cup T) \subseteq M_2(S) \cup M_2(T)$. There could, in fact, be $y \in (M_1(S) \cup M_1(T)) \setminus M_1(S \cup T)$ such that $(y,z) \in P_2$ for some $z \in M_1(S) \cup M_1(T)$, while for all $y' \in M_1(S \cup T)$ it is the case that $(y',z) \notin P_2$. So, if it were $(z,x) \in P_3$, x could not be chosen from $S \cup T$.

(b) Weak WARP. Let $x = \gamma(xy) = \gamma(S)$, $y \in S$. Then $x = \gamma(xy)$ implies that $(x,y) \in P_1 \cup P_2$. If $(x,y) \in P_1$, then the desired conclusion follows immediately. Suppose, then, that $(x,y) \in P_2$. The fact that $x = \gamma(S)$ implies that for all $z \in S$ it is the case that $(z,x) \notin P_1$. Therefore, $x \in \max(R; P_1)$ for all $R \subset S$ for which $x \in R$. Since $(x,y) \in P_2$, then $y \notin \max(\max(R; P_1); P_2)$ for all such R, and thus $y \neq \gamma(R)$.

Sufficiency: Suppose that γ satisfies the axioms. We construct the rationales explicitly. Define

$$P_1 = \{(x,y) \in X \times X \mid \text{there exists no } S \in \mathcal{P}(X) \text{ such that } y = \gamma(S) \text{ and } x \in S\}.$$

Define $(x,y) \in P_2$ if and only if $x = \gamma(xy)$.

Observe that P_1 and P_2 are asymmetric: if $(x,y) \in P_1$ and $(y,x) \in P_1$ then, in particular, $\gamma(xy) \neq x, y$, which is not possible; and P_2 is consistent with the binary choices.

To check that P_1 and P_2 rationalize γ, take any $S \in \mathcal{P}(X)$ and let $x = \gamma(S)$. First, we show that all alternatives that are chosen over x in binary choice are eliminated in the first round. Second, we show that x survives both rounds, and that it eliminates all remaining alternatives in the second round.

Let $z \in S$ be such that $z = \gamma(xz)$. Suppose by contradiction that for all $y \in S \backslash z$ there exists $T_{yz} \ni y, z$ such that $z = \gamma(T_{yz})$. Then by Expansion $z = \gamma(\cup_{y \in S\backslash z} T_{yz})$. If $S = \cup_{y \in S\backslash z} T_{yz}$ we have an immediate contradiction. If $S \subset \cup_{y\in S\backslash z} T_{yz}$, by Weak WARP $x \neq \gamma(S)$, a contradiction. Thus for all such z there exists $y_z \in S$ such that $(y_z, z) \in P_1$.

Clearly x is not eliminated by either P_1 or P_2: for $y \in S$, if $(y,x) \in P_1$, then, it could not be $x = \gamma(S)$, whereas if $(y,x) \in P_2$ by the argument in the previous paragraph, y would have been eliminated by the application of P_1 before P_2 can be applied.

Finally for all $z \in \max(S, P_1)$, with $z \neq x$, such that $x = \gamma(xz)$, we have $(x,z) \in P_2$.

As discussed above, the strength of this characterization lies in the fact that it connects what would be traditionally considered highly "irrational" choice patterns to easy-to-check rationality properties. The only relaxation from standard tests is to allow a limited form of *menu-dependence* in the Weak WARP axiom.

In Appendix A, we establish by means of examples that the set of axioms in Theorem 1 is tight.

REMARK 1: *There isn't a unique way to construct the rationales. One algorithm that performs the task is the following:* (*i*) *if an alternative x is never chosen when y is present, then assign* (y,x) *to the first rationale* P_1; (*ii*) *if x "beats" y in pairwise comparison, then assign* (y,x) *to the second rationale* P_2.[9]

Theorem 1 can be extended to choice functions on any subdomain $\Sigma \subset \mathcal{P}(X)$. The following property, which we use to this effect below, combines in a single property Weak WARP and Expansion.

WWE: If $x = \gamma(S_i)$ *in a class and* $x = \gamma(xy)$, *then* $y \neq \gamma(R)$ *for all* $R \in \mathcal{P}(X)$ *with* $\{x,y\} \subset R \subseteq \cup_i S_i$.

WWE says that if you choose pizza over steak when only pizza and steak are available, then you don't choose steak from a menu containing pizza and some other items, all taken from menus from which pizza is chosen. The previous RSM characterization in terms of Expansion and Weak WARP may not work on restricted domains due to the possible lack of closure under set union of these domains.[10] However, WWE solves this difficulty. For any subdomain $\Sigma \subset \mathcal{P}(X)$, we refer to a function $\gamma : \Sigma \rightarrow X$ as a choice function on Σ. By following essentially the same argument of the proof of the main theorem, it is easy to show the following.

[9] Note that in this construction there is a one-to-one relationship between violations of WARP and differences between the two rationales. In fact, if $(x,y) \in P_1$, then clearly, by definition, $(x,y) \notin P_2$. Therefore, the only possible difference between the two rationales is when there are two alternatives x and y such that $(x,y), (y,x) \notin P_1$ and $(x,y) \notin P_2$. This is a violation of WARP. We are grateful to a referee for pointing this out to us.

[10] To be more precise, we may not be able to carry out the step in which we assert that since $y = \gamma(T_z)$ for a class of sets $\{T_z\}$, then $y = \gamma(\cup_z T_z)$, since $\cup_z T_z$ may not be in the domain.

COROLLARY 1: *A choice function* γ *on* $\Sigma \subset \mathcal{P}(X)$ *is an RSM if and only if it satisfies WWE.*

To conclude this section, we note that, were one to allow a decision maker to apply two rationales in a variable order, depending on the problem, then many more choice functions could be rationalized. In other words, it would be interesting to consider the following definition.

Say that a choice function γ is a *menu dependent RSM* if there exists a pair of rationales P_1, P_2 such that

$$\{\gamma(S)\} \in \{\max(\max(S; P_1); P_2), \max(\max(S; P_2); P_1)\} \quad \text{for all } S \in \Sigma.$$

We do not know at present which choice functions can be rationalized in this way.

We recall the result by Gil Kalai, Ariel Rubinstein, and Rani Spiegler (2002), in whose model one single rationalizing relation is used on each choice set, but the relation may vary from one choice set to another. Each relation is assumed to be an order (so it is complete and transitive), and several relations are in general needed to rationalize a choice function.

II. Beyond Two Rationales

A. *Sequential Rationalizability*

The concept of an RSM suggests an immediate generalization. Instead of using only two rationales, the decision maker might use a larger number of them. For example, in the routes scenario of the previous section, one can conceive that the decision maker uses not only traffic and length, but also scenery, as criteria for choice. This leads us to the following definition.

DEFINITION 2: *A choice function* γ *is* ***sequentially rationalizable*** *whenever there exists an ordered list* $P_1, \ldots, P_K$ *of asymmetric relations, with* $P_i \subseteq X \times X$ *for* $i = 1 \ldots K$*, such that, defining recursively,*

$$M_0(S) = S,$$

$$M_i(S) = \max(M_{i-1}(S; P_i)),\ i = 1, \ldots, K,$$

we have

$$\{\gamma(S)\} = M_K(S) \text{ for all } S \in \mathcal{P}(X).$$

In that case, we say that $(P_1, \ldots, P_K)$ *sequentially rationalize* γ*. We call each* P_i *a* ***rationale****. If we want to emphasize the fact that no more than K rationales are needed, we call the choice function* ***K-sequentially rationalizable****.*

So the choice from each S can be constructed through sequential rounds of elimination of alternatives. At each round, only the elements that are maximal according to a round-specific rationale survive. Like for RSMs (which can now be viewed as special sequentially rationalizable choice functions where only two rationales are used), the rationales and the sequence are invariant with respect to the choice set.

Are there choices that are not sequentially rationalizable? At first sight, it may seem that if we are free to use as many rationales as we like, *any* choice can be rationalized by a sufficiently large number of rationales. On the contrary, the answer may be negative even for very simple choice functions (on a domain X with as few as three alternatives). Examples are provided in Appendix A.

B. *Violations of Economic Rationality Are of Only Two Types*

To delve deeper into the notion of sequential rationalizability, let us recall another well-known property of choice.

Independence of Irrelevant Alternatives.—[11] If an alternative is chosen from a set, it remains chosen when some rejected alternatives are discarded from the set. Formally, for all $S, T \in \mathcal{P}(X)$: $[\gamma(T) \in S, S \subset T] \Rightarrow [\gamma(S) = \gamma(T)]$.

Recall that, at least for the finite case, Independence of Irrelevant Alternatives is equivalent to

[11] For single-valued choice functions, this conflates several properties of correspondences such as Chernoff's

WARP and therefore is a necessary and sufficient condition for rationalizability with a single ordering.[12]

What types of boundedly rational behavior does sequential rationalizability allow? To answer this question consider the following two very basic rationality requirements. The first one requires that if an alternative "beats" all others in a set in binary choices, then this same alternative is chosen from the set—this is obviously a weakening of Expansion. The second property requires that there are no pairwise cycles of choice—this is a weakening of Independence of Irrelevant Alternatives and WARP:

Always Chosen.—If an alternative is chosen in pairwise choices over all other alternatives in a set, then it is chosen from the set. Formally, for all $S \in \mathcal{P}(X)$: $[x = \gamma(xy)$ for all $y \in S \backslash x] \Rightarrow [x = \gamma(S)]$.

No Binary Cycles.—There are no pairwise cycles of choice. Formally, for all $x_1, \ldots, x_{n+1} \in X$: $[\gamma(x_i x_{i+1}) = x_i,\ i = 1, \ldots, n] \Rightarrow [x_1 = \gamma(x_1 x_{n+1})]$.

The reason for highlighting these two properties is that the class of choice functions that do not satisfy WARP (i.e., that are not rationalizable by a single standard economic preference relation) can be classified very simply. They are partitioned into just three subclasses: the choice functions that violate exactly one of No Binary Cycles or Always Chosen, and those that violate both. This is established in the Proposition 1, which is of independent interest.

PROPOSITION 1: *A choice function that violates WARP also violates Always Chosen or No Binary Cycles.*

PROOF:

It is easier to conduct the proof in terms of Independence of Irrelevant Alternatives rather than the equivalent property WARP. Let γ be a choice function on X. We argue by induction on the cardinality of X. Let $X = \{x, y, z\}$. Suppose that $x = \gamma(X)$ and $y = \gamma(xy)$, so that Independence of Irrelevant Alternatives is violated. There are two possibilities: if $y = \gamma(yz)$, then Always Chosen is violated; if, instead, $z = \gamma(yz)$, then either Always Chosen is violated (if $z = \gamma(xz)$), or No Binary Cycles is violated (if $x = \gamma(xz)$, so that $x = \gamma(xz)$, $z = \gamma(yz)$, $y = \gamma(yx)$).

Assume now that the statement holds for all sets X with $|X| \leq n$. Take X' such that $|X'| = n + 1$. Suppose that $x = \gamma(X')$ but there exists $\{x, y\} \subseteq S \subset X'$ such that $y = \gamma(S)$. If the restriction of γ to S violates Independence of Irrelevant Alternatives, then the result follows by the inductive hypothesis. Suppose, then, that the restriction of γ to S satisfies Independence of Irrelevant Alternatives. Consider the set $V = X' \backslash S$. Obviously, $V \neq \varnothing$, and let $z = \gamma(V)$.

If the restriction of γ to V violates Independence of Irrelevant Alternatives, then the result follows by the inductive hypothesis. Suppose it satisfies Independence of Irrelevant Alternatives. Then, $z = \gamma(vz)$ for all $v \in V \backslash z$.

Suppose that $z = \gamma(yz)$. If $z = \gamma(sz)$ for all $s \in S$, then Always Chosen is violated. If there exists some $t \in S$ such that $t = \gamma(tz)$, then this generates the cycle $t = \gamma(tz)$, $z = \gamma(yz)$, $y = \gamma(ty)$, where the last relation follows from Independence of Irrelevant Alternatives on S.

Suppose, alternatively, that $y = \gamma(yz)$. If $y = \gamma(sy)$ for all $s \in V$, then Always Chosen is violated. If there exists some $t \in V$ such that $t = \gamma(ty)$, then this generates the cycle $t = \gamma(ty)$, $y = \gamma(yz)$, $z = \gamma(tz)$, where the last relation follows from Independence of Irrelevant Alternatives on V.

C. *Sequential Rationalizability Excludes One Type of Irrational Behavior*

Next, we show that sequential rationalizability restricts violations of the two basic rationality properties introduced in this section.

LEMMA 1: *If a choice function is sequentially rationalizable, it satisfies Always Chosen.*

PROOF:

Let γ on X be sequentially rationalizable by the rationales $P_1, P_2 \ldots P_K$. For any two alternatives

property ($S \subset T \Rightarrow \gamma(T) \cap S \subseteq \gamma(S)$) and Arrow's condition ($S \subset T$, $\gamma(T) \cap S \neq \emptyset \Rightarrow \gamma(S) = \gamma(T) \cap S$).

[12] See, e.g., Moulin (1985) and Suzumura (1983).

$a,b \in X$, let $i(a,b)$ be the smallest i such that P_i relates a and b, that is

$$i(a,b) = \min\{i \in \{1,\dots,K\} \mid (a,b) \in P_i \text{ or } (b,a) \in P_i\}.$$

Given $S \subseteq X$ and $x \in S$, let $x = \gamma(xy)$ for all $y \in S\backslash x$. For each $y \in S\backslash x$, we must have $(x,y) \in P_{i(x,y)}$, so that the successive application of the rationales eliminates all $y \in S\backslash x$, and no rationale can eliminate x. Therefore, $x = \gamma(S)$, as desired.

Our partial characterization result shows the equivalence of WARP and No Binary Cycles on the domain of sequentially rationalizable choice functions; it follows from Proposition 1 and Lemma 1 by observing that WARP is violated if there is a binary cycle.

THEOREM 2: *A sequentially rationalizable choice function violates WARP, if and only if it exhibits binary cycles.*

Thus, the results in this section generalize the message of the basic "routes" example of the previous section. We have established that, in general, and not only in that example, *all* violations of "rationality" can be traced back to two elementary pathologies of choice, corresponding to Case 2 and Case 3 of the routes example: violations of Always Chosen and No Binary Cycles. Like RSMs, even the more general notion of sequential rationalizability is intimately connected with pairwise cycles of choice, and cannot possibly explain the other pathology.

D. *A Recursion Lemma*

In this section and the next, we provide conditions on observable choices that fully characterize 3-rationalizable choice functions. In the course of doing this, we also provide a recursive result that permits one to move from any given characterization of $(K-2)$–rationalizable choices to K–rationalizable choices, thus providing a basis for a general characterization of sequential rationalizability.

To this aim, we need to extend some of the previous definitions to *choice correspondences*. A choice correspondence on X selects a set of alternatives from each possible element of $\mathcal{P}(X)$: so it is a set-valued map $\gamma : \mathcal{P}(X) \to X$ with $\gamma(S) \subseteq S$ for all $S \in \mathcal{P}(X)$. The definitions of sequential rationalizability and RSM extend in the obvious way.

Any choice correspondence γ on X defines naturally a subdomain $\Sigma(\gamma)$, defined as follows:

$$\Sigma(\gamma) = \{S \in \mathcal{P}(X): S = \gamma(T) \text{ for some } T \in \mathcal{P}(X)\}.$$

In words, $\Sigma(\gamma)$ contains all the sets in the full domain $\mathcal{P}(X)$ that coincide with the choice that γ produces from some element of the full domain.

Now we are ready to state our key result.

RECURSION LEMMA: *A choice function γ is K–sequentially rationalizable, if and only if there exists a $(K-2)$–sequentially rationalizable choice correspondence γ^* on X such that:*

(*i*) $\gamma(\gamma^*(S)) = \gamma(S)$ *for all* $S \in \mathcal{P}(X)$;

(*ii*) *the restriction of γ to $\Sigma(\gamma^*)$ satisfies WWE.*

This result shows that the process of selection for a sequentially rationalizable choice function γ can be recursively broken down into two steps. First, a sequentially rationalizable "preselection" is made, described as a choice correspondence γ^* which contains the chosen alternative for each set. This choice correspondence is sequentially rationalizable with two fewer rationales than the given choice function. In the second step, a choice function is applied to the preselected sets. This choice function satisfies WWE on that domain and is just the restriction of the given choice function γ to the preselected sets.

PROOF OF THE RECURSION LEMMA:

Let γ be K–sequentially rationalizable by $P_1 \dots, P_K$. The sequential application of $P_1,\dots,P_{K-2}$ defines a $(K-2)$–sequentially rationalizable choice correspondence on X, say γ^*. It must be $\gamma(\gamma^*(S)) = \gamma(S)$ for all $S \in \mathcal{P}(X)$, since both the left-hand side and the right-hand side are obtained by applying exactly the same rationales, exactly in the same sequence. The restriction of γ to $\Sigma(\gamma^*)$ is an RSM with rationales P_{K-1} and P_K, since by definition of Σ

(γ^*) the first $K - 2$ rationales produce no effect when applied to any element of $\Sigma(\gamma^*)$ (as they have already been used), and only the rationales P_{K-1} and P_K will be effective. Then the statement follows by Corollary 1.

That γ is sequentially rationalizable (say by $P_1, \ldots, P_K$) if the conditions of the statement hold is obvious: the first $K - 2$ rationales are those that rationalize γ^*, while P_{K-1} and P_k are the rationales that rationalize the restriction of γ to $\Sigma(\gamma^*)$.

The Recursion Lemma is useful as an observable test for sequential rationalizability, *provided* one also has observable conditions that characterize sequential rationalizability of a lower order for choice *correspondences*. In general, we still lack such conditions for general correspondences, except for the case where γ^* is rationalizable by just one rationale, as shown in the next section.

E. *A Characterization of 3-Rationalizability*

A classical result of choice theory uses the following condition on choice correspondences (e.g., Moulin 1985; Suzumura 1983).

Binariness.—For all $S \in \mathcal{P}(X) : x \in \gamma(S)$, if and only if $x \in \gamma(xy)$ for all $y \in S$.

Binariness says that an alternative is chosen from a set, if and only if it is chosen in binary contests with any other alternative in the set. This means that the choice function is determined entirely by its behavior on binary sets. Amartya Sen (1970) proved that a choice correspondence γ on $\mathcal{P}(X)$ is rationalized by a binary relation P, if and only if γ satisfies binariness.[13] Thanks to this fact, we can "solve" the case of 3-rationalizability.

THEOREM 3: *A choice function γ is 3-sequentially rationalizable if and only if there exists a choice correspondence γ^* on X such that:*

(*i*) $\gamma(\gamma^*(S)) = \gamma(S)$ *for all* $S \in \mathcal{P}(X)$;

(*ii*) *the restriction of γ to $\Sigma(\gamma^*)$ satisfies WWE;*

(*iii*) γ^* *satisfies binariness.*

PROOF:

The result follows directly from Corollary 1 and Sen's theorem, with one observation. If there is a P as in Sen's theorem (necessarily complete since $\gamma(xy)$ is well defined for all $x, y \in X$), then there is an *asymmetric* relation P' such that $(y, \gamma(S)) \in P'$ for no $S \in \mathcal{P}(X)$ and $y \in S$ (i.e., γ maximizes P'). The relation P' is just the asymmetric part of P.

This result provides a characterization of 3-rationalizability exclusively in terms of conditions on observed choice. It involves checking an axiom of the standard expansion-contraction type for a *set* of choice functions rather than just for the original one. In practice, the result defines an algorithm that uses the choice data provided by γ, as follows:

Step 1: Consider all the possible choice correspondences γ^* defined only on binary sets, and such that $\gamma(xy) \in \gamma^*(xy)$ for all $x, y \in X$.

Step 2: Fix a γ^* from step 1, extend it to $\mathcal{P}(X)$ (if possible), with the following formula: $x \in \gamma^*(S)$, if and only if $x \in \gamma^*(xy)$ for all $y \in S \backslash x$, and $\gamma(S) \in \gamma^*(S)$. If the extension is not possible (i.e., it yields an empty set), pick a different γ^* from step 1, and repeat.

Step 3: Check if γ on $\Sigma(\gamma^*)$ satisfies WWE. If it does, move to step 4. If not, repeat step 2 with a different γ^*.

Step 4: Check if $\gamma(\gamma^*(S)) = \gamma(S)$ for all $S \in \mathcal{P}(X)$. If it does, the original choice function is 3-rationalizable. If not, repeat step 2 with a different γ^*. If the answer is negative for all choice correspondences, then the original choice function is not 3-rationalizable.

An example of an application of this algorithm, also illustrating some practical shortcuts, is given in Appendix B.

III. Rational Shortlist Methods and Choice over Time

Throughout the paper, we have focused on general violations of rationality. However, we believe that RSMs can prove very useful to explain other

[13] That is, for all $S \in \mathcal{P}(X)$, $\gamma(S) = \{x \in S : (x,y) \in P$ for all $y \in S\}$.

choice anomalies in specific contexts, in which certain rationales can suggest themselves. Here, we consider an application to choice over time.

The standard model of choice over time is the exponential discounting model. It has been observed that actual choices in experimental settings consistently violate its predictions. The most notable violation is possibly preference reversal. Let P_γ refer to observed pairwise choices over date-outcome pairs $(x,t) \in X \times T$, where X is a set of monetary outcomes and T is a set of dates. In this context, preference reversal is the shorthand for the following situation: $(x,t_x)\ P_\gamma\,(y,t_y)$ and $(y,t_y+t)\ P_\gamma\,(x,t_x+t)$. This violates stationarity of time preferences, a premise on which the exponential discounting model is constructed.

This choice pattern can be easily accounted for by interpreting γ as an RSM with rationales P_1 and P_2 defined as follows. For some function $u: X \times T \rightarrow \mathbb{R}$ and number $\sigma > 0$, $(x,t_x)P_1(y,t_y)$, if and only if $u(x,t_x) > u(y,t_y) + \sigma$, and $(x,t_x)P_2(y,t_y)$, if and only if $u(y,t_y) \leq u(x,t_x) \leq u(y,t_y) + \sigma$, and either $x > y$, or $x = y$ and $t_x < t_y$. That is, the decision maker looks first at discounted value, and chooses one alternative over the other if it exceeds the discounted value of the latter by an amount of at least σ. Otherwise he looks first at the outcome dimension and, if this is not decisive, at the time dimension.

This is compatible with preference reversal, even with an exponential discounting type of u function. Let $x < y$, $t_x < t_y$ and $u(x,t_x) = x\delta^{t_x}$ for $\delta \in (0,1)$. Suppose that $x\delta^{t_x} > y\delta^{t_y} + \sigma$ so that (x,t_x) is chosen over (y,t_y) by application of P_1. Given σ, if t is sufficiently large it will be $x\delta^{t_x+t} < y\delta^{t_y+t} + \sigma$, so that the two date-outcome pairs (x,t_x+t) and (y,t_y+t) are not comparable via P_1. However, the application of P_2 yields the choice of (y,t_y+t) over (x,t_x+t), thus "reversing the (revealed) preference."

Obviously, P_γ could also be sequentially rationalized by using three rationales, where the outcome and time dimension comparisons are used in two separate P_i.

The same model can explain cyclical intertemporal choices and other "anomalies" (see Manzini and Mariotti 2006a, and bibliography therein, notably Rubinstein 2003, who proposes a multistage procedure based on similarity relations).

Our model differs from that in Efe A. Ok and Yusufcan Masatlioglu (forthcoming), who consider a *complete* binary preference relation B over a set of date-outcome pairs. They axiomatize the following representation class: $(x,t)\ B\,(y,s)$, if and only if $U(x) \geq U(y) + \varphi(s,t)$, where U is interpreted as an instantaneous utility function, while φ captures the effect of time delay. Unlike our setup, this representation is not an interval order, and the "contributions" of outcome and time to the agent's utility are separated. In Ok and Masatlioglu's approach, cycles can be accounted for without resorting to a second partial order. Our view is different: the first partial order represents the "rational" though incomplete component of decision making; hence we assume it transitive. In our approach, intransitivities arise as the by-product of resorting to the "tie-breaking" second rationale.

IV. Concluding Remarks

We have proposed an economic, "revealed preference" approach to the type of decision-making procedures often promoted by psychologists. For example Gigerenzer and Todd (1999) in their work on "fast and frugal" heuristics observe, "One way to select a single option from multiple alternatives is to follow the simple principle of elimination: successive cues are used to eliminate more and more alternatives and thereby reduce the set of remaining options, until a single option can be decided upon." Such heuristics focus mostly on the simplicity of cues used to narrow down possible candidates for choice. Simplicity is an essential virtue in a world in which time is limited. An overarching preference relation—let alone a utility function—is not a cognitively simple object, and as a consequence these authors stress the difference from heuristics-based reasoning and the "unlimited demonic or supernatural reasoning" relied upon in economics.[14] Yet, in this paper we have shown that the standard tools, concepts, and properties of revealed preference theory can be used to formalize and infer the use of such heuristics. A seemingly limited form of menu-dependence (encapsulated in our Weak WARP and Expansion properties) is equivalent to the use of a two-stage

[14] See Gigerenzer and Todd (1999).

procedure that may generate economically "irrational" choice behavior.

Our way of incorporating bounded rationality is to translate the psychological notion of "cues" into a set of *not necessarily complete* binary relations. Rationality for us is the *consistent* application of a sequence of rationales. The order in which they are applied may be hardwired and may depend on the specific context and on the type of decision maker,[15] but it should be the same in a relevant class of decision problems. Each single rationale in itself need not exhibit any other strong property, such as completeness or transitivity.

The usefulness of elimination heuristics in practical decision making is self-evident[16] and widely spread in disparate fields, from clinical medicine[17] to marketing and management. In this perspective, the *sequentiality* in the application of rationales, which lies at the core of our analysis, is an appealing feature of our rationalization results. Our approach may be contrasted with the recent contribution by Kalai, Rubinstein, and Spiegler (2002) and José Apesteguia and Miguel A. Ballester (2005). They use multiple rationales to explain choices, but each rationale is applied to a subset of the domain of choice. This results in all choice functions being rationalizable, and the focus becomes that of "counting" the minimum number of rationales necessary to explain choices. One could imagine adapting a similar approach in our framework, by making the order of application of the rationales dependent on the set to which they are applied. Whether this would reduce the number of rationales needed to explain choices is still an open problem.

A different and intriguing approach to the theme of "simplifying" choice problems is pursued by Yuval Salant (2003), who shows how a rational choice function can be viewed as being minimally complicated from a computational-theoretic point of view.

Recently, Rubinstein and Salant (2006) have also discussed the use of the revealed preference approach to explain "behavioral" phenomena, and they provide an alternative characterization of RSMs in a different framework. In this same spirit, Masatlioglu and Ok (2003) characterize the phenomenon of status quo dependence in terms of axioms on observable choice data. And Kfir Eliaz and Ok (2006) weaken WARP for choice correspondences to characterize the rationalization by a not necessarily complete preference relation.[18]

In Manzini and Mariotti (2006b), we consider a two-stage elimination procedure in which, in the first stage, the relation is applied to *sets* of alternatives instead of to the alternatives themselves. The interpretation is that, in the first stage, alternatives are grouped by "similarity" and the elimination is between "similarity groups." Interestingly, that procedure is characterized by Weak WARP alone, and therefore it can explain even those choices that violate Always Chosen, besides exhibiting pairwise cycles.[19]

We should also mention the work by Ok (2004), which characterizes the choice correspondences satisfying Independence of Irrelevant Alternatives by means of a two-stage procedure. Unlike this paper, in the second stage of Ok's procedure, elimination of alternatives does not occur on the basis of a relation, but rather on the information contained in the entire feasible set.

To conclude, we observe that recently Lars Ehlers and Yves Sprumont (2006) and Michele

[15] For example, in order to "choose" whether to stay or flee in the presence of a bird, a rabbit may use as its first rationale the fact that the bird is gliding, which would identify a predator. Conversely, a human decision maker may well look first at size or shape in order to recognize the bird.

[16] As put very effectively by Gigerenzer and Todd (1999), "If we can decide quickly and with few cues whether an approaching person or bear is interested in fighting, playing, or courting, we will have more time to prepare and act accordingly (though in the case of the bear all three intentions may be equally unappealing)."

[17] As an example, the online self-help guide of the UK National Health Service (http://www.nhsdirect.nhs.uk/SelfHelp/symptoms/) helps users recognize an ailment by giving yes/no answers along a sequence of symptoms. This presumably formalizes the mental process of a trained doctor.

[18] They propose the following Weak Axiom of Revealed Non-Inferiority (WARNI): for any $y \in S$, if for every $x \in \gamma(S)$ there exists a choice set T such that $y \in \gamma(T)$ and $x \in T$, then $y \in \gamma(S)$. They prove that WARNI is equivalent to rationalization by a single but possibly incomplete preference relation. Our WEE seems reminiscent of WARNI. As Case 2 of the route example shows, however, there are (single-valued) choice correspondences that satisfy WWE but violate WARNI: so, not all RSMs can be rationalized by a single and possibly incomplete preference relation.

[19] We also present experimental evidence to show that this type of choices is empirically relevant in certain contexts.

Lombardi (forthcoming) have studied rationalization of choice functions by a tournament. The first two authors use expansion-contraction axioms to characterize (necessarily multivalued) choice functions, which are the top cycle of the tournament, where the tournament coincides with the base relation. Lombardi (2006) characterizes choice functions which are the uncovered set of the tournament. One can show that sequentially rationalizable choice functions refine the top cycle (of the base relation) in each choice set. In other words, a sequentially rationalizable choice function picks an element of the top cycle, so that the choice beats in an arbitrary number of steps any other feasible alternative.

Appendix A

We establish by means of examples that the set of axioms in Theorem 1 is tight. In order to describe choice functions compactly in examples, we use the following notation: given $x \in X$, let $C_\gamma(x) = \{S \in \mathcal{P}(X) \mid x = \gamma(S)\}$.[20]

Example 1. Expansion but not Weak WARP:

$$X = \{x, y, w, z\}$$

$$C_\gamma(w) = \{wx\}$$

$$C_\gamma(x) = \{xy, xz, xyz, wxy, wxyz\}$$

$$C_\gamma(y) = \{wy, yz, wyz\}$$

$$C_\gamma(z) = \{wz, \mathrm{w}xz\}$$

Binary choices are visualized in Figure 2, where $a \rightarrow b$ stands for $a = \gamma(ab)$. It is straightforward to verify that this choice function satisfies Expansion, but not Weak WARP (e.g., $x = \gamma(X)$ and $x = \gamma(xz)$ but $z = \gamma(wxz)$). This choice function is not an RSM. To see this, suppose $(w, x) \in P_1$. Then $x = \gamma(X)$ cannot be rationalized. Suppose, then, that $(w, x) \in P_2$. Then $z = \gamma(wxz)$ cannot be rationalized, for x will eliminate z regardless of whether $(x, z) \in P_2$ or $(x, z) \in P_1$.

[20] In this notation, the Expansion axiom says that, for all $x \in X$, $C_\gamma(x)$ is closed under set union.

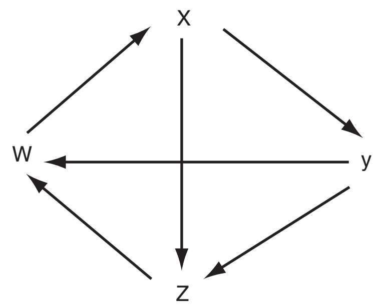

Figure 2

Example 2. Weak WARP but not Expansion:

$$X = \{x, y, z\}$$

$$C_\gamma(x) = \{xy, xz\}$$

$$C_\gamma(y) = \{yz, xyz\}$$

$$C_\gamma(z) = \{\varnothing\}$$

Binary choices are visualized in Figure 3. While this choice function satisfies Weak WARP (trivially, as the premise of Weak WARP does not apply), it fails Expansion. This choice function is not an RSM. Indeed, it is not sequentially rationalizable. As before, for any two alternatives $a, b \in X$, let $i(a, b)$ be the smallest i such that P_i relates a and b. Suppose by contradiction that γ were sequentially rationalizable by $P_1, \ldots, P_K$. Since $x = \gamma(xy)$, it must be $(x, y) \in P_{i(x,y)}$. Given this, $y = \gamma(xyz)$ can hold only if $(z, x) \in P_{i(x,z)}$, which contradicts $x = \gamma(xz)$.

The examples above can be used to make two additional points. First, there are choice functions that are not RSMs but are sequentially rationalizable. Namely, γ in Example 1 is 3-rationalizable, as shown in Appendix B. Second, the notion of sequential rationalizability is not vacuous, in the sense that there exist choice functions thar are not sequentially rationalizable (Example 2).[21]

[21] The violations of Always Chosen shown in this example appear in other notable examples of plausible choice procedures introduced in the literature, which are therefore *not* sequentially rationalizable. Let $X = \{x, y, z\}$, and consider the following refinement of the *choose the median* procedure: There is a "fundamental" order B on X (e.g., given by ideology from left to right) such that $(z, y), (y, x) \in B$.

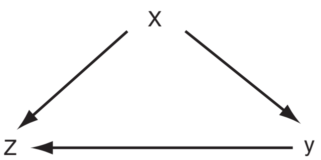

FIGURE 3

APPENDIX B

The task of verifying the 3-rationalizability of a given choice function is much more manageable, even "by hand," that one could fear. We illustrate this with an example. Take the choice function of Example 1 from Appendix A.

Step 1: To construct the family of choice correspondences on binary sets such that $\gamma(ab) \in \gamma^*(ab)$ for all $a, b \in X$, observe that this requirement restricts γ^* as follows: for any $\{a, b\}$, either $\gamma^*(ab) = \{\gamma(ab)\}$, or $\gamma^*(ab) = \{a, b\}$. Thus, admissible choice correspondences are given by the resulting combinations. In our example, this would generate $2^6 = 64$ choice correspondences on the binary sets to start with.

Step 2: The number of choice correspondences allowed in Step 1 can be greatly reduced by observing that if $a \in \gamma(S)$ for some S which includes a pair $\{a, b\}$, it must be that $a \in \gamma^*(a, b)$, for otherwise if a is excluded from γ^*, the latter could never be extended as required in this step. In our example, then, this requires $x \in \gamma^*(wx)$ (otherwise $x = \gamma(X) \notin \gamma^*(X)$), and $z \in \gamma^*(xz)$ (otherwise $z = \gamma(wxz) \notin \gamma^*(wxz)$), so that it must be $\gamma^*(wx) = \{w, x\}$ and $\gamma^*(xz) = \{x, z\}$, reducing the number of eligible starting choice correspondences to $2^4 = 16$, depending on the behavior of γ^* in the remaining binary sets.[22]

As for the extension to nonbinary sets, recall that, according to our algorithm, the extension satisfies $a \in \gamma^*(S)$, if and only if $a \in \gamma^*(ab)$ for all $a \in S$, and $\gamma(S) \in \gamma^*(S)$. Consequently, it is very easy to extend the domain to nonbinary sets: simply drop from each of these larger sets S any alternative that is rejected by γ^* in a binary set whose other alternative is in S. It obviously makes sense to start checking whether WWE is satisfied on the choice correspondence that generates the least number of additional sets. We start from the most "parsimonius" γ^*. Then, take $\gamma^*(ab) = \{\gamma(ab)\}$ for all remaining binary sets, i.e., $\gamma^*(wy) = \{y\}$, $\gamma^*(wz) = \{z\}$, $\gamma^*(xy) = \{x\}$ and $\gamma^*(yz) = \{y\}$, so that the most alternatives are dropped in the extension. For this γ^*, we derive the following extension:

$$\gamma^*(wxy) = \{x\},\ \gamma^*(wxz) = \{x, z\},$$

$$\gamma^*(wyz) = \{y\},\ \gamma^*(xyz) = \{x\},$$

and $$\gamma^*(X) = \{x\}.$$

Step 3: WWE holds trivially. We can move to Step 4.

Step 4: With the current choice correspondence, the algorithm sends us back to Step 2, as $\gamma(\gamma^*(wxz)) = x \neq z = \gamma(wxz)$. This failure, however, alerts us to the fact that we cannot leave z "alone" with x, suggesting that it might make sense to have $\gamma^*(wz) = \{w, z\}$. With this single modification to our correspondence in Step 2, the extension changes to

$$\gamma^*(wxy) = \{x\},\ \gamma^*(wxz) = \{w, x, z\},$$

$$\gamma^*(wyz) = \{y\},\ \gamma^*(xyz) = \{x\},$$

and $$\gamma^*(X) = \{x\}.$$

The decision maker chooses the median according to B, breaking ties by picking the highest element in the set of median elements. We have $z = \gamma(xz) = \gamma(yz)$ and yet $y = \gamma(xyz)$, violating Always Chosen. The same choice pattern is consistent with the *never choose the uniquely largest* procedure (e.g., a hungry polite guest refrains from picking the largest piece of cake from the tray). Formally, there is again a fundamental order B on alternatives (e.g., size) and the chosen alternative must not be the unique maximizer of B. However, to interpret the choice pattern $z = \gamma(xz) = \gamma(yz)$ and $y = \gamma(xyz)$ in this way, the fundamental ordering must be exactly the reverse of the one used for the "choose the median" procedure, namely, $(x, y), (y, z) \in B$. Nick Baigent and Wulf Gaertner (1996) and Gaertner and Yongsheng Xu (1999a, b) have axiomatized this type of procedure.

[22] Indeed, this is generally true for all choice functions that are sequentially rationalizable. Since any such function satisfies Always Chosen, then: either no choice from a nonbinary set is "beaten" pairwise by some other alternative in that set, in which case the choice function is rationalizable in the standard way; or the converse is true, in which case the number of choice correspondences on binary sets is reduced by a factor of at least two, and possibly more.

In Step 3, again WWE holds trivially. This time, though, in Step 4 it is easy to check that $\gamma(\gamma^*(S)) = \gamma(S)$ for all $S \in \mathcal{P}(X)$. We conclude that the choice function is 3-rationalizable.

Note that the algorithm also provides an indication of how the rationales can be constructed: the single valuedness of the choice correspondence on the three binary sets $\{w,y\}$, $\{x,y\}$, and $\{y,z\}$ suggests that (y,w), (x,y), $(y,z) \in P_1$. Using the construction from the proof of Theorem 1 on the subdomain $\Sigma(\gamma^*) = \{wx, wz, xz, wxz\}$, we have, $(w,x), (z,w) \in P_2$ while $(w,x), (z,w), (x,z) \in P_3$.

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