# SER Performance Analysis for Physical Layer Network Coding over AWGN Channel


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***Abstract*—While original network coding is proposed over data link layer, recent work suggests that it can also be implemented on physical layer. In fact it is more natural for wireless networks because of its omnidirectional transmission. In this paper, we investigate the *symbol-error-rate* (SER) for *binary phase-shift keying* (BPSK) and *quadrature phase-shift keying* (QPSK), but the approaches can be generalized to other constellation schemes. The closed-form SER results are derived for physical layer network coding over AWGN channels. The theoretical analysis is also validated by numerical simulations.**


## I. Introduction

One of the key characteristics of network coding [1] is the function of relay nodes. In traditional network systems, relay nodes only copy and forward the received information to their neighbors. However in network coding system, relay nodes will first combine received information from different sources through a simple bitwise-XOR operation and then forward the summation to next receivers. By which, the throughput of the whole network is increased since less information exchange is needed. Recent progress on the analysis of capacity with network coding can be found in [2], [3]. A practical wireless network coding system is developed in [4].

While original design of network coding was implemented in data link layer, the information manipulation can also be implemented over physical layer, which in fact is more natural in wireless networks because of the omnidirectional broadcast. In this case, the relay node try to detect combined signals from different source nodes at the same time over the same frequency band. For traditional systems, if there are more than one signal over the same channel, all of them are interweaved and interfere with others. But for network coding systems, the interest of relay nodes is just the summation, not the value of individual signals. As shown in [5], the capacity of such a system is doubled for the 2-way relay networks. Also, the authors implemented an analog network coding system by using software defined radio. In [6] a similar scheme, physical layer network coding, was discussed from information theoretic aspect. In [7] the authors presented the throughput analysis based on a bi-directional traffic model with network coding over physical layer.

In this paper, we investigate the SER performance for physical layer network coding systems. Unlike canonical point-to-point transmission where one error occurs when the received symbol is different with the transmitted one, the symbol error is defined as the summation difference between the source (different transmitters) and the sink (the relay node). The analytical results derived in this paper can be directly applied to the analysis of network throughput. In [6], the authors presented bit-error-rate (BER) analysis for BPSK modulation. In this work, we derive a closed-form SER for BPSK and QPSK with the techniques discussed in [8], but the approaches developed in this work can be generalized to other modulation schemes. More importantly, our results include the effect of different signal-to-noise ratio (SNR), which is a more practical model since different signals experience different path loss and fading degradation.

The rest of this paper is organized as the follows. In Section II, we present the wireless system model with physical layer network coding. In Section III, we derive the SER for BPSK and QPSK over additive white Gaussian noise (AWGN) channels. In Section IV, we show simulation results to validate our analysis. Finally, Section IV concludes the paper.

## II. System Model

We consider the same system model as that in [6] [7], where a three-node network is discussed to demonstrate the throughput improvement over other schemes. As shown in Fig. 1, nodes $S_1$ will send information $x_1$ to node $S_2$, and $S_2$ send $x_2$ to $S_1$. Both transmission will be relayed by node $R$. This is an typical scenario in WLAN, where relay $R$ is the access point (AP) and $S_1$ and $S_2$ are the nodes of a basic service set (BSS).

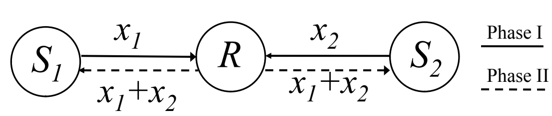


Fig. 1. System Model

The traditional relay schemes may take four steps to finish the information exchange:

1) $S_1 \mapsto R : x_1$;
2) $R \mapsto S_2 : x_1$;
3) $S_2 \mapsto R : x_2$;
4) $R \mapsto S_1 : x_2$;

However, the network coding relay may take three steps. Let the information symbols $x_m$ be from some finite field with size $M$.

1) $S_1 \mapsto R : x_1$;
2) $S_2 \mapsto R : x_2$;
3) $R \mapsto \{S_1, S_2\} : x_R = x_1 \bigoplus x_2$;

where $\bigoplus$ denotes the summation in modulo to $M$. When $M = 2$, $\bigoplus$ is equivalent to the XOR operation for the bit level information. Since node $S_1$ has the *a priori* information of $x_1$, $S_1$ can decode $x_2$ through the modulo operation $x_2 = x_R \bigoplus x_1$. Similarly, $S_2$ can extract $x_1$ through $x_1 = x_R \bigoplus x_2$. The network coding scheme not only reduce one time slot for the information exchange, but also fully exploit the broadcast benefits of wireless channel, which is always ignored in previous designs.

Further improvement is achieved through physical layer network coding, as shown in [5], [6], [7], where the information can be exchanged within two steps:

1) $S_1 \mapsto R : x_1$, $S_2 \mapsto R : x_2$;
2) $R \mapsto \{S_1, S_2\} : x_R = x_1 \bigoplus x_2$;

where $S_1$ and $S_2$ will transmit their information simultaneously to the relay $R$ at the first time slot. The relay node will broadcast the summation of $x_1$ and $x_2$ to both $S_1$ and $S_2$. While there are two approaches for this forward: amplify-and-forward (AF) and detect-and-forward (DF), in this paper we focus on the performance of DF. We will derive the SER for the links in the first time slot.

Please note that there are several practical issues for the implementation of physical layer network coding. In the model shown in Fig. 1, $S_1$ and $S_2$ have their own oscillators, which may lead to carrier and phase offset. Also a global synchronization is required among the three-node network. To facilitate the analysis, we assume a perfect synchronization available through other techniques and the effect of imperfect synchronization is considered as a degradation of SNR.

An example of received signal at relay node is shown in Fig. 2 where both $S_1$ and $S_2$ transmit QPSK symbols to relay. While there are four constellation points for QPSK (solid circles), we can observe that there are a total of nine (9) nodes for the constellations of the combined signals (empty circles).

## III. Performance Analysis

In this section, we present the analysis for BPSK and QPSK. But the analysis can be generalized to any two-dimensional constellations.

To simplify the analysis we assume AWGN channel. We focus on the impact of the attenuation factor and Gaussian noise. To avoid confusion, hereafter we use $\rho$ to indicate the received signal power at the relay node $R$. Furthermore, we consider a general model that the two signals have different power levels and $\rho_1 \geq \rho_2$, regardless the sender. This is reasonable because, in DF scheme, the relay node does not need to decode the content of individual messages. Instead, it only needs to decode the combination of the two signals. Therefore, it does not matter where are the sources of $S_1$ and $S_2$.

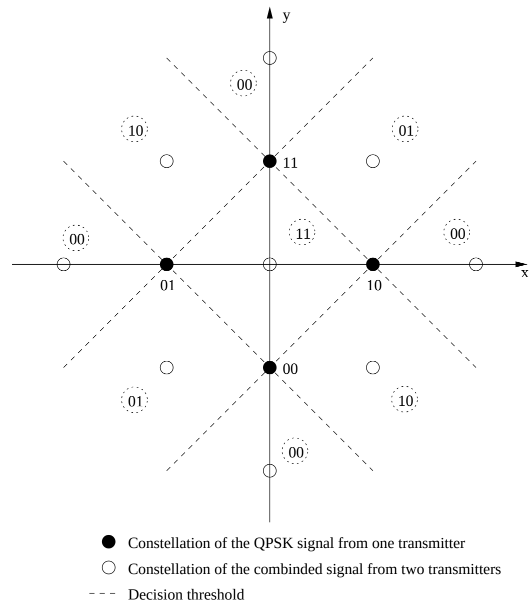


Fig. 2. Received constellation at relay node for QPSK.

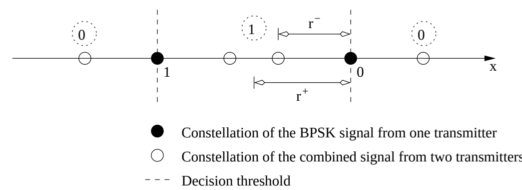


Fig. 3. Received constellation at relay node for BPSK with the same constellation.

Since the two power levels are known to the receiver, the analysis will follow the exact expression of error probability presented in [8]. In general, the received signal can be expressed by

$$y = \sqrt{\rho_1} x_1 + \sqrt{\rho_2} x_2 + N, \tag{1}$$

where $N$ represents Gaussian noise with zero mean and variance $\sigma^2$. Again it is important to note that it is $x_1 \bigoplus x_2$ to be decoded at the receiver.

### A. BPSK with the Same Constellation

We first consider that A and B use the same BPSK constellation ($\pm 1$) to modulate the signals. In addition, both of them use the same constellation. Therefore, the received constellation can be shown in Fig. 3. To simplify the notation, in Fig. 3, $r^+ = \sqrt{\rho_1}$ and $r^- = \sqrt{\rho_2}$, respectively.

As shown in Fig. 3, there are four nodes in the constellation (illustrated by small empty cycles), since $r^+ \geq r^-$. Therefore, we have the decision thresholds $\pm r^+$, which are shown by the dash lines in Fig. 3. In particular, we have

$$Y = \begin{cases} 0, & Y < -r^+ \\ 1, & -r^+ \leq Y < r^+ \\ 0, & Y \geq r^+. \end{cases} \tag{2}$$

As shown in [6], the decision threshold in Eq. (2) is not optimal. However, the simulation results based on this

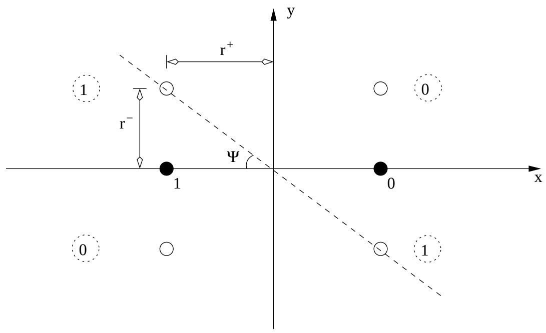


Fig. 4. Received constellation at relay node for BPSK with orthoganal constellation.

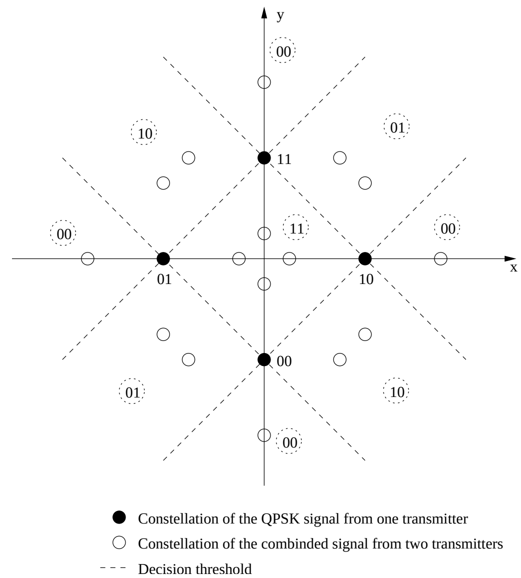


Fig. 5. Received constellation at relay node for QPSK with the same constellation.

approximation are very close to the performance from optimal threshold. We derive the symbol error performance from the above decision regions. Let $F_1$ be the probability of error if the combination of $x_1$ and $x_2$ is 1, and let $F_0$ be the probability of error if the combination of $x_1$ and $x_2$ is $0$. We can then derive

$$\begin{aligned} F_1 &= \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{\rho_2}{2\sigma^2\sin^2(\phi)}\right]d\phi \\ &\quad + \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{(2\sqrt{\rho_1}-\sqrt{\rho_2})^2}{2\sigma^2\sin^2(\phi)}\right]d\phi \end{aligned} \tag{3}$$

$$\begin{aligned} F_0 &= \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{\rho_2}{2\sigma^2\sin^2(\phi)}\right]d\phi \\ &\quad - \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{(2\sqrt{\rho_1}+\sqrt{\rho_2})^2}{2\sigma^2\sin^2(\phi)}\right]d\phi. \end{aligned} \tag{4}$$

Notice that in Eq. (4), the second term represents the probability that the error is large enough such that the received signal is located in another region for $0$, which will lead to a correct decision.

Clearly, the average SER for BPSK with same constellation is

$$F^S_{BPSK} = \frac{1}{2}(F_0 + F_1). \tag{5}$$

### *B. BPSK with Orthogonal Constellation*

In the previous subsection, we have studied the performance of BPSK with the same constellation from two end nodes. It is interesting to compare the performance when the two nodes send BPSK with orthogonal constellation, i.e., one has points $\pm 1$ and the other has points $\pm j$. The constellations for the combined signals are shown in Fig. 4. In Fig. 4, the decision thresholds overlap with the $x$-axis and $y$-axis.

From the constellation shown in Fig. 4, we note that all nodes in the constellation have the same performance. Therefore, we can derive the SER performance as

$$\begin{aligned} F^O_{BPSK} &= \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{\rho_1}{2\sigma^2\sin^2(\phi)}\right]d\phi \\ &\quad + \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{\rho_2}{2\sigma^2\sin^2(\phi)}\right]d\phi \\ &\quad - \frac{1}{\pi}\int_0^{\pi/2-\Psi} \exp\left[-\frac{\rho_1}{2\sigma^2\sin^2(\phi)}\right]d\phi \\ &\quad - \frac{1}{\pi}\int_0^{\Psi} \exp\left[-\frac{\rho_2}{2\sigma^2\sin^2(\phi)}\right]d\phi \end{aligned} \tag{6}$$

where

$$\Psi = \tan^{-1}\frac{\sqrt{\rho_2}}{\sqrt{\rho_1}}.$$

### *C. QPSK with the Same Constellation*

In this subsection, we study the performance of QPSK in physical-layer network coding scenario. We assume that the two end nodes use the same constellation, so we have the constellation shown in Fig. 5. From this figure we can observe that, the performance is the same for all nodes that represent $00$. Similarly, we can calculate the error performance of $11$ by focusing a single node in the graph. In addition, the performance of $01$ and $10$ are the same. In the rest of this subsection, we consider the above three scenarios.

*1) Performance of decoding* $00$*:* To understand the decoding process for $00$, we can draw a graph in Fig. 6. From this figure we can see that the error can occur if $Y$ is not located in the region bounded by the two red lines (to the right). However, the decision will be correct if $Y$ is in the region bounded by the two blue lines (to the left). In addition, the decision is correct if $Y$ is in the other two regions for $00$, which are symmetric to the $x$-axis. Clearly, any of the two regions can be consider as the region bounded by two purple

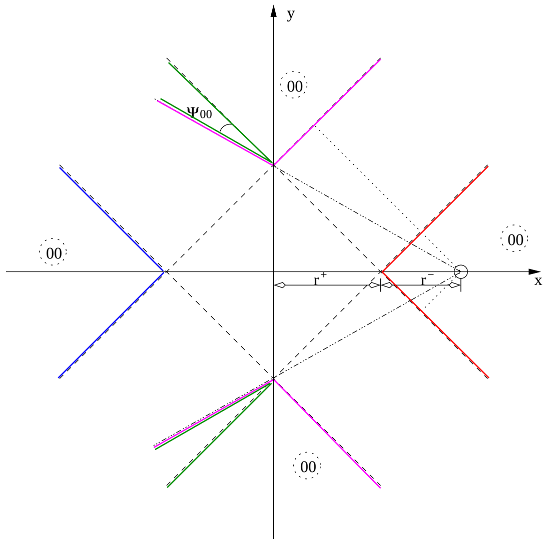


Fig. 6. Illustration for decoding 00.

lines minus the region bounded by two green lines, as shown in Fig. 6. With the above analysis, we can derive the error performance for decoding 00 by

$$\begin{aligned} F_{00} &= \frac{1}{\pi}\int_0^{3\pi/4}\exp\left[-\frac{\rho_2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &-\frac{1}{\pi}\int_0^{\pi/4}\exp\left[-\frac{(2\sqrt{\rho_1}+\sqrt{\rho_2})^2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &-\frac{1}{\pi}\int_0^{\pi/2+\Psi_{00}}\exp\left[-\frac{(2\sqrt{\rho_1}+\sqrt{\rho_2})^2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &+\frac{1}{\pi}\int_0^{\Psi_{00}}\exp\left[-\frac{\rho_2}{4\sigma^2\sin^2(\phi)}\right]d\phi \end{aligned} \quad (7)$$

where

$$\Psi_{00} = \frac{\pi}{4} - \tan^{-1}\left(\frac{\sqrt{\rho_1}}{\sqrt{\rho_1}+\sqrt{\rho_2}}\right)$$

Similarly, we can obtain the error performance for decoding 11 by

$$\begin{aligned} F_{11} &= \frac{1}{\pi}\int_0^{3\pi/4}\exp\left[-\frac{\rho_2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &+\frac{1}{\pi}\int_0^{3\pi/4}\exp\left[-\frac{(2\sqrt{\rho_1}-\sqrt{\rho_2})^2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &-\frac{1}{\pi}\int_0^{\Psi_{11}}\exp\left[-\frac{\rho_2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &-\frac{1}{\pi}\int_0^{\pi/2-\Psi_{11}}\exp\left[-\frac{(2\sqrt{\rho_1}-\sqrt{\rho_2})^2}{4\sigma^2\sin^2(\phi)}\right]d\phi \end{aligned} \quad (8)$$

where

$$\Psi_{11} = \frac{\pi}{4} - \tan^{-1}\left(\frac{\sqrt{\rho_1}-\sqrt{\rho_2}}{\sqrt{\rho_1}}\right)$$

The error performance for decoding 10 and 01 is

$$\begin{aligned} F_{10} = F_{01} &= \frac{1}{\pi}\int_0^{3\pi/4}\exp\left[-\frac{\rho_2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &+\frac{1}{\pi}\int_0^{\pi/2}\exp\left[-\frac{(2\sqrt{\rho_1}-\sqrt{\rho_2})^2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &-\frac{1}{2\pi}\int_0^{\Psi_{11}}\exp\left[-\frac{\rho_2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &-\frac{1}{2\pi}\int_0^{\pi/2-\Psi_{11}}\exp\left[-\frac{(2\sqrt{\rho_1}-\sqrt{\rho_2})^2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &+\frac{1}{\pi}\int_0^{\pi/2}\exp\left[-\frac{(2\sqrt{\rho_1}+\sqrt{\rho_2})^2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &-\frac{1}{2\pi}\int_0^{\Psi_{01}}\exp\left[-\frac{(2\sqrt{\rho_1}-\sqrt{\rho_2})^2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &-\frac{1}{2\pi}\int_0^{\pi/2-\Psi_{01}}\exp\left[-\frac{(2\sqrt{\rho_1}+\sqrt{\rho_2})^2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &-\frac{1}{2\pi}\int_0^{\Psi_{00}}\exp\left[-\frac{\rho_2}{4\sigma^2\sin^2(\phi)}\right]d\phi \\ &-\frac{1}{2\pi}\int_0^{\pi/2-\Psi_{00}}\exp\left[-\frac{(2\sqrt{\rho_1}+\sqrt{\rho_2})^2}{4\sigma^2\sin^2(\phi)}\right]d\phi \end{aligned} \quad (9)$$

where

$$\Psi_{01} = \frac{\pi}{4} - \tan^{-1}\left(\frac{\sqrt{\rho_2}}{2\sqrt{\rho_1}}\right)$$

Finally, the performance for QPSK is

$$F^S_{QPSK} = \frac{1}{4}(F_{00}+F_{11}+2F_{01}). \quad (10)$$

## IV. Numerical Results

In this section, we present numerical results to validate the analysis in Section III. In our study, we consider the SER performance versus the SNR, where SNR is for the stronger signal at the receiver of relay node. In this paper, this is the SNR for information $x_1$. To provide insight for the design, we focus on the SER performance at the physical layer. In other words, we do not consider channel coding or retransmission, which may be utilized in practice.

Fig. 7 shows the performance of BPSK with the same constellation from nodes $S_1$ and $S_2$ to relay node $R$. As we can see from this figure, our analysis is consistent with the simulation results for scheme with different arrival power. Note that, in Fig. 7, the SNR difference means the difference between $\rho_1$ and $\rho_2$ in terms of decibel (dB). For example, "Diff 1dB" denotes that $\rho_1 - \rho_2 = 1dB$.

From Fig. 7, we can also observe that, for the same SNR at the receiver, the performance of physical layer network coding with the same arrival power is very close to the performance of point-to-point transmission with BPSK. In fact, according to Eqs. (3)-(5), we can estimate that, if $\rho_1 = \rho_2 = \rho$, then

$$F^S_{BPSK} = 0.5(F_0+F_1) \le \frac{3}{2\pi}\int_0^{\pi/2}\exp\left[-\frac{\rho}{2\sigma^2\sin^2(\phi)}\right]d\phi$$

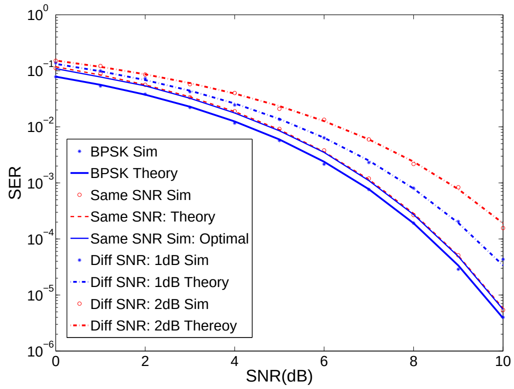


Fig. 7. Performance of BPSK with the same constellation.

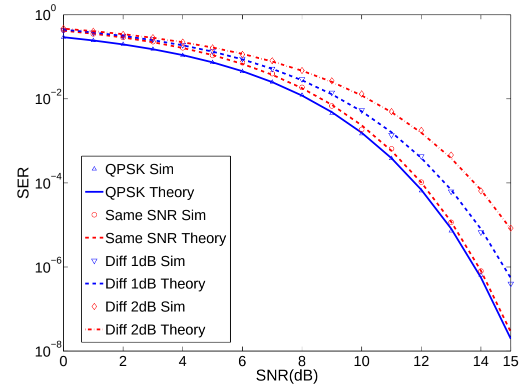


Fig. 9. Performance of QPSK with the same constellations.

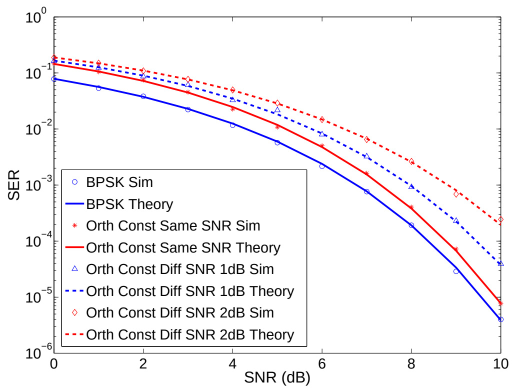


Fig. 8. Performance of BPSK with orthogonal constellations.

Since the SER performance of BPSK is

$$P_{BPSK} = \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{\rho}{2\sigma^2 \sin^2(\phi)}\right] d\phi,$$

we can see that $F^S_{BPSK} \le 1.5 \times P_{BPSK}$, given the same SNR.

It is interesting to note that the performance decreases with the increase of SNR difference. And 1dB difference between received signal leads to 1dB difference in SER performance. This result indicates that senders $S_1$ and $S_2$ should adjust the transmission power such that they have the same SNR at the relay nodes. In Fig. 7 the curve "Same SNR Sim: Optimal" shows the simulation result with optimal threshold when $S_1$ and $S_2$ have the same SNR. There is only 0.2dB difference between the approximation and optimal results.

Fig. 8 compares the performance of BPSK modulation with orthogonal constellations, i.e., $S_1$ sends symbols $\pm 1$ but $S_2$ sends $\pm j$. We can see clearly the match between the analysis and simulation results. Also, the performance with same SNR is better than the cases with SNR difference.

By comparing Fig. 7 and Fig. 8 we can see that the performance of the same constellation is better than the one with different constellations, which suggests that $S_1$ and $S_2$ may coordinate the transmission to choose the same constellation.

Fig. 9 shows the performance comparison for QPSK and it demonstrates similar properties as that of BPSK.

## V. Conclusions

In this paper, we study SER performance for a system with physical layer network coding over AWGN channels. A closed-form representation is derived for BPSK and QPSK modulations, but the techniques presented here can be generalized to other two-dimensional constellations. As shown by both theoretical analysis and simulation results, the SER performance at the relay node to detect the summation of two signals depends on the balance of two SNRs. The optimal SER performance is achieved when the two signals have the same SNR at the receiver. We also discuss the effect of different constellations on the performance. The SER performance of same constellation is better than that of different constellations. In this context, coordination on SNRs and transmission constellation is important for physical layer network coding system.

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