SERIAL CORRELATION AND QUADRATIC FORMS IN NORMAL VARIABLES[1]

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1. **Estimation problems of stochastical processes.** In regression analysis of economic time series a situation often arises in which a certain observed quantity represents a "dependent" variable at one time and an "independent" variable at a later time. For instance, the following relations may exist between the price x_t and the supply y_t of hogs at any time t:

$$(1) \qquad \begin{aligned} x_t &= \alpha - \beta y_t + z_t' \\ y_t &= \gamma + \delta x_{t-1} + z_t''. \end{aligned}$$

The first of these equations expresses the price-depressing influence of large supplies. The second equation expresses the supply-stimulating influence of high prices one time unit (in the case of hogs, about 18 months) earlier. The terms z_t' and z_t'' represent influences of additional variables and/or random disturbances. Elimination of y_t leads to

$$(2) \qquad x_t = \epsilon - \zeta x_{t-1} + z_t .$$

The statistical estimation of the parameters ϵ and ζ of such an equation is usually attempted by the ordinary least squares method, disregarding the fact that the observation x_t is both a dependent variable at time t and an independent variable at time $t + 1$. The following simple example shows that this may lead to erroneous results particularly in small samples. Suppose that $\epsilon = 0$, $\zeta = -1$, and that z_t is a purely random variable with mean 0, while only three successive observations are available. The least squares estimate of η is then given by the slope of the straight line connecting the points (x_1, x_2) and (x_2, x_3) in the plane of x_{t-1} and x_t. This slope, however, has an expected value 0, because according to our assumptions the conditional expectation of x_3 for a prescribed value of x_2 is equal to x_2, whatever value that is. Thus the least squares estimate of $\zeta = -1$ has an expected value 0 showing an important bias.

Mathematical business cycle theories utilize systems of equations much more complicated than the example considered [1]. The common feature of these equation systems is, however, that they reduce fluctuations in a set of economic variables to

1. earlier fluctuations in the same set of variables,
2. changes in given non-economic or external variables, and
3. random disturbances.

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[1] This investigation was carried out at the Local and State Government Section (Princeton Surveys) of the School for Public and International Affairs of Princeton University. The main results were presented to the Chicago meeting of the Institute of Mathematical Statistics in September 1941.

An equation system of this type has been said to define a *stochastical process* in a number of variables [2]. The statistical testing of mathematical business cycle theories accordingly requires a theory of estimation of the parameters of stochastical processes. The operation of stochastical processes is also apparent in meteorological data. Assuming a normal distribution for the random disturbances, it will be seen that the mathematical prerequisite for an estimation theory of stochastical processes is the study of joint distributions of certain quadratic forms in normal variables.

In this article only the very simplest problem of this class will be treated, namely that of testing the significance of ζ in equation (2) if it is known that $|\zeta| < 1$ and that ϵ is equal to zero. This is the problem of testing the significance of single serial regression, or of single serial correlation, because the distinction between single regression and correlation coefficients disappears in this simple case for coefficients absolutely smaller than unity.

In the next section the problem of estimating single serial correlation if the mean is known will be stated and the difficulties involved will be discussed. In section 3 a conditional distribution of a quadratic form in normal variables will be derived. The proof in section 3 covers only forms in five or more variables, but another proof covering any number of variables is given in section 4. This distribution is then applied to devise a test of significance of serial correlation in section 5. The reading of section 4 is not necessary for the understanding of section 5. Readers desiring to locate only the main results can read those from equations (3), (11), (16), (21), (36), (61), (62), (74), (79), (82), (92), and (96).

2. **The estimation of serial correlation.** In the stochastical process

$$x_t = \rho x_{t-1} + z_t, \tag{3}$$

where the z_t are independent drawings from a normal distribution with mean 0 and standard deviation σ, the parameter ρ may have any positive or negative values. The process will only be a stationary one if

$$|\rho| < 1. \tag{4}$$

For, since

$$Ex_t = Ex_{t-1} = Ez_t = 0, \qquad Ez_t^2 = \sigma^2, \tag{5}$$

and

$$Ex_t^2 = \rho^2 Ex_{t-1}^2 + \sigma^2, \tag{6}$$

a variance of x_t independent of t will be possible only if (4) is satisfied, in which case

$$Ex_t^2 = \frac{\sigma^2}{1 - \rho^2}. \tag{7}$$

If (4) is not satisfied, however, Ex_t^2 will be an increasing function of t tending to infinity in approximately geometric progression if t exceeds any limit. In this article the limitation (4) will be imposed a priori.

It follows from (3), (7), and the assumptions regarding z_t, that the joint distribution of the quantities $x_1, z_2, z_3 \cdots z_T$ is given by

$$(8) \qquad \left(\frac{1-\rho^2}{2\pi\sigma^2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}(1-\rho^2)x_1^2/\sigma^2} \cdot \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}(T-1)} e^{-\frac{1}{2}\sum_{t=2}^{T} z_t^2/\sigma^2} \cdot dx_1\, dz_2 \cdots dz_T.$$

Since the Jacobian of the transformation (3) from the variables $x_1, z_2 \cdots z_T$ to the variables $x_1, x_2 \cdots x_T$ equals unity, the joint distribution function of the T successive observations $x_1, x_2 \cdots x_T$ that make up a sample is found simply by replacing the z_t in (8) by the corresponding expressions in the x_t. This leads to the distribution

$$(9) \qquad \frac{(1-\rho^2)^{\frac{1}{2}}}{(2\pi\sigma^2)^{\frac{1}{2}T}} e^{-\frac{1}{2}[l-2\rho m+(1+\rho^2)n]/\sigma^2} dx_1\, dx_2 \cdots dx_T,$$

in which the three quadratic forms

$$(10) \qquad \begin{aligned} l &= x_1^2 + x_T^2, \\ m &= x_1x_2 + x_2x_3 + \cdots + x_{T-1}x_T, \\ n &= x_2^2 + x_3^2 + \cdots + x_{T-1}^2, \end{aligned}$$

are the only characteristics of the sample that enter. In other words, l, m and n are jointly sufficient statistics for the estimation of ρ and σ. It may be noted that these statistics remain the same if the series of observations is taken in inverse order.

It seems natural to attempt maximum likelihood estimation of ρ and σ, even if the usual optimal properties of estimates so obtained have so far not been proved for stochastical processes. Straightforward calculations lead to the following third-degree equation for the maximum likelihood estimate $\hat{\rho}$ of ρ:

$$(11) \qquad (m - \hat{\rho}n)(1 - \hat{\rho}^2) - \frac{\hat{\rho}}{T}[l - 2\hat{\rho}m + (1 + \hat{\rho}^2)n] = 0.$$

Of course the root asymptotically approaching m/n has to be selected. The corresponding maximum likelihood estimate $\hat{\sigma}$ of σ is given by

$$(12) \qquad \hat{\sigma}^2 = \frac{1}{T}[l - 2\hat{\rho}m + (1 + \hat{\rho}^2)n].$$

In view of the complicated definition of $\hat{\rho}$ it seems desirable as a first step to derive from (9) the joint probability distribution of l, m and n. This requires a transformation of the volume element $dx_1 \cdots dx_T$ in (9) to the form

$$(13) \qquad \phi(l, m, n)\, dl\, dm\, dn,$$

which it assumes after integration over $T - 3$ other coordinates the variation of which does not change l, m and n.

Since this is purely a problem of integration completely defined by the expressions (10), the resulting function $\phi(l, m, n)$ is independent of ρ and σ. The joint distribution

$$(14) \qquad \frac{(1 - \rho^2)^{\frac{1}{2}}}{(2\pi\sigma^2)^{\frac{1}{2}T}} e^{-\frac{1}{2}[l - 2\rho m + (1+\rho^2)n]/\sigma^2} \phi(l, m, n)\, dl\, dm\, dn$$

of l, m and n will thus be known for any values of ρ and σ as soon as it is known for two particular values.

If as particular values we choose $\rho = 0$ and $\sigma = 1$, the x_t become identical with the z_t, and the problem is that of finding the joint distribution of the quadratic forms (10) in independent normal variables with mean 0 and variance 1. Even if so simplified, the problem is a complicated one. While there are infinitely many common sets of principal axis of the forms l and n, none of these sets of axis has a single axis in common with m.

Although no solution is offered for this problem, the following suggestion may be ventured. Once $\phi(l, m, n)$ is known, the mathematically simplest procedure for interval estimation of ρ might well be one that confines attention to samples having the same values of l and n as the sample actually obtained. Suitably chosen percentiles of the conditional distribution of m with l and n fixed at the observed values, would be convertible into confidence limits for ρ with the help of (11).

A simpler mathematical problem is encountered in testing whether the existence of a difference between ρ and 0 can be established, or, in other words, in testing the significance of serial correlation. If $\rho = 0$, the distribution function in (9) depends only on $p = l + n$, not on l or n separately, and exact significance limits for m can be derived from the joint distribution

$$(15) \qquad (2\pi\sigma^2)^{-\frac{1}{2}T} e^{-\frac{1}{2}p/\sigma^2} \psi(p, m)\, dp\, dm$$

of p and m only. This distribution will be studied in the next three sections. It is hoped that the methods there applied will provide a useful starting point in the treatment of other problems of the class described in section 1.

3. **Distribution of a quadratic form in normal variables on the unit sphere.** Consider two quadratic forms in T independent normal variables with mean 0 and variance 1,

$$(16) \qquad \begin{aligned} p &= x_1^2 + x_2^2 + \cdots + x_T^2 , \\ q &= \kappa_1 x_1^2 + \kappa_2 x_2^2 + \cdots + \kappa_T x_T^2 . \end{aligned}$$

While the characteristic values of the form p are all coincident with the value 1, the characteristic values κ_t of q are provisionally supposed to be different from each other, so that they can be arranged in decreasing order:

(17) $$\kappa_1 > \kappa_2 > \cdots > \kappa_T .$$

The probability density

(18) $$(2\pi)^{-\frac{1}{2}T}e^{-\frac{1}{2}p}$$

in the space of the variables is constant on any sphere

(19) $$p = p_0 = \text{constant},$$

while the distribution function $g(p)$ of p is that of the χ^2-distribution with T degrees of freedom

(20) $$g(p) = \frac{p^{\frac{1}{2}T-1}e^{-\frac{1}{2}p}}{2^{\frac{1}{2}T}\Gamma(\frac{1}{2}T)}.$$

The hyper-surfaces on which the ratio

(21) $$r = \frac{q}{p}$$

of q to p is constant are cones with the origin as vertex dissecting the same proportion of the metric "surface" of each sphere (19). It follows that the conditional distribution function of r for a prescribed value p_0 of p is independent of that value p_0, and is therefore equal to the unrestricted distribution function $h(r)$ of r. In other words, p and r are independently distributed. Their joint distribution being

(22) $$g(p)h(r)\,dp\,dr,$$

the joint distribution of p and $q = rp$ is found to be

(23) $$f(p, q)\,dp\,dq = g(p)h\left(\frac{q}{p}\right)dp\frac{dq}{p} = \frac{g(p)}{p}\,h\left(\frac{q}{p}\right)dp\,dq.$$

The function $h(\ )$ may therefore also be described as the conditional distribution function of q on the unit sphere

(24) $$p = 1.$$

Since κ_1 and κ_T are the extreme values of q under the condition (24), the function $h(r)$ vanishes outside these limits.

We shall now derive an expression for $h(r)$ by comparing (23) with an expression for $f(p, q)$ obtained through the inversion theorem of characteristic functions. The characteristic function $F(\eta, \theta)$ corresponding to the variables p and q is

(25) $$F(\eta, \theta) = (2\pi)^{-\frac{1}{2}T}\int e^{-\frac{1}{2}p+i(\eta p+\theta q)}\,dx_1 \cdots dx_T = D^{-\frac{1}{2}}(\eta, \theta),$$

where, according to (16), the polynomial $D(\eta, \theta)$ is given by

(26) $$D(\eta, \theta) = \prod_{t=1}^{T}(1 - 2i\eta - 2i\theta\kappa_t).$$

It follows from the inversion theorem that

$$f(p, q) = (2\pi)^{-2} \iint_{-\infty}^{\infty} e^{-i(\eta p+\theta q)} D^{-\frac{1}{2}}(\eta, \theta)\, d\eta\, d\theta, \tag{27}$$

the order of integration over η and θ being immaterial.

Any elementary factor of $D(\eta, \theta)$ may be written

$$d_t(\eta, \theta) = 1 - 2i\eta - 2i\theta\kappa_t = (1 - 2i\eta)\left(1 - \frac{2i\theta\kappa_t}{1 - 2i\eta}\right). \tag{28}$$

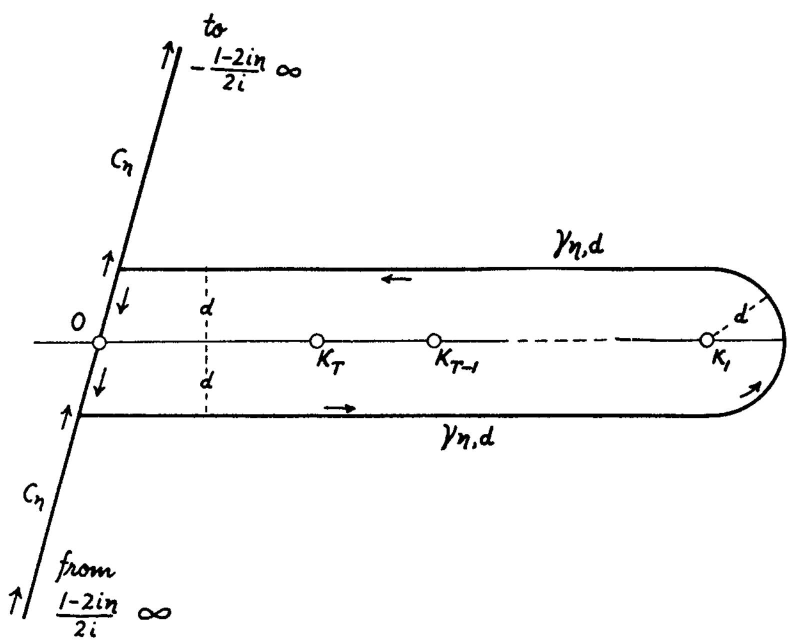

FIGURE 1. Paths of integration in the κ-plane

First considering the integration over θ (while η has some fixed value), we may instead of θ use

$$\kappa = \frac{1 - 2i\eta}{2i\theta} \tag{29}$$

as an integration variable. The path of integration c_η in the κ-plane then is a straight line from 0 to $-\frac{1 - 2i\eta}{2i}\infty$ and another straight line from $\frac{1 - 2i\eta}{2i}\infty$ back to 0, as indicated in Figure 1, and the transformed integral (27) runs

$$f(p, q) = (2\pi)^{-2} \int_{-\infty}^{\infty} \left[e^{-i\eta p}(1 - 2i\eta)^{-\frac{1}{2}T+1} \int_{c_\eta} e^{-(1-2i\eta)q/2\kappa} \cdot \left\{ \prod_{t=1}^{T} \left(1 - \frac{\kappa_t}{\kappa}\right) \right\}^{-\frac{1}{2}} \left(-\frac{1}{2i\kappa^2}\right) d\kappa \right] d\eta.$$

The integrand

$$e^{-(1-2i\eta)q/2\kappa}\left\{\prod_t\left(1-\frac{\kappa_t}{\kappa}\right)\right\}^{-\frac{1}{2}}\left(-\frac{1}{2i\kappa^2}\right) \tag{31}$$

for the integration over κ has singularities only in the points $\kappa = 0$ and $\kappa = \kappa_t$, $t = 1, 2 \cdots T$. In order to simplify the argument we shall suppose that the quadratic form q is positive definite, or, in connection with (17), that $\kappa_T > 0$. The location of the singularities is then as pictured in Figure 1. At $\kappa = \infty$ the integrand (31) is regular and of the order of magnitude of κ^{-2}. Consequently a curve integral of (31) along the whole or any part of the circle $|\kappa| = R$ will tend to 0 if R tends to infinity. Using a theorem of Cauchy, it is therefore permissible in (30) to replace the described path c_η by another path $c_{\eta,R}$ which starts out along c_η from 0 up to $-\frac{1-2i\eta}{2i}R$, from there follows the circle $|\kappa| = R$ to the right over an angle π up to the point $\frac{1-2i\eta}{2i}R$, and from there returns to 0 along c_η—provided that $R > \kappa_1$. After reversing the direction in which the path is followed in order to do away with the negative sign in (31), the path so obtained can again be replaced by the path $\gamma_{\eta,d}$ shown in Figure 1, which coincides with c_η only up to a small distance d from the real axis, and encircles all singularities κ_t while retaining a distance d from the part of the real axis to the left of and up to κ_1. Finally, a path of integration γ' independent of the value of η is obtained by going to the limit in which $d = 0$. This is an integration twice along the part of the real axis between 0 and κ_1, integrating from 0 to κ_1 that branch of the integrand which is obtained by passing "under" each singularity, and going back from κ_1 to 0 with the branch obtained by passing "around" κ_1 and "over" each other singularity[2]. The integral so obtained converges at each singularity. This is also true for the singularity $\kappa = 0$ because we are dealing only with positive values of q, which makes the exponential factor in (31) tend to 0 if κ approaches zero. We shall now show that if in (30) the path γ' is substituted for c_η (with a change in sign), the order of integration over κ and η can be reversed if $T \geqq 5$.

The integral over η, taken from (30),

$$I = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-i\eta(p-q/\kappa)}(1-2i\eta)^{-\frac{1}{2}T+1}\,d\eta, \tag{32}$$

(in which κ is now a positive real number), is by the substitution $\chi^2 = p - q/\kappa$ transformed to the integral encountered in the derivation of the χ^2-distribution (with $T - 2$ degrees of freedom) by the inversion theorem of characteristic functions. It may be quoted without proof (see [3] p. 42) that it equals

$$\begin{cases} I = \dfrac{(p-q/\kappa)^{\frac{1}{2}T-2}e^{-\frac{1}{2}(p-q/\kappa)}}{2^{\frac{1}{2}T-1}\Gamma(\frac{1}{2}T-1)}, & \text{if } p - q/\kappa \geqq 0, \text{ or } \kappa \geqq r, \\ I = 0, & \text{if } p - q/\kappa \leqq 0, \text{ or } \kappa \leqq r. \end{cases} \tag{33}$$

[2] For even values of T the parts of γ' for which $\kappa < \kappa_T$ can be disregarded, because on these parts the same branch of the integrand is integrated in opposite directions.

It is necessary to observe, however, that the integral I converges uniformly for all real values of κ whenever $T \geqq 5$, because then

$$(34) \qquad \int_{-\infty}^{\infty} |\, 1 - 2i\eta \,|^{-\frac{1}{2}T+1}\, d\eta,$$

is convergent. Because of this property, the reversal of the order of integration is allowed for $T \geqq 5$.

If now in (30) we first carry out the integration over η and use (33), we are left with

$$(35) \qquad f(p, q) = \frac{e^{-\frac{1}{2}p}}{2^{\frac{1}{2}T}\Gamma(\frac{1}{2}T - 1)} \int_{\gamma_r} \left(p - \frac{q}{\kappa}\right)^{\frac{1}{2}T-2} \left\{\prod_t \left(1 - \frac{\kappa_t}{\kappa}\right)\right\}^{-\frac{1}{2}} \frac{d\kappa}{2i\kappa^2},$$

where γ_r now is any curve proceeding from $\kappa = r$ into the lower half-plane, crossing the real axis at a point $\kappa > \kappa_1$, and returning to $\kappa = r$ through the upper half-plane, as indicated in Figure 2. (The path directly obtained is a path γ_r' consisting of twice the real axis between r and κ_1, the branches of the integrand being taken as indicated by γ_r). Comparing (35) with (23) and (20), using (21)

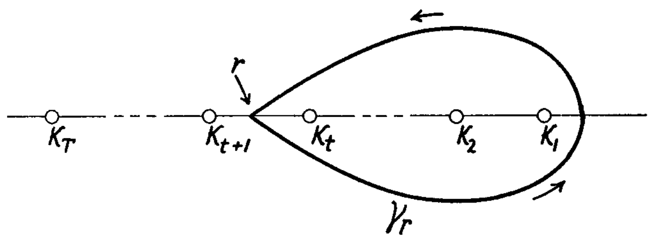

FIGURE 2. The integration path γ_r

and the well-known formula $\Gamma(x) = (x - 1)\Gamma(x - 1)$, we find the following expression for the distribution function of r:

$$(36) \qquad h(r) = \frac{\frac{1}{2}T - 1}{2\pi i} \int_{\gamma_r} \frac{(\kappa - r)^{\frac{1}{2}T-2}}{\prod_t (\kappa - \kappa_t)^{\frac{1}{2}}}\, d\kappa.$$

This function vanishes for $r \geqq \kappa_1$. In order to arrive at a positive distribution function for $\kappa_T < r < \kappa_1$ that branch of the integrand must be selected which is positive for real values of κ exceeding κ_1.

It is worth noting that the degree in κ of the numerator of the integrand is two less than that of the denominator. Owing to this fact, indeed, the distribution function $h(r)$ satisfies the two obvious conditions:

$$(37) \qquad h(r) = 0 \quad \text{for} \quad r \leqq \kappa_T, \qquad \int_{\kappa_T}^{\kappa_1} h(r)\, dr = 1.$$

For $r \leqq \kappa_T$ the path of integration in (36) can be replaced by any closed contour enclosing all the singularities $r, \kappa_T, \cdots, \kappa_1$ (r is a singularity only if T is odd). Taking as such a contour the circle $|\, \kappa \,| = R$ with R tending to infinity, we find that $h(r) = 0$ because the integrand is of an order κ^{-2} at $\kappa = \infty$. Further, if γ_r is again replaced by γ_r' which runs entirely along the real axis,

$$
(38)\quad \begin{cases} \displaystyle\int_{\kappa_T}^{\kappa_1} h(r)\,dr = \frac{\frac{1}{2}T-1}{2\pi i}\int_{\kappa_T}^{\kappa_1}\left[\int_{\gamma_r'}\frac{(\kappa-r)^{\frac{1}{2}T-2}}{\prod_t(\kappa-\kappa_t)^{\frac{1}{2}}}\,d\kappa\right]dr \\[2ex] \displaystyle\qquad = \frac{\frac{1}{2}T-1}{2\pi i}\int_{\gamma'_{\kappa_T}}\left[\prod_t(\kappa-\kappa_t)^{-\frac{1}{2}}\int_{\kappa_T}^{\kappa}(\kappa-r)^{\frac{1}{2}T-2}\,dr\right]d\kappa \\[2ex] \displaystyle\qquad = \frac{1}{2\pi i}\int_{\gamma'_{\kappa_T}}\frac{(\kappa-r)^{\frac{1}{2}T-1}}{\prod_t(\kappa-\kappa_t)^{\frac{1}{2}}}\,d\kappa = 1, \end{cases}
$$

because the integrand in the last integral is of the order of κ^{-1} at the point $\kappa = \infty$.

The quantities r and κ_t enter into the right hand member of (36) only in the form of differences from the integration variable κ. The addition of a constant ϵ to both r and the κ_t will therefore merely result in a change of location of the distribution on the r-axis without a change in form:

$$
(39)\qquad h^*(r+\epsilon) = h(r).
$$

This could be expected since such a transformation means the addition of ϵp to the quadratic form q studied. It follows that the validity of (36) is not limited to positive definite quadratic forms q, since any other quadratic form can be transformed to a positive definite form by this operation if a sufficiently large value of ϵ is taken.

The function $h(r)$ is a different analytic function between any two different successive characteristic values κ_t and κ_{t-1}. The expression (36) holds for even and for odd values of T, and is also valid for any number of coincidences in the set of characteristic values κ_t. It is true that integration along the paths γ' or γ_r' entirely coincident with the real axis, such as has been introduced in intermediate stages of the above proof, cannot be done if two or more of the κ_t coincide, because of divergence of the integral. Once (36) has been established for distinct characteristic values, however, it follows from considerations of continuity that this result holds good also if coincidences occur in the set κ_t.

The function $h(r)$ has been studied by von Neumann [4] by an entirely different and very ingeneous method for the special case that T is even while no two characteristic values are equal, and for the case that the characteristic values are equal two by two but otherwise different. The properties established by von Neumann, and some generalizations of these properties, can be derived from (36). If T is even, the derivative of $h(r)$ of order $\frac{1}{2}T - 1$ is

$$
(39)\quad \left(\frac{d}{dr}\right)^{\frac{1}{2}T-1} h(r) \begin{cases} \displaystyle = \frac{(\frac{1}{2}T-1)!}{\pi}\cdot\frac{(-1)^{\frac{1}{2}(T-t-1)}}{\prod_{s=1}^{T}|r-\kappa_s|^{\frac{1}{2}}} & \text{if } \kappa_{t+1} < r < \kappa_t \text{ and } t \text{ odd}, \\ \text{does not exist for } r = \kappa_t,\ t = 1, 2 \cdots T, \\ = 0 \quad \text{for all other values of } r. \end{cases}
$$

If all characteristic values are distinct, all derivatives of an order lower than $\frac{1}{2}T - 1$ exist and are continuous everywhere. Generally, whether T be even or odd, at a point where k characteristic values coincide $\left(\frac{d}{dr}\right)^j h(r)$ will exist and will be continuous if $j \leqq \frac{1}{2}(T - k) - \frac{3}{2}$, and will not exist if $j \geqq \frac{1}{2}(T - k) - 1$.

If the characteristic values are pairwise equal,

$$\kappa_{2s-1} = \kappa_{2s} = \lambda_s , \qquad s = 1, 2, \cdots S, \tag{40}$$

but otherwise distinct, their total number $T = 2S$ must be even, and the only singularities of the integrand in (36) are poles at the points $\kappa = \lambda_s$. Accordingly the path of integration γ_r can be considered as a closed curve, and the integral in (36) can be replaced by the sum of the residuals of the integrand at all poles inside the curve:

$$h(r) = (S - 1) \sum_{s=1}^{s_r} \frac{(\lambda_s - r)^{S-2}}{P'(\lambda_s)}, \quad \text{if } \lambda_{s_r+1} < r < \lambda_{s_r}. \tag{41}$$

Here $P'(\lambda)$ is the derivative of

$$P(\lambda) = \prod_{\| 1s}^{S} (\lambda - \lambda_s), \tag{42}$$

its value in the point $\lambda = \lambda_s$ being

$$P'(\lambda_s) = \left[\frac{P(\lambda)}{\lambda - \lambda_s}\right]_{\lambda=\lambda_s} = \prod_{\substack{u=1 \\ (u \neq s)}}^{S} (\lambda_s - \lambda_u). \tag{43}$$

For $S = 2$ this is simply the rectangular distribution

$$h(r) = \frac{1}{\lambda_1 - \lambda_2}, \qquad \lambda_2 < r < \lambda_1. \tag{44}$$

The numerical calculation of the distribution (36) with distinct characteristic values is extremely cumbersome except for very small values of T. If the characteristic values κ_t follow some definite pattern, however, it may be possible in some instances to work out a reasonable approximation formula. Two examples of this type will be shown in section 5.

4. **Another proof that covers also cases with $T < 5$.** The proof of (36) given above holds only for $T \geqq 5$. Once the form of (36) is known or presumed, however, another proof of its validity is available, which has mathematical interest in itself, and covers all cases from $T = 2$ upwards. This is a proof by complete induction, based on the proposition that, if (36) holds for T variables, then it also holds for $T + 1$ variables. This proposition again rests on the recurrent relation

$$h_{T+1}(r') = \frac{\Gamma(\frac{1}{2}T + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}T)} (r' - \kappa_{T+1})^{\frac{1}{2}T-1} \int_{r'}^{\kappa_1} (r - r')^{-\frac{1}{2}}(r - \kappa_{T+1})^{-\frac{1}{2}T+\frac{1}{2}} h_T(r)\, dr, \tag{45}$$

$$\text{if } \kappa_{T+1} < r' < \kappa_1 \text{ and } \kappa_{T+1} < \kappa_T,$$

proved elsewhere in this issue by von Neumann[3] [5]. It connects the distribution function $h_T(r)$ for T variables with the function $h_{T+1}(r')$ obtained by the addition of one variable x_{T+1} and one characteristic value κ_{T+1}.

We shall substitute the "presumed" expression (36) for $h_T(r)$ with $T \geqq 3$ in (45) in order to show that the result for $h_{T+1}(r)$ is the same expression with T increased by one. In this proof if has for simplicity's sake been assumed that the new characteristic value κ_{T+1} is smaller than any of those already present, and that no two of the κ_t are equal. It is then possible again to select in (36) the path of integration γ_r' which proceeds along the real axis from r to κ_1 and returns along the real axis to r, passing each singularity in the same way as γ_r does. If the integral (36) is substituted in (45) in this form, the order of integration over κ and r can be reversed, the result being

$$\begin{aligned} h_{T+1}(r') &= \frac{\frac{1}{2}T - 1}{2\pi i} \frac{\Gamma(\frac{1}{2}T + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}T)} (r' - \kappa_{T+1})^{\frac{1}{2}T-1} \\ &\cdot \int_{\gamma_{r'}'} \left[\left\{ \prod_{t=1}^{T} (\kappa - \kappa_t) \right\}^{-\frac{1}{2}} \int_{r'}^{\kappa} (r - r')^{-\frac{1}{2}}(r - \kappa_{T+1})^{-\frac{1}{2}T+\frac{1}{2}}(\kappa - r)^{\frac{1}{2}T-2}\, dr \right] d\kappa. \end{aligned} \tag{46}$$

Writing for greater clarity $\kappa_{T+1} = a$, $r' = b$, $\kappa = c$, $r = z$, we have to evaluate the integral

$$I_T(a, b, c) = \int_b^c (z - a)^{-\frac{1}{2}T+\frac{1}{2}}(z - b)^{-\frac{1}{2}}(c - z)^{\frac{1}{2}T-2}\, dz, \tag{47}$$

$$a < b < c,\ T \geqq 3,$$

with the positive square roots taken if z is real and $b < z < c$. Suppose first that $T = 2S + 1$ or odd. Then the integrand

$$\phi_{2S+1}(z) = (z - a)^{-S}(z - b)^{-\frac{1}{2}}(c - z)^{S-\frac{3}{2}} \tag{48}$$

has singularities at a, b and c, of which only those at b and c are of a type such that $\phi_{2S+1}(z)$ changes its sign if the argument z is turned once around the singularity. It follows that

$$2I_{2S+1} = \int_\delta \phi_{2S+1}(z)\, dz, \tag{49}$$

the path of integration δ being as indicated in Figure 3. For if the curve δ is contracted so as to run entirely along the real axis, from b to c and back to b, the two parts of the curve will each yield a contribution equal to I_{2S+1}, the understanding being that positive square roots are taken when going from b to c.

The integrand $\phi_{2S+1}(z)$ is regular at $z = \infty$ and of order z^{-2} in a neighborhood of that point. It follows that

[3] I am greatly indebted to Professor von Neumann for communicating this relation to me before its publication.

$$(50) \qquad -2I_{2S+1} = \int_{\epsilon} \phi_{2S+1}(z)\, dz,$$

where ϵ, as in Figure 3, encloses the only singularity not enclosed by δ. In a neighborhood of $z = a$ the following expansion of $\phi_{2S+1}(z)$ holds:

$$(51) \qquad \phi_{2S+1}(z) = \sum_{s=0}^{\infty} \frac{(z-a)^{-S+s}}{s!} \left[\left(\frac{\partial}{\partial z}\right)^{s} (z-b)^{-\frac{1}{2}}(c-z)^{S-\frac{1}{2}} \right]_{z=a}$$

The only term contributing to (50) is that with $-S + s = -1$. Since we selected a branch of $\phi_{2S+1}(z)$ such that $(z-b)^{-\frac{1}{2}}(c-z)^{S-\frac{1}{2}}$ falls on the positive pure imaginary axis for real values of z below b, this term can be written

$$(52) \qquad (z-a)^{-1} \frac{i}{(S-1)!} \left(\frac{\partial}{\partial a}\right)^{S-1} (b-a)^{-\frac{1}{2}}(c-a)^{S-\frac{1}{2}},$$

where positive square roots should now be taken. The contribution of this term in (50) is $2\pi i$ times the coefficient of $(z-a)^{-1}$, and therefore

$$I_{2S+1} = \frac{\pi}{(S-1)!} \left(\frac{\partial}{\partial a}\right)^{S-1} (b-a)^{-\frac{1}{2}}(c-a)^{S-\frac{1}{2}}$$

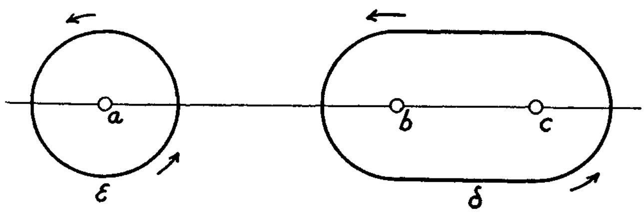

FIGURE 3. The integration paths δ and ϵ

$$(53) \qquad = \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot \cdots (S-\frac{3}{2})\pi}{(S-1)!} (-1)^{S-1} \sum_{s=0}^{S-1} \binom{S-1}{s} (-1)^{s} (b-a)^{-\frac{1}{2}-s}(c-a)^{-\frac{1}{2}+s}$$

$$= \frac{\Gamma(S-\frac{1}{2})\pi}{\Gamma(\frac{1}{2})\Gamma(S)} (b-a)^{-S+\frac{1}{2}}(c-a)^{-\frac{1}{2}}(c-b)^{S-1}.$$

Since $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$, it follows that for odd values of T

$$(54) \qquad I_T = \frac{\Gamma(\frac{1}{2}T - 1)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}T - \frac{1}{2})} (b-a)^{-\frac{1}{2}T+1}(c-a)^{-\frac{1}{2}}(c-b)^{\frac{1}{2}(T-3)}.$$

It is easily seen that the same relation holds good if $T = 2S$ is even. In that case it follows from (47) that

$$(55) \qquad \left(\frac{\partial}{\partial c}\right)^{S-1} I_{2S} = \frac{\partial}{\partial c}(S-2)! \int_b^c (z-a)^{-S+\frac{1}{2}}(z-b)^{-\frac{1}{2}}\, dz$$

$$= (S-2)!(c-a)^{-S+\frac{1}{2}}(c-b)^{-\frac{1}{2}}.$$

In a manner similar to the transformations in (53) it can likewise be proved that the right hand member in (54) has the same derivative of order $S - 1$

with respect to c. It follows that the two members of (54) differ by a polynomial $Q(c)$ in c of a degree at most equal to $S - 2$, the coefficients of which may depend on a and b. However, both members of (54) as well as their first $S - 2$ derivatives with respect to c vanish if $c = b$. Therefore $Q(c)$ vanishes identically, and (54) holds for any integral values of T not smaller than 3.

Finally, if (54) is inserted in (46) an expression for $h_{T+1}(r')$ is obtained which corresponds to (36) with T replaced by $T + 1$.

It remains to prove (36) for some initial value of T. For $T = 2$ the integral in (36) is divergent, but the form of $h(r)$ is easily found directly. Writing

$$p = x_1^2 + x_2^2, \qquad r = \frac{q}{p} = \frac{\kappa_1 x_1^2 + \kappa_2 x_2^2}{x_1^2 + x_2^2}, \tag{56}$$

we find that

$$\begin{aligned} \frac{\partial(x_1, x_2)}{\partial(p, r)} &= \left[\frac{\partial(p, r)}{\partial(x_1, x_2)}\right]^{-1} = \begin{vmatrix} 2x_1 & 2x_2 \\ \frac{2x_1}{p}(\kappa_1 - r) & \frac{2x_2}{p}(\kappa_2 - r) \end{vmatrix}^{-1} \\ &= -\frac{p}{4x_1 x_2(\kappa_1 - \kappa_2)} = -\frac{1}{4(\kappa_1 - r)^{\frac{1}{2}}(r - \kappa_2)^{\frac{1}{2}}}. \end{aligned} \tag{57}$$

The probability density in the x_1-x_2-plane is, of course, $(2\pi)^{-1}e^{-\frac{1}{2}p}$, but in making the transformation (57) a factor 4 must be applied to account for the fact that to given values of p and r correspond 4 sets of values of x_1 and x_2, differing in the signs only. This leads to the joint distribution of p and r

$$\frac{1}{2\pi} e^{-\frac{1}{2}p} \frac{dp\, dr}{(\kappa_1 - r)^{\frac{1}{2}}(r - \kappa_2)^{\frac{1}{2}}}, \tag{58}$$

and, after integration over p, to

$$\begin{aligned} h_2(r) &= \frac{1}{\pi(\kappa_1 - r)^{\frac{1}{2}}(r - \kappa_2)^{\frac{1}{2}}}, \quad \text{if } \kappa_2 < r < \kappa_1, \\ &= 0, \quad \text{if } r < \kappa_2 \quad \text{or} \quad \kappa_1 < r, \end{aligned} \tag{59}$$

in accordance with (39).

Finally, if (59) is inserted in (45) with $T = 2$, the result is

$$h_3(r') = \frac{1}{2\pi} \int_{[\kappa_2, r']}^{\kappa_1} \frac{(r - r')^{-\frac{1}{2}}}{(\kappa_1 - r)^{\frac{1}{2}}(r - \kappa_2)^{\frac{1}{2}}(r - \kappa_3)^{\frac{1}{2}}} dr, \qquad \kappa_3 < r' < \kappa_1, \tag{60}$$

if $[\kappa_2, r']$ denotes the largest of κ_2 and r'. Writing κ for r, we find that this integral is equivalent to that in (36) for $T = 3$, taking into account the rule established for selecting the branch of the integrand in (36). For, taking the path of integration γ_r' coincident with the real axis, the equal contributions from the two parts of the path between κ_2 and κ_1 reinforce each other, while for $r' < \kappa_2$ the remaining contributions (intervals between r' and κ_2) add up to zero. This completes the second proof of (36).

5. **Application to serial correlation.** We shall now derive the characteristic values κ_t in the case that

$$q = m = x_1x_2 + x_2x_3 + \cdots + x_{T-1}x_T . \tag{61}$$

It will be of interest to compare this case with the slightly modified case of the quadratic form

$$\bar{m} = x_1x_2 + x_2x_3 + \cdots + x_{T-1}x_T + x_Tx_1 , \tag{62}$$

which contains an additional term x_Tx_1 accomplishing a circular arrangement of the variables. This modification was originally suggested by Hotelling in order to simplify the characteristic polynomial. Other simplifications arising out of the circular arrangement will appear below. It is possible, of course, that the power of the test of significance of serial correlation is slightly affected by the substitution of $\bar{m}$ for m, but this presumption needs corroboration by a study of power functions.

The characteristic values of m are those values of κ for which the determinant of order T

$$\Delta_T = \begin{vmatrix} -\kappa & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & -\kappa & \frac{1}{2} & \cdots & 0 \\ 0 & \frac{1}{2} & -\kappa & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & -\kappa \end{vmatrix} = 0. \tag{63}$$

By development according to elements of the first row we find that

$$\Delta_T = -\kappa\Delta_{T-1} - \tfrac{1}{4}\Delta_{T-2} , \tag{64}$$

from which it follows that

$$\Delta_T = c_1\xi_1^T + c_2\xi_2^T , \tag{65}$$

if ξ_1 and ξ_2 are the roots of

$$\xi^2 + \kappa\xi + \tfrac{1}{4} = 0, \tag{66}$$

satisfying

$$\xi_1 + \xi_2 = -\kappa, \qquad \xi_1\xi_2 = \tfrac{1}{4}. \tag{67}$$

By inserting the known values of Δ_1 and Δ_2 in (65), the values of c_1 and c_2 are easily found to be such that

$$\Delta_T = \frac{\xi_1^{T+1} - \xi_2^{T+1}}{\xi_1 - \xi_2} . \tag{68}$$

Although as a polynomial in κ this is a rather complicated expression, the implicit form (68) will suffice for finding the roots of (63). Expressing all other variables in terms of one new variable ω,

$$\xi_1 = -\frac{\omega}{2}, \qquad \xi_2 = -\frac{1}{2\omega}, \qquad \kappa = \tfrac{1}{2}(\omega + \omega^{-1}), \tag{69}$$

we find for (68),

$$\Delta_T = (-\tfrac{1}{2})^T \frac{\omega^{T+1} - \omega^{-T-1}}{\omega - \omega^{-1}} = \left(-\frac{1}{2\omega}\right)^T \frac{\omega^{2(T+1)} - 1}{\omega^2 - 1}. \tag{70}$$

The only values of ω for which this expression vanishes are the roots of

$$\omega^{2(T+1)} = 1, \tag{71}$$

excepting those that are also roots of

$$\omega^2 = 1. \tag{72}$$

This leaves us with

$$\omega = e^{\pm \pi i t/(T+1)}, \qquad t = 1, 2, \cdots T. \tag{73}$$

The corresponding characteristic values are

$$\kappa_t = \cos \frac{\pi t}{T+1}, \qquad t = 1, 2 \cdots T, \tag{74}$$

because the same value of κ_t is obtained whether the positive or the negative sign is taken in (73). These are T different values κ_t, and hence each one is a single root of (63).

The characteristic values of $\bar{m}$ can now be derived from (68), although a simple straightforward method based on the properties of circulants is also available (see [6], p. 13). Writing

$$\bar{\Delta}_T = \begin{vmatrix} -\kappa & \frac{1}{2} & 0 & \cdots & \frac{1}{2} \\ \frac{1}{2} & -\kappa & \frac{1}{2} & \cdots & 0 \\ 0 & \frac{1}{2} & -\kappa & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2} & 0 & 0 & \cdots & -\kappa \end{vmatrix} = \Delta_T + 2(-1)^{T-1}(\tfrac{1}{2})^T - \tfrac{1}{4}\Delta_{T-2}, \tag{75}$$

we find easily from (70) that

$$\begin{aligned} \bar{\Delta}_T &= (-\tfrac{1}{2})^T \left(\frac{\omega^{T+1} - \omega^{-T+1}}{\omega - \omega^{-1}} - 2 - \frac{\omega^{T-1} - \omega^{-T+1}}{\omega - \omega^{-1}} \right) \\ &= (-\tfrac{1}{2})^T(\omega^T + \omega^{-T} - 2) = -(-\tfrac{1}{2})^{T-1}(\cos T\alpha - 1), \end{aligned} \tag{76}$$

if

$$\omega = e^{i\alpha} \tag{77}$$

A complete set of the values ω_t for which $\bar{\Delta}_T$ vanishes is found from

$$\alpha_t = \frac{2\pi t}{T}, \qquad t = 1, 2 \cdots T, \tag{78}$$

and the corresponding characteristic values[4] $\bar{\kappa}_t$ are, according to (69),

[4] In order to simplify the formulae, the numbering of characteristic values according to decreasing size has been abandoned in (79).

$$(79) \qquad \bar{\kappa}_t = \cos \alpha_t = \cos \frac{2\pi t}{T}, \qquad t = 1, 2 \cdots T.$$

In contradistinction to the case without circular arrangement, the characteristic values with indices t and $T - t$ now coincide, such that all characteristic values are double except one ($\bar{\kappa}_T = 1$) if T is odd, and except two ($\bar{\kappa}_T = 1$, $\bar{\kappa}_{\frac{1}{2}T} = -1$) if T is even.

Taking advantage of the duplicity of almost all characteristic values, Anderson [6] has derived expressions equivalent to (36) for this case, using methods that depend on this particular condition. On the basis of these results he has computed 99- and 95-percentiles in the distribution of $\bar{r} = \frac{\bar{m}}{p}$ for the values $T = 2, 3, 4, 5, 6, 7, 9, 11, 13, 15, 25, 45$, interpolating the percentiles for intermediate values of T. The 95-percentile for $T = 45$ is 0.240, as compared with 0.261 for the normal distribution that provides an asymptotic approximation.

Whereas on this showing the normal approximation is slow in becoming accurate with increasing T, a method for obtaining a much closer approximation is available, which works out simplest with respect to $\bar{r}$, but can also be applied to r. The principle of this method is applicable whenever the characteristic values follow a definite mathematical pattern.

The method consists in replacing the finite number of discrete values $\bar{\kappa}_t$ in (36) by a continuous variable λ, distributed according to a density function suggested by, and as closely as possible approximating to, the scatter of the values $\bar{\kappa}_t$. According to (79) the values $\bar{\kappa}_t$ are ordinates of the cosine function at equidistant points spaced out so as to cover one complete period 2π of that function. It is natural to approximate this scatter by the density function

$$(80) \qquad \bar{\chi}(\lambda) = \frac{T}{\pi(1 - \lambda^2)^{\frac{1}{2}}},$$

of the cosine $\lambda = \cos \frac{2\pi\bar{\tau}}{T}$ of an expression in which the variable $\bar{\tau}$ has a rectangular distribution between 0 and T. The numerical factor in (80) is such that

$$(81) \qquad \int_{-1}^{1} \bar{\chi}(\lambda)\, d\lambda = T$$

equals the total number of characteristic values to be replaced by a density function. The idea underlying the substitution of $\bar{\chi}(\lambda)$ for the $\bar{\kappa}_t$ is to obtain what intuitively seems to be in some sense the closest approximation to the exact distribution function $\bar{h}(\bar{r})$ that has continuous derivatives of any order in any point except the two points ($\bar{r} = -1$ and $\bar{r} = +1$) that limit its range.

The factor in the integrand in (36) which involves the $\bar{\kappa}_t$ is approximated as follows:

$$(82) \qquad \prod_{t=1}^{T} (\kappa - \bar{\kappa}_t)^{-\frac{1}{2}} = e^{-\frac{1}{2}\sum_{t=1}^{T} \log(\kappa - \bar{\kappa}_t)} \sim \exp\left[-\frac{T}{2\pi} \int_{-1}^{1} \frac{\log(\kappa - \lambda)}{(1 - \lambda^2)^{\frac{1}{2}}}\, d\lambda \right].$$

In order to evaluate the integral

$$J = \int_{-1}^{1} \frac{\log(\kappa - \lambda)}{(1 - \lambda^2)^{\frac{1}{2}}} d\lambda \tag{83}$$

we shall first prove that its real part is independent of κ, or that

$$\Re \int_{-1}^{1} \frac{\log(\kappa - \lambda) - \log(-\lambda)}{(1 - \lambda^2)^{\frac{1}{2}}} d\lambda = 0, \tag{84}$$

if $\Re$ denotes "the real part of". The integrand in (84) has singularities at the points $\lambda = -1, 0, \kappa, 1$. These are of two types. The singularities $\lambda = \pm 1$ are introduced by the denominator and make the integrand change its sign if the argument λ is turned once around either singularity. If starting from a point on the real axis we turn the argument λ once around either of the other singularities, $\lambda = 0$ and $\lambda = \kappa$, introduced by the numerator, then the real part of the integrand is not affected, while $2\pi i$ or $-2\pi i$ is added to the imaginary part of the numerator, depending on the sense (clockwise or anti-clockwise) of the rotation *and* on the sign of the logarithm in (84) responsible for the singular-

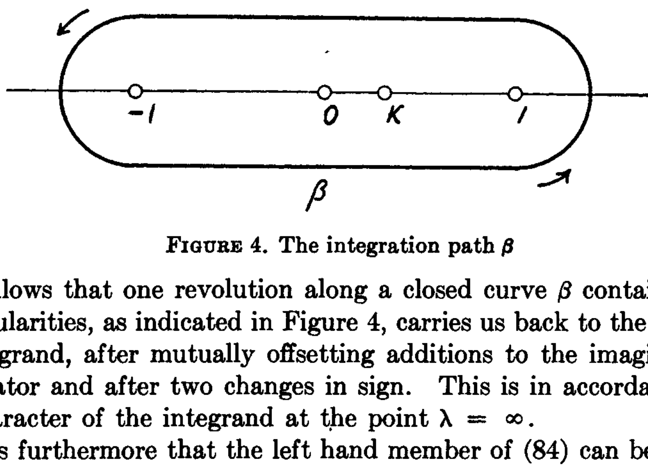

FIGURE 4. The integration path β

ity. It follows that one revolution along a closed curve β containing all four of the singularities, as indicated in Figure 4, carries us back to the same branch of the integrand, after mutually offsetting additions to the imaginary part of the numerator and after two changes in sign. This is in accordance with the regular character of the integrand at the point $\lambda = \infty$.

It follows furthermore that the left hand member of (84) can be replaced by

$$\tfrac{1}{2}\Re \int_{\beta} \frac{\log(\kappa - \lambda) - \log(-\lambda)}{(1 - \lambda^2)^{\frac{1}{2}}} d\lambda. \tag{85}$$

For, if the curve β is constricted to a path β' running along the real axis from -1 to $+1$ and back to -1, the contributions of the two halves of the path will be equal to each other, also with respect to sign. This is also true for the parts of the path β' between 0 and κ, because the behavior of the real part

$$\log|\kappa - \lambda| - \log|\lambda|$$

of the numerator in passing either of the points 0 and κ is independent of the side along which the singularity is passed[5].

[5] For the same reason it is not necessary to specify in (84) on what sides these singularities are passed, although this is necessary with respect to κ in (83) where the imaginary part has not been eliminated.

Finally, if β in (85) is replaced by a large circle $|\lambda| = R$, the validity of (84) follows from the fact that (85) tends to zero if R tends to infinity because the integrand is of the order of magnitude of λ^{-2}.

The real part of the integral in (83) accordingly is

(86) $$\Re J = \int_{-1}^{1} \frac{\log|\lambda|}{(1-\lambda^2)^{\frac{1}{2}}}\, d\lambda = 2\int_0^1 \frac{\log \lambda}{(1-\lambda^2)^{\frac{1}{2}}}\, d\lambda,$$

or, by the transformation $\lambda = \sin x$,

(87) $$\begin{aligned}\Re J &= 2\int_0^{\frac{\pi}{2}} \log \sin x\, dx = 2\int_0^{\frac{\pi}{2}} \log \cos x\, dx = \int_0^{\frac{\pi}{2}} \log(\sin x \cos x)\, dx \\ &= \int_0^{\frac{\pi}{2}} \log\left(\tfrac{1}{2}\sin 2x\right) dx = \frac{\pi}{2}\log\tfrac{1}{2} + \tfrac{1}{2}\Re J,\end{aligned}$$

so that

(88) $$\Re J = -\pi \log 2.$$

In order to evaluate the imaginary part $\Im J$ of (83), it is necessary to specify on which side the singularity κ is passed by the integration variable λ. In fact, both cases need to be considered; the passage of λ "over" κ for values of κ on the first part of the path of integration γ_r' of κ in (36), where κ goes along the real axis from r to 1; and the passage of λ "under" κ for values of κ on the second part of its path γ_r', from 1 back to r. If the upper sign in the following formulae relates to the first of these two cases, we have

(89) $$\Im J = \mp \pi i \int_\kappa^1 \frac{d\lambda}{(1-\lambda^2)^{\frac{1}{2}}} = \mp \pi i \operatorname{arc\,cos} \kappa,$$

and, from (88) and (89), we find for the last member in (82)

(90) $$e^{-\frac{1}{2}TJ/\pi} = 2^{\frac{1}{2}T} e^{\pm\frac{1}{2}Ti \operatorname{arc\,cos} \kappa}$$

Writing

(91) $$\operatorname{arc\,cos} \kappa = \alpha, \quad \kappa = \cos\alpha, \quad e^{\frac{1}{2}Ti\alpha} - e^{-\frac{1}{2}Ti\alpha} = 2i \sin \tfrac{1}{2}T\alpha,$$

we find the following approximation for $\bar{h}(\bar{r})$ by inserting (90) in (36) as indicated in (82):

(92) $$\bar{h}(\bar{r}) \sim \frac{(\frac{1}{2}T-1)2^{\frac{1}{2}T}}{\pi} \int_0^{\operatorname{arc\,cos} \bar{r}} (\cos\alpha - \bar{r})^{\frac{1}{2}T-2} \sin\tfrac{1}{2}T\alpha \sin\alpha\, d\alpha$$

Calculations of the distribution function and of its percentiles will be much simpler for this approximation than for the exact function.

In the case of $r = m/p$ in which no circular arrangement is made a slight complication arises. The characteristic values κ_t given in (74) are again ordinates of the cosine function at equidistant points, but they do not cover a complete period or half-period of this function. Probably the most accurate procedure would be to replace the limits of integration in (83) by $\cos [(\frac{1}{2}\pi)/(T + 1)]$ and $\cos [(T + \frac{1}{2})\pi/(T + 1)]$, so as to have each discrete integral value of t in (74) contribute an interval $(t - \frac{1}{2}, t + \frac{1}{2})$ of unit length to the range of the rectangularly distributed variable τ now defining the distribution of $\lambda = \cos [\pi\tau/(T + 1)]$; while making such an adjustment in the numerical factor in (80) that the equivalent of (81) with the new limits of integration is satisfied. However, the evaluation of (83) and the simplicity of the result essentially rest on the fact that the limits of integration coincide with singularities of the integrand. In these circumstances a rather simple result can again be obtained by introducing two further changes which very nearly compensate each other. The first change is the arbitrary extension of the limits of integration to what they are in (83), while increasing the numerical factor in (80) in such a manner that the integral in (81) will be $T + 1$ instead of T. This leaves the described contributions of the discrete values of t in (74) to the range of τ unaffected, but adds to that range the two intervals $(0, \frac{1}{2})$ and $(T + \frac{1}{2}, T)$ of half a unit length not representing anything that was already present. This can be largely offset by introducing two additional discrete values $t = 0$ and $t = T + 1$, each with the negative weight $-\frac{1}{2}$, if the weight of all other discrete values is considered to be $+1$. Instead of (82) we then have

$$e^{-\frac{1}{2}\sum_{t=1}^{T}\log(\kappa-\kappa_t)} \tag{93}$$
$$\sim \exp\left[-\frac{T+1}{2\pi}\int_{-1}^{1}\frac{\log(\kappa-\lambda)}{(1-\lambda^2)^{\frac{1}{2}}}d\lambda + \tfrac{1}{4}\log(\kappa-1) + \tfrac{1}{4}\log(\kappa+1)\right]$$

If this expression is inserted in (36) with γ_r constricted to γ_r', the argument of

$$e^{(\frac{1}{4})\log(\kappa-1)} = (\kappa - 1)^{1/4} \tag{94}$$

is $-\pi i/4$ when κ goes from r to 1, and $\pi i/4$ when κ returns from 1 to r. On account of

$$(1 - \kappa^2)^{1/4} = \sin^{\frac{1}{2}} \alpha, \tag{95}$$
$$e^{\frac{1}{2}(T+1)i\alpha-\pi i/4} - e^{-\frac{1}{2}(T+1)i\alpha+\pi i/4} = 2i \sin [\tfrac{1}{2}(T + 1)\alpha - \pi/4],$$

the result now is

$$h(r) \sim \frac{(\frac{1}{2}T - 1)2^{\frac{1}{2}T+1}}{\pi} \tag{96}$$
$$\cdot\int_0^{\text{arc cos } r} (\cos \alpha - r)^{\frac{1}{2}T-2} \sin [\tfrac{1}{2}(T + 1)\alpha - \pi/4] \sin^{3/2} \alpha \, d\alpha.$$

It is not necessary to prove by direct integration that the conditions equivalent to (37) are satisfied by the approximate expressions (92) and (96). This follows from the fact that the difference of 2 between the degrees in κ of the numerator and the denominator in (36) is preserved by the substitutions (82) and (93);

that the numerical value of the limit for $\kappa \to \infty$ of κ^2 times the integrand in (36) is not changed; and that no singularities outside the segment $-1 \leqq \kappa \leqq 1$ of the real axis are introduced.

There is, of course, a certain degree of distortion involved in replacing the exact distribution functions by the smooth approximations derived. Such distortion is most serious in so far as it occurs at the tails of the distribution, where the usual significance limits are located. For instance, the exact distribution of $\bar{r}$ is asymmetric if T is odd, and ranges from $\cos [(T - 1)\pi/T]$ to $+1$, whereas the smooth approximation is symmetric and ranges from -1 to $+1$. In the case of r both the exact distribution and the approximation are symmetric, but the former ranges from $\cos [T\pi/(T + 1)]$ to $\cos [\pi/(T + 1)]$, the latter from -1 to $+1$. However, this difference is to some extent compensated by a curious anomaly in the function (96). This function actually dips below zero on symmetrically placed small intervals adjoining -1 and $+1$, the length of which is of the order of the difference $1 - \cos [\pi/(T + 1)]$ between unity and the highest characteristic value. Percentiles must therefore be counted on both sides from two points absolutely smaller than unity, defined by requiring that the small parts of the area "under" the curve (95) outside these points are algebraically zero each.

These distortions have importance only for small values of T. Anderson finds ([6] p. 52) that the exact function $\bar{h}(\bar{r})$ is symmetrical within three-decimal accuracy for all values of $T \geqq 11$ (the modal value $\bar{h}(0)$ for $T = 11$ is about 1.27). There are in the case of $\bar{r}$ three characteristic values $\bar{\kappa}_t$ exceeding the 95-percentile as given by Anderson for $T = 7$; 5 for $T = 13$; 11 for $T = 25$. Corresponding numbers for the 99-percentile are 3 for $T = 13$; 9 for $T = 25$; 17 for $T = 45$. These numbers suggest that the approximations (92) and (96) will provide good significance limits long before the normal approximation is acceptable. Accurate calculations will be needed to find out from what value of T onward the approximations can safely be substituted for the exact distributions.

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