

## SERIES-PARALLEL GROUNDED TWO-PORTS\*

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**1. Introduction.** The synthesis of grounded two-ports without mutual inductance has been an outstanding unsolved problem of electric network theory for many years. The principal structure that has been utilized in the study of this problem has been the series-parallel structure. The present paper derives a number of properties of series-parallel two-ports. In particular a new realizability condition, independent of the residue and coefficient conditions, is obtained for *RC* series-parallel grounded two-ports. These results are described below.

As is well known, common factors play a decisive role in the synthesis of *RC* transfer functions when the reduced numerators of the transfer functions have some negative coefficients. Similarly, in two-port theory, a special position is occupied by those two-ports—called factorable networks—for which the unsimplified numerators and denominators of all the admittance functions and impedance functions have a common factor. These networks are discussed in §4. The main purpose of the paper is to study the effect on a given *RC* factorable network of two basic operations: (i) removal of an impedance which is series connected to the rest of the network at one of its external nodes (ii) decomposition of the network into two parallel networks. This is accomplished in §5 and §6.

Now every series-parallel network may be simplified at each stage of its decomposition by at least one of the above methods. Hence the results of §5 and §6 may be applied to these networks. This is done in §7. By this means the following theorem is proved (Theorem 7.1): *For an RC series-parallel factorable, grounded two-port, with common factor  $f(s)$ , the ratios  $\Delta_{12}/f$ ,  $(\Delta_{11} - \Delta_{12})/f$ ,  $(\Delta_{22} - \Delta_{12})/f$  are polynomials with non-negative coefficients.* Here  $\Delta_{11}$ ,  $\Delta_{12}$ ,  $\Delta_{22}$  are the cofactors of the network admittance determinant. Some of the consequences of this new realizability criterion are explored. Thus, for example, we prove the result (Theorem 7.2): *Let the admittance functions  $Y_{11} = B^*/G^*$ ,  $Y_{22}^* = C^*/G^*$ ,  $-Y_{12}^* = D^*/G^*$  of an RC series-parallel, grounded two-port be written so that  $B^*$ ,  $C^*$ ,  $D^*$ ,  $G^*$  have no common factor and let  $G_2$  be the factor of  $G$  which contains all non-compact poles of the network. Then  $D^*G_2$  is a polynomial with non-negative coefficients.*

These results are independent of the residue and coefficient conditions and constitute further necessary properties of *RC* series-parallel networks. However, as indicated in §8, even all of these conditions do not constitute a set which is sufficient for series-parallel realizability. However if  $D^*$ , in Theorem 7.2 above, has non-negative coefficients, a method of series-parallel synthesis of  $Y_{11}$ ,  $Y_{22}$ ,  $KY_{12}$  is given for sufficiently small  $K$ . As is well known, the results for *RC* networks may be modified so as to apply to networks containing any two kinds of elements only.

**2. Preliminaries.** Let  $\Gamma$  be a general *RLC* transformerless network. The nodes of  $\Gamma$

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are identified so that each branch consists of an  $R$ ,  $L$ , and  $C$  in parallel. Consequently the admittance  $y_{ij}$  ( $i \neq j$ ) of the branch between nodes  $i$  and  $j$  is of the form  $a + bs + c/s$  where  $s$  is the complex frequency variable,  $a$ ,  $b$  and  $c$  are all non-negative and  $y_{ii} = y_{ji}$ . If  $\Gamma$  contains  $t + 1$  nodes, also write

$$y_{ii} = \sum_{\substack{i=0 \\ i \neq i}}^t y_{ii}, \quad (2.1)$$

and introduce the notation

$$(ii) = y_{ii}, \quad (ij) = -y_{ij}. \quad (i \neq j) \quad i, j = 0, 1, 2, \dots, t$$

Let  $I_i$  be the current impressed by the driving sources upon node  $i$  and let  $E_i$  be the voltage from any fixed node (for example, node 0) taken as reference node to node  $i$ . The equations of the nodal system may be written as

$$I_i = \sum_{j=0}^t (ij)E_j, \quad i = 0, 1, 2, \dots, t$$

where, of course,  $E_0 \equiv 0$ . The network determinant  $\mathfrak{D}$  is

$$\mathfrak{D} = |(ij)|, \quad i, j = 0, 1, 2, \dots, t$$

and equals zero, since (2.1) is equivalent to  $\sum_{i=0}^t (ij) = 0$ . Let  $\mathfrak{D}_{ij}$  be the cofactor of  $(ij)$  in  $\mathfrak{D}$ ,  $\mathfrak{D}_{ijkl}$  be the cofactor of  $(kl)$  in  $\mathfrak{D}_{ij}$ , etc. Then, as shown in\* [3, p. 58], all the  $\mathfrak{D}_{ij}$  are equal. Also,

$$\mathfrak{D}_{iijj} = \mathfrak{D}_{iijk} + \mathfrak{D}_{jijk}. \quad i, j, k = 0, 1, 2, \dots, t \quad (2.2)$$

Throughout the remainder of the paper, we suppose that the network  $\Gamma$  has only three external nodes which we take to be nodes 0, 1 and 2. To promote economy of language, we shall continue to refer to this three terminal network or grounded two-port simply as a network. In this case, we introduce the following additional notation:

$$\begin{aligned} A &= \mathfrak{D}_{1122}, & B &= \mathfrak{D}_{0022} = \Delta_{22}, & C &= \mathfrak{D}_{0011} = \Delta_{11}, \\ D &= \mathfrak{D}_{0012} = \Delta_{12}, & E &= \mathfrak{D}_{1102}, & F &= \mathfrak{D}_{2201} & G &= \mathfrak{D}_{001122} = \Delta_{1122}. \end{aligned} \quad (2.3)$$

It follows from (2.2) that

$$A = E + F, \quad B = D + F, \quad C = D + E \quad (2.4)$$

From (2.4) it is immediate that the three functions remaining from among  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$  may be determined by any triplet of independent functions such as  $A$ ,  $B$ ,  $C$  or  $D$ ,  $E$ ,  $F$  or  $B$ ,  $C$ ,  $D$  or  $D$ ,  $E$ ,  $A$ . Furthermore, Jacobi's theorem applied to the cofactors of  $\Delta$  yields  $BC - D^2 = G\Delta$ . After use of (2.4), this last equation may be written in the equivalent forms

$$\begin{aligned} BC - D^2 &= CA - E^2 = AB - F^2 = G\Delta = DE + EF + FD \\ &= \frac{1}{4} \{ [A^2 + B^2 + C^2] - [(A - B)^2 + (B - C)^2 + (C - A)^2] \} \end{aligned} \quad (2.5)$$

The external behavior of  $\Gamma$  is usually characterized by the triplet of impedance functions

\*The brackets refer to the references at the end of the paper.

$$Z_{11} = \frac{C}{\Delta}, \quad Z_{22} = \frac{B}{\Delta}, \quad Z_{12} = \frac{D}{\Delta} \tag{2.6}$$

or, equivalently, by the triplet of admittance functions

$$Y_{11} = \frac{B}{G}, \quad Y_{22} = \frac{C}{G}, \quad Y_{12} = -\frac{D}{G} \tag{2.7}$$

In both of these characterizations, node 0 occupies a special position. (The two-port  $\Gamma$  has node 0 as common to both the input and output terminals). However, the theory of the network with three external nodes may also be given a symmetric formulation by basing it on the quotients of  $A, B, C$  or alternatively, of  $D, E, F$  with either  $\Delta$  or  $G$ .

**3. Series-parallel networks.** Every series-parallel network (and only these) is capable of further simplification at each stage either by parallel decomposition into two simpler networks of the same kind or by the series removal of an impedance at an external node leaving a simpler network of the same kind. We now consider the effect of these two operations upon the network determinant and its (repeated) cofactors.

First suppose that  $\Gamma$  (with external nodes 0, 1, 2) consists of the series connection of the admittance  $y$  joining node 0 to node 3 and the network  $\Gamma'$ , whose external nodes are 1, 2, 3. Here node 3 in  $\Gamma'$  will also be written as node  $0'$ , since its role is analogous to that of node 0 in  $\Gamma$ . The network determinants of  $\Gamma$  and  $\Gamma'$  are

$$\mathfrak{D} = \begin{vmatrix} y & 0 & 0 & -y & \cdots & 0 & \cdots & 0 \\ 0 & (11) & (12) & (13) & & (14) & \cdots & (1t) \\ 0 & (21) & (22) & (23) & & (24) & \cdots & (2t) \\ -y & (31) & (32) & (33) + y & & (34) & \cdots & (3t) \\ 0 & (41) & (42) & (43) & & (44) & \cdots & (4t) \\ & & & & \cdots & & & \\ 0 & (t1) & (t2) & (t3) & & (t4) & \cdots & (tt) \end{vmatrix}, \tag{3.1}$$

and

$$\mathfrak{D}' = \begin{vmatrix} (33) & (31) & (32) & (34) & \cdots & (3t) \\ (13) & (11) & (12) & (14) & \cdots & (1t) \\ (23) & (21) & (22) & (24) & \cdots & (2t) \\ (43) & (41) & (42) & (44) & \cdots & (4t) \\ & & & \cdots & & \\ (t3) & (t1) & (t2) & (t4) & \cdots & (tt) \end{vmatrix}. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\Delta = \begin{vmatrix} (11) & (12) & (13) & (14) & \cdots & (1t) \\ (21) & (22) & (23) & (24) & \cdots & (2t) \\ (31) & (32) & (33) + y & (34) & \cdots & (3t) \\ (41) & (42) & (43) & (44) & \cdots & (4t) \\ & & & \cdots & & \\ (t1) & (t2) & (t3) & (t4) & \cdots & (tt) \end{vmatrix} = \mathfrak{D}' + y\mathfrak{D}'_{33},$$

or, since  $\mathfrak{D}' = 0$ ,  $\mathfrak{D}'_{33} \equiv \mathfrak{D}'_{00} = \Delta'$ , that

$$\Delta = y\Delta'. \tag{3.3}$$

Similarly,

$$\mathfrak{D}_{00\alpha\beta} = \mathfrak{D}'_{\alpha\beta} + y\mathfrak{D}'_{\alpha\beta 33}, \quad \alpha, \beta = 1, 2, 4, 5, \dots, t$$

or, since  $\mathfrak{D}'_{\alpha\beta} = \Delta'$ ,  $\mathfrak{D}'_{\alpha\beta 33} \equiv \mathfrak{D}'_{00\alpha\beta}$

$$\mathfrak{D}_{00\alpha\beta} = \Delta' + y\mathfrak{D}'_{00\alpha\beta}, \quad \alpha, \beta = 1, 2, 4, 5, \dots, t \tag{3.4}$$

Also,

$$\mathfrak{D}_{001122} = \mathfrak{D}'_{1122} + y\mathfrak{D}'_{112233},$$

or

$$\mathfrak{D}_{001122} = \mathfrak{D}'_{1122} + y\mathfrak{D}'_{001122}. \tag{3.5}$$

From (2.2) and (3.4), we find that

$$\mathfrak{D}_{\alpha\alpha 0\beta} = y\mathfrak{D}'_{\alpha\alpha 0\beta} \tag{3.6}$$

$$\mathfrak{D}_{\alpha\alpha\beta\beta} = y\mathfrak{D}'_{\alpha\alpha\beta\beta}, \quad \alpha, \beta = 1, 2, 4, 5, \dots, t$$

In the notation of (2.3), equations (3.3), (3.4), (3.5), (3.6) become

$$\begin{aligned} \Delta &= y\Delta', & A &= yA', & E &= yE', & F &= yF', & B &= \Delta' + yB', \\ C &= \Delta' + yC', & D &= \Delta' + yD', & G &= A' + yG'. \end{aligned} \tag{3.7}$$

If the admittance  $y$  is removed at nodes 1 or 2, the corresponding results are obtained by interchanging subscripts in equations (3.3) to (3.6).

Now assume that  $\Gamma$  consists of the parallel connection of two networks  $\Gamma'$  and  $\Gamma''$ . If the network determinants of  $\Gamma'$  and  $\Gamma''$  are written as

$$\begin{aligned} \mathfrak{D}' &= |(pq)'|, & p, q &= 0, 1, 2, 3, \dots, a \\ \mathfrak{D}'' &= |(uv)''|, & u, v &= 0, 1, 2, a + 1, a + 2, \dots, t \end{aligned}$$

then the corresponding determinant of  $\Gamma$  is

$$\mathfrak{D} = |(ij)|, \quad i, j = 0, 1, 2, 3, \dots, t$$

where the  $(ij)$  are given by

$$\begin{aligned} (\alpha\beta) &= (\alpha\beta)' + (\alpha\beta)'', & (hl) &= 0, \\ (\alpha k) &= (\alpha k)', & (hk) &= (hk)', \\ (\alpha l) &= (\alpha l)'', & (lm) &= (lm)''. \end{aligned} \tag{3.8}$$

Here  $\alpha, \beta; h, k; l, m$  have the restricted ranges  $0, 1, 2; 3, 4, \dots, a; a + 1, a + 2, \dots, t$  respectively, while  $i, j$  have the total range  $0, 1, 2, \dots, t$  and  $(ij) = (ji)$ . A Laplace expansion of the determinants defining  $B, C, D$  and  $G$ , after using (3.8), yields the results

$$\begin{aligned} B &= B'G'' + B''G', & C &= C'G'' + C''G', \\ D &= D'G'' + D''G', & G &= G'G'', \\ \Delta &= \Delta'G'' + \Delta''G' + B'C'' + B''C' - 2D'D'', \end{aligned} \tag{3.9}$$





*Example 2.* [3, p. 64]  $C, D, G, \Delta$  have common factor  $s + 1$ .  $D/(s + 1) = 4s^2 - s + 4$ .

$$\Delta = \begin{vmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & s + 1 & 0 & -1 & -s \\ -1 & 0 & 2(s + 1) & -1 & -s \\ 0 & -1 & -1 & 4s + 3 & 0 \\ 0 & -s & -s & 0 & 3s + 4 \end{vmatrix}.$$

*Example 3.*  $B, C, D, G$  have common factor  $s + 1$ .  $B = C = (s + 1)(32s^3 + 116s^2 + 85s + 12)$ ,  $D = (s + 1)(4s^2 - s + 4)$ ,  $G = (s + 1)^2(32s^2 + 104s + 32)$ .

$$\Delta = \begin{vmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2(s + 1) & 0 & -1 & -s \\ 0 & -1 & 0 & 2(s + 1) & -1 & -s \\ 0 & 0 & -1 & -1 & 4s + 3 & 0 \\ 0 & 0 & -s & -s & 0 & 3s + 4 \end{vmatrix}.$$

These three examples include all essentially different combinations of four network functions (including one transfer admittance function) to which a common factor may be assigned. This suggests the question whether negative coefficients can occur when  $f$  is a common factor of more than four network functions. Otherwise, does a factorable  $RC$  network exist with common factor  $f$  such that  $D/f, E/f$  or  $F/f$  has negative coefficients? We investigate aspects of this problem in the sequel.

**5.  $RC$  series connected factorable networks.** We now consider an  $RC$  network  $\Gamma$  which consists of an admittance  $y$  at node 0 in series with a network  $\Gamma'$ . Since the  $RC$  impedance  $1/y$  may be considered as a series connection of simpler elements, each element being an  $R$  and  $C$  in parallel, we may assume without loss of generality that our  $y$  corresponds to the first of these elements, incorporating the others into  $\Gamma'$ , and write  $y = a + bs$ . Also assume that  $\Gamma$  is factorable, so that the network functions  $A, B, C, D, E, F, G, \Delta$  of  $\Gamma$  have a (non-constant) greatest common factor

$$f(s) = \prod_i x_i^{p_i}$$

where the  $x_i$  are the distinct factors of  $f(s)$ . The relation between the network functions of  $\Gamma$  and  $\Gamma'$  is given by (3.7). Concerning these networks we now prove the theorem:

**THEOREM 5.1.** *Let  $RC$  network  $\Gamma$ , consisting of admittance  $y = a + bs$  in series with  $\Gamma'$  be factorable with common factor  $f(s)$ . Then the reduced network  $\Gamma'$  is factorable with common factor  $f'(s) = f(s)/g$ , where  $g$  is the greatest common divisor of  $f(s)$  and  $y$ .*

We consider two cases for each distinct factor  $x_i$ :

(i)  $x_i$  does not divide  $y$ . Then it follows from the first line of (3.7) that  $A', E', F', \Delta'$  are divisible by  $x_i^{p_i}$  and then from the remaining equations of (3.7) that  $B', C', D', G'$  are also divisible by  $x_i^{p_i}$ .

(ii)  $x_i$  divides  $y$ . Then, since  $y = kx_i$ , where  $k$  is a constant, it follows from (3.7) that  $A', E, F', \Delta'$  are all divisible by  $x_i^{q_i}$ , where  $q_i = p_i - 1$ . Also, from (3.7),

$$\frac{B}{\Delta} = \frac{1}{y} + \frac{B'}{\Delta'}$$

Now  $B/\Delta$ ,  $1/y$  and  $B'/\Delta'$  are all  $RC$  impedances. Also the residue of  $1/y$  at  $x_i = 0$  is positive and the corresponding residue of  $B'/\Delta'$  cannot be negative. Hence  $B/\Delta$  has a pole at  $x_i = 0$ . Since  $B$  is divisible by  $x_i^{p_i}$ ,  $\Delta$  must be divisible by  $x_i^{p_i+1}$ . Then, from (3.7),  $\Delta'$  is divisible by  $x_i^{q_i}$ . It follows from (3.7) that  $B', C', D'$  are all divisible by  $x_i^{q_i}$ . Finally, since

$$\frac{G}{A} = \frac{1}{y} + \frac{G'}{A'}$$

we find in a similar manner that  $G/A$  has a pole at  $x_i = 0$ , so that  $A$  must be divisible by  $x_i^{p_i+1}$ . Then, from (3.7), we must have that  $G'$  is divisible by  $x_i^{q_i}$ . Consequently all the network functions of  $\Gamma'$  have the common factor  $x_i^{q_i}$  in this case.

The results obtained in cases (i) and (ii) also hold if the role of node 0 is played by either node 1 or node 2 since only the names of the nodes are involved. This completes the proof of Theorem 5.1.

The preceding discussion also shows that, when  $y$  is removed at node 0,  $\Delta'$  and  $A'$  are each divisible by  $f(s)$ . Consequently, we find from (3.7) that

$$\begin{aligned} \frac{E}{f} &= \frac{y}{g} \frac{E'}{f'} & \frac{F}{f} &= \frac{y}{g} \frac{F'}{f'} \\ \frac{D}{f} &= \frac{\Delta'}{f} + \frac{y}{g} \frac{D'}{f'} \end{aligned}$$

Now  $y/g$  and  $\Delta'/f$  are polynomials in  $s$  with positive coefficients. (The same is true for  $A'/f, B'/f', C'/f', G'/f,$ ). Then the preceding equations show that  $D/f, E/f$  or  $F/f$  can have a negative coefficient only if the same is true of the corresponding quotient  $D'/f', E'/f'$  or  $F'/f'$  respectively. We state this result in

**THEOREM 5.2.** *The presence of a negative coefficient in any of the quotients  $D/f, E/f$  or  $F/f$  of an  $RC$  series connected factorable network  $\Gamma$  implies the presence of a negative coefficient in the corresponding quotient  $D'/f', E'/f'$  or  $F'/f'$  respectively of the reduced network  $\Gamma'$ .*

**6.  $RC$  parallel connected factorable networks.** We now consider an  $RC$  network  $\Gamma$  which is the parallel connection of two networks  $\Gamma'$  and  $\Gamma''$ . Also assume that  $\Gamma$  is factorable so that its network functions  $A, B, C, D, E, F, G, \Delta$  have a (non constant) greatest common factor

$$f(s) = \prod_i x_i^{\gamma_i}, \quad \gamma_i > 0 \tag{6.1}$$

where the  $x_i$  are the distinct factors of  $f(s)$ . The dependence of the network functions of  $\Gamma$  on those of  $\Gamma'$  and  $\Gamma''$  (which we indicate by one and two primes respectively) is given by (3.9). In particular, since  $G = G'G''$ ,  $f(s)$  must divide  $G'G''$  so that we may write

$$G' = G_1 \cdot \prod x_i^{\alpha_i}, \quad G'' = G_2 \cdot \prod x_i^{\beta_i}, \tag{6.2}$$



where  $G_1$  and  $G_2$  are each relatively prime to  $f(s)$  and

$$\alpha_i + \beta_i \geq \gamma_i, \quad \alpha_i \geq 0, \quad \beta_i \geq 0 \quad (6.3)$$

Now define the functions

$$\phi(s) = \prod x_i^{p_i}, \quad \psi(s) = \prod x_i^{q_i}, \quad \zeta(s) = \prod x_i^{r_i}, \quad (6.4)$$

where the definition of the non-negative exponents  $p_i, q_i, r_i$  for each value of  $i$  depends upon which of the following mutually exclusive, exhaustive inequalities holds for that value of  $i$ :

$$\alpha_i \geq \gamma_i, \quad \beta_i \geq \gamma_i, \quad (6.5)_1$$

$$\alpha_i \geq \gamma_i > \beta_i, \quad (6.5)_2$$

$$\beta_i \geq \gamma_i > \alpha_i, \quad (6.5)_3$$

$$\gamma_i > \alpha_i, \quad \gamma_i > \beta_i. \quad (6.5)_4$$

In these four cases, the corresponding values of  $p_i, q_i, r_i$  are:

$$p_i = q_i = 0, \quad r_i = \gamma_i, \quad (6.6)_1$$

$$p_i = \gamma_i - \beta_i, \quad q_i = 0, \quad r_i = \beta_i, \quad (6.6)_2$$

$$p_i = 0, \quad q_i = \gamma_i - \alpha_i, \quad r_i = \alpha_i, \quad (6.6)_3$$

$$p_i = \gamma_i - \beta_i, \quad q_i = \gamma_i - \alpha_i, \quad r_i = \alpha_i + \beta_i - \gamma_i, \quad (6.6)_4$$

respectively. It is clear from (6.1), (6.4) and (6.6) that

$$f = \phi\psi\zeta. \quad (6.7)$$

Also, as follows from (6.2), (6.4), (6.5) and (6.6),  $G'$  and  $G''$  are divisible by  $\zeta\phi$  and  $\zeta\psi$  respectively, so that we may write

$$G' = \zeta\phi G'^*, \quad G'' = \zeta\psi G''*. \quad (6.8)$$

In the sequel, we shall have occasion to develop proofs which subdivide according to the different inequalities (6.5). When this occurs, (6.5)<sub>3</sub> will be omitted from the discussion since any proof for case (6.5)<sub>2</sub> is valid for (6.5)<sub>3</sub> after interchange of the roles of  $\alpha_i$  and  $\beta_i$ .

We shall prove the following theorem concerning the parallel connected network  $\Gamma$ :

**THEOREM 6.1.** *Let RC network  $\Gamma$  consisting of the parallel connection of networks  $\Gamma'$  and  $\Gamma''$  be factorable with common factor  $f(s)$ . Then the component networks  $\Gamma', \Gamma''$  are also factorable with common factors  $\phi(s), \psi(s)$  respectively, where\*  $f = \phi\psi\zeta$ .*

We first prove that the network functions  $B', C', D'; B'', C'', D''$  have the common factors  $\phi(s), \psi(s)$  respectively; that is

$$\begin{aligned} B' &= \phi B'^*, & C' &= \phi C'^*, & D' &= \phi D'^*, \\ B'' &= \psi B''*, & C'' &= \psi C''*, & D'' &= \psi D''*. \end{aligned} \quad (6.9)$$

Since all the factors  $x_i$  are relatively prime, it will suffice to show that the functions  $B',$

\*The factor functions  $\phi, \psi, \zeta$  are defined by (6.4), (6.5) and (6.6).

$C', D'$  and  $B'', C'', D''$  are divisible by the factors  $x_i^{p_i}$  and  $x_i^{q_i}$  respectively for each value of  $i$ ; that is,

$$\begin{aligned} B' &= x_i^{p_i+\sigma_i} B'_i, & B'' &= x_i^{q_i+\tau_i} B''_i, \\ C' &= x_i^{p_i+\sigma_i} C'_i, & C'' &= x_i^{q_i+\tau_i} C''_i, \\ D' &= x_i^{p_i+\sigma_i} D'_i, & D'' &= x_i^{q_i+\tau_i} D''_i, \end{aligned} \tag{6.10}$$

where  $B'_i, C'_i, D'_i, B''_i, C''_i, D''_i$  are all relatively prime to  $x_i$  and

$$\sigma_i \geq 0, \quad \tau_i \geq 0. \quad j = 1, 2, 3 \tag{6.11}$$

This is immediate for factors arising under (6.5)<sub>1</sub>. To discuss the remaining cases, we begin with  $B = B'G'' + B''G'$  from (3.9) which, in virtue of (6.1) and (6.2), we may write as

$$x_i^{\gamma_i} B_i = x_i^{\beta_i} B'G'_i + x_i^{\alpha_i} B''G'_i \tag{6.12}$$

where

$$B = x_i^{\gamma_i} B_i, \quad G' = x_i^{\alpha_i} G'_i, \quad G'' = x_i^{\beta_i} G''_i \tag{6.13}$$

and  $G'_i, G''_i$  are relatively prime to  $x_i$ .

For a factor satisfying (6.5)<sub>2</sub>, division of both members of (6.12) by  $x_i^{\beta_i}$  shows that  $B'G'_i$  is divisible by  $x_i^{\gamma_i-\beta_i}$ . Since  $G'_i$  is prime to  $x_i$ , it follows from (6.6)<sub>2</sub> that  $B', B''$  obey (6.10), (6.11) in this case. If we begin with the equation of (3.9) for  $C$  or  $D$  instead of the one for  $B$ , we find similarly that  $C', D'$  and  $C'', D''$  satisfy (6.10), (6.11) in case (6.5)<sub>2</sub>.

To fix our ideas in case (6.5)<sub>4</sub>, we assume without loss of generality that  $\alpha_i \geq \beta_i$ . Then division of (6.12) by  $x_i^{\beta_i}$  leads to the conclusion

$$B' = x_i^{\alpha_i-\beta_i+\delta_1} B'_i, \quad B'' = x_i^{\epsilon_1} B''_i, \tag{6.14}$$

where  $\delta_1 \geq 0, \epsilon_1 \geq 0$  and  $B'_i, B''_i$  are prime to  $x_i$ . Similarly,

$$\begin{aligned} C' &= x_i^{\alpha_i-\beta_i+\delta_2} C'_i, & C'' &= x_i^{\epsilon_2} C''_i, \\ D' &= x_i^{\alpha_i-\beta_i+\delta_3} D'_i, & D'' &= x_i^{\epsilon_3} D''_i, \end{aligned} \tag{6.15}$$

where  $\delta_2, \delta_3, \epsilon_2, \epsilon_3$  are also non-negative and  $C'_i, C''_i, D'_i, D''_i$  are all prime to  $x_i$ . Now the ratios

$$\begin{aligned} \frac{B'}{G'} &= x_i^{\delta_1-\beta_i} \frac{B'_i}{G'_i}, & \frac{C'}{G'} &= x_i^{\delta_1-\beta_i} \frac{C'_i}{G'_i}, \\ \frac{B''}{G''} &= x_i^{\epsilon_1-\beta_i} \frac{B''_i}{G''_i}, & \frac{C''}{G''} &= x_i^{\epsilon_1-\beta_i} \frac{C''_i}{G''_i}, \end{aligned} \tag{6.16}$$

are all  $RC$  driving point admittances and the ratios

$$\frac{D'}{G'} = x_i^{\delta_2-\beta_i} \frac{D'_i}{G'_i}, \quad \frac{D''}{G''} = x_i^{\epsilon_2-\beta_i} \frac{D''_i}{G''_i}, \tag{6.17}$$

are  $RC$  transfer admittances. It is known [1, 4] that the poles of such functions must be simple and, for driving point admittances, the zeros must also be simple. Consequently

$$|\delta_1 - \beta_i| \leq 1, \quad |\delta_2 - \beta_i| \leq 1, \quad |\epsilon_1 - \beta_i| \leq 1, \quad |\epsilon_2 - \beta_i| \leq 1, \\ \delta_3 \geq \beta_i - 1, \quad \epsilon_3 \geq \beta_i - 1.$$

Since these inequalities imply  $\delta_j \geq \beta_i - 1, \epsilon_j \geq \beta_i - 1$  where  $j = 1, 2, 3$ , it follows from (6.3) that

$$\delta_j \geq \gamma_i - \alpha_i, \quad \epsilon_j \geq \gamma_i - \alpha_i, \quad j = 1, 2, 3 \tag{6.18}$$

unless  $\gamma_i - \alpha_i = \beta_i$ . Suppose that  $\gamma_i - \alpha_i = \beta_i$ . Now, from (6.13), the *RC* admittance

$$\frac{B}{G} = x_i^{\gamma_i - \alpha_i - \beta_i} \frac{B_i}{G_i' G_i''} = \frac{B_i}{G_i' G_i''},$$

has no pole at  $x_i$ . Also, from (3.9),

$$\frac{B}{G} = \frac{B'}{G'} + \frac{B''}{G''}$$

so that  $B'/G'$  and  $B''/G''$  each cannot have a pole at  $x_i$  since the residue at a pole must be positive. It follows from (6.16) that  $\delta_1 \geq \beta_i = \gamma_i - \alpha_i, \epsilon_1 \geq \beta_i = \gamma_i - \alpha_i$ . Similarly,  $C'/G'$  has no pole at  $x_i$  and, from (6.16),  $\delta_2 \geq \gamma_i - \alpha_i, \epsilon_2 \geq \gamma_i - \alpha_i$ . Finally, it follows from the residue condition [1, 4] that  $D'/G'$  has no pole at  $x_i$  and, from (6.17) that  $\delta_3 \geq \gamma_i - \alpha_i, \epsilon_3 \geq \gamma_i - \alpha_i$ . This means that (6.18) is true without restrictions. Then equations (6.14) and (6.18) are equivalent to (6.10), (6.11), after account is taken of (6.6)<sub>4</sub>. This completes the proof of (6.9) or, alternatively of (6.10), (6.11) for all possible cases. Of course, as follows from (2.4) and (6.9),  $A', E', F'; A'', E'', F''$  also have the common factors  $\phi(s), \psi(s)$  respectively.

In the sequel, we prove that  $\Delta', \Delta''$  are divisible by  $\phi(s), \psi(s)$  respectively; that is

$$\Delta' = \phi \Delta'^*, \quad \Delta'' = \psi \Delta''*. \tag{6.19}$$

If we write

$$\Delta' = x_i^{u_i} \Delta'_i, \quad \Delta'' = x_i^{v_i} \Delta''_i \tag{6.20}$$

where  $u_i \geq 0, v_i \geq 0$  and  $\Delta'_i, \Delta''_i$  are each relatively prime to  $x_i$ , then (6.19) is equivalent to

$$u_i \geq p_i, \quad v_i \geq q_i, \tag{6.21}$$

for all values of  $i$ . Clearly (6.21) is true for those values of  $i$  where (6.5)<sub>1</sub> is satisfied, since (6.6)<sub>1</sub> holds.

Also, from (6.10), (6.13), (6.20)

$$\frac{B'}{G'} = x_i^{p_i + \sigma_1 - \alpha_i} \frac{B'_i}{G'_i} \quad \frac{B''}{G''} = x_i^{q_i + \tau_1 - \beta_i} \frac{B''_i}{G''_i} \\ \frac{B'}{\Delta'} = x_i^{p_i + \sigma_1 - u_i} \frac{B'_i}{\Delta'_i} \quad \frac{B''}{\Delta''} = x_i^{q_i + \tau_1 - v_i} \frac{B''_i}{\Delta''_i}$$

and analogous equations obtain with  $C', D'; C'', D''$  replacing  $B', B''$  respectively. Now these ratios are all *RC* admittances or *RC* impedances and consequently have simple poles and, for driving point functions, simple zeros. Consequently

$$|\alpha_i - p_i - \sigma_i| \leq 1, \quad |\beta_i - q_i - \tau_i| \leq 1, \quad j = 1, 2 \tag{6.22} \\ \sigma_3 \geq \alpha_i - p_i - 1, \quad \tau_3 \geq \beta_i - q_i - 1,$$

and

$$u_i \geq p_i + \sigma_j - 1, \quad v_i \geq q_i + \tau_j - 1. \quad j = 1, 2 \tag{6.23}$$

We restrict the subsequent discussion to the case

$$\sigma_j = 0, \quad \tau_j = 0, \quad j = 1, 2 \tag{6.24}$$

since otherwise (6.23) would establish (6.21) and attain our objective. From (6.20), (6.22), (6.23) and (6.24),

$$\Delta' = x_i^{p_i-1+w_1} \Delta'_i, \quad \Delta'' = x_i^{q_i-1+w_2} \Delta''_i, \quad w_1 \geq 0, \quad w_2 \geq 0 \tag{6.25}$$

$$|\alpha_i - p_i| \leq 1, \quad |\beta_i - q_i| \leq 1, \tag{6.26}$$

are true for those factors  $x_i$  still to be considered.

If (6.5)<sub>2</sub> is true for some  $i$ , then from (6.3) and (6.6)<sub>2</sub>, we find that (6.26) becomes

$$\alpha_i + \beta_i - \gamma_i = 0 \text{ or } 1, \quad \beta_i = 0 \text{ or } 1. \tag{6.27}$$

If,  $\beta_i = 0$ , then the first equation (6.27) becomes  $\alpha_i = \gamma_i$  since (6.5)<sub>2</sub> holds. Then the last equation of (3.9), after simplification by (6.10) and (6.13), shows that  $\Delta'$  is divisible by  $x_i^{\gamma_i}$ , proving (6.21) for this subcase. Consequently we may restrict our consideration of (6.27) to  $\beta_i = 1$ . After using (6.5)<sub>2</sub>, the remaining condition (6.27) becomes

$$\beta_i = 1, \quad \gamma_i = \alpha_i, \tag{6.28}$$

for case (6.5)<sub>2</sub>.

If (6.5)<sub>4</sub> is true for some  $i$ , then (6.3) and (6.6)<sub>4</sub> transform (6.26) into the equivalent condition  $\alpha_i + \beta_i - \gamma_i = 0$  or 1. However, if  $\alpha_i + \beta_i - \gamma_i = 0$ , then the equations

$$B'C' - D'^2 = G'\Delta', \quad B''C'' - D''^2 = G''\Delta'',$$

which are analogous to (2.5), readily yield (6.21) after substitution from (6.6)<sub>4</sub>, (6.10), (6.13), (6.20) and (6.24). Consequently we restrict our discussion of case (6.5)<sub>4</sub> to the remaining condition

$$\alpha_i + \beta_i - \gamma_i = 1. \tag{6.29}$$

Returning to case (6.5)<sub>2</sub>, subject to (6.28), we find that the last equation of (3.9) becomes

$$x_i \Delta^+ = x_i^{w_1} \Delta'_i G''_i + x_i^{1+w_2} \Delta''_i G'_i + (B'_i C''_i + B''_i C'_i - 2D'^+ D''^+) \tag{6.30}$$

after using

$$\Delta = x_i^{\gamma_i} \Delta^+, \quad D' = x_i^{p_i} D'^+, \quad D'' = x_i^{q_i} D''^+, \tag{6.31}$$

and (6.6)<sub>2</sub>, (6.10), (6.13), (6.24) and (6.25) and then dividing by  $x_i^{\gamma_i-1}$ . Similarly, for case (6.5)<sub>4</sub>, after use of (6.6)<sub>4</sub>, (6.10), (6.13), (6.24), (6.25) and (6.29), the last equation of (3.9) becomes

$$x_i \Delta^+ = x_i^{w_1} \Delta'_i G''_i + x_i^{w_2} \Delta''_i G'_i + (B'_i C''_i + B''_i C'_i - 2D'^+ D''^+). \tag{6.32}$$

Now, for case (6.5)<sub>2</sub>, using (6.6)<sub>2</sub>, (6.10), (6.13), (6.24) and (6.28)

$$\frac{B'}{G'} = \frac{B'_i}{x_i G'_i} = \frac{g'(s)}{G'_i} - \frac{b'}{x_i}, \tag{6.33}$$

where the last expression is the partial fraction expansion and  $b' > 0$  is the residue of the admittance  $B'/G'$  at the pole  $x_i = 0$ . Equation (6.33) is also found to be true for case (6.5)<sub>4</sub>, after use of (6.6)<sub>4</sub>, (6.10), (6.13), (6.24) and (6.29). Also, in the same way for both cases, we find

$$\frac{B''}{G''} = \frac{B'_i}{x_i G''_i} = \frac{g''(s)}{G''_i} - \frac{b''}{x_i}, \tag{6.34}$$

where  $b'' > 0$  is the residue of the admittance  $B''/G''$  at the pole  $x_i = 0$ . Equations similar to (6.33), (6.34) also obtain for  $C'/G'$ ,  $D'/G'$ ;  $C''/G''$ ,  $D''/G''$  respectively involving the residues  $c'$ ,  $d'$ ;  $c''$ ,  $d''$  respectively.

For case (6.5)<sub>2</sub>,

$$B'C' - D'^2 = G'\Delta' = x_i^{\alpha_i} G'_i x_i^{\gamma_i - 2 + w_i} \Delta'_i = x_i^{2\gamma_i - 2 + w_i} G'_i \Delta'_i$$

Division by  $G'^2$  yields, after use of equations like (6.33),

$$\frac{x_i^{w_i} \Delta'_i}{x_i^2 G'_i} = \frac{B'C' - D'^2}{G'^2} = \frac{h(s)}{x_i G'_i} + \frac{(b'c' - d'^2)}{x_i^2},$$

and finally

$$\frac{x_i^{w_i} \Delta'_i}{x_i} = h(s) + \frac{(b'c' - d'^2)G'_i}{x_i} = k(s) + \frac{(b'c' - d'^2)(G'_i)_0}{x_i}, \tag{6.35}$$

where  $h(s)$ ,  $k(s)$  are polynomials and  $(G'_i)_0$  is the value of  $G'_i$  for  $x_i = 0$ . In a similar manner

$$\begin{aligned} \frac{B'C'' + B''C' - 2D'D''}{G'G''} &= \frac{x_i^{\gamma_i - 1} (B'_i C''_i + B''_i C'_i - 2D'^+ D''^+)}{x_i^{\alpha_i + 1} G'_i G''_i} \\ &= \frac{B'_i C''_i + B''_i C'_i - 2D'^+ D''^+}{x_i^2 G'_i G''_i}. \end{aligned}$$

Use of equations like (6.33) and (6.34) in the above equation yields

$$\frac{B'_i C''_i + B''_i C'_i - 2D'^+ D''^+}{x_i} = l(s) + \frac{(b'c'' + b''c' - 2d'd'')(G'_i)_0 (G''_i)_0}{x_i} \tag{6.36}$$

where  $l(s)$  is a polynomial and  $(G''_i)_0$  is the value of  $G''_i$  for  $x_i = 0$ . For case (6.5)<sub>4</sub>, parallel calculations to the above (but involving different details) establish the validity of the same equations (6.35) and (6.36) as well as of

$$\frac{x_i^{w_i} \Delta''_i}{x_i} = m(s) + \frac{(b'c'' - d''^2)(G''_i)_0}{x_i} \tag{6.37}$$

where  $m(s)$  is a polynomial.

For case (6.5)<sub>2</sub>, we substitute (6.35) and (6.36) in (6.30). This yields

$$\Delta^+ - x_i^{w_i} \Delta'_i G'_i = H(s) + [(b'c' - d'^2) + (b'c'' + b''c' - 2d'd'')](G'_i)_0 (G''_i)_0 / x_i$$

where  $H(s)$  is a polynomial. Since  $(G'_i)_0 (G''_i)_0 \neq 0$ , we must have

$$(b'c' - d'^2) + (b'c'' + b''c' - 2d'd'') = 0. \tag{6.38}$$

Now  $(b'c'' - b''c')^2 \geq 0$  implies  $b'^2 c''^2 + 2b'b''c'c'' + b''^2 c'^2 \geq 4b'b''c'c''$ . Since  $b'$ ,  $b''$ ,

$c', c''$  are all non-negative, this last inequality implies  $b'c'' + b''c' \geq 2\sqrt{b'b''c'c''} \geq 2d'd''$ . The final inequality follows from the residue conditions for RC admittance functions,  $b'c' - d'^2 \geq 0, b''c'' - d''^2 \geq 0$ . Consequently (6.38) implies that

$$b'c' - d'^2 = 0, \quad b'c'' + b''c' - 2d'd'' = 0 \tag{6.39}$$

Substitution of (6.39) in (6.35) shows that  $w_1 \geq 1$ , so that from (6.25), the truth of (6.21) is established for case (6.5)<sub>2</sub>.

For case (6.5)<sub>4</sub>, we substitute (6.35), (6.36) and (6.37) in (6.32). Consequently  $\Delta^+ = K(s) + [(b'c' - d'^2) + (b''c'' - d''^2) + (b'c'' + b''c' - 2d'd'')](G'_i)_0(G''_i)_0/x_i$ , where  $K(s)$  is a polynomial. Therefore

$$(b'c' - d'^2) + (b''c'' - d''^2) + (b'c'' + b''c' - 2d'd'') = 0. \tag{6.40}$$

The discussion following (6.38) shows that (6.40) implies (6.39) as well as

$$b''c'' - d''^2 = 0 \tag{6.41}$$

Substitution of (6.39) and (6.41) in (6.35) and (6.37) indicates that  $w_1 \geq 1, w_2 \geq 1$ . Then, from (6.25), the validity of (6.21) is established for case (6.5)<sub>4</sub>. This completes the proof of (6.19). The conclusions (6.8), (6.9) and (6.19), which have been proved above, establish the validity of Theorem 6.1.

According to (3.9),

$$D = D'G'' + D''G'.$$

If we write  $D = f(s)D^*$  and use (6.7), (6.8) and (6.9), the preceding equation becomes

$$D^* = D'^*G''^* + D''^*G'^*$$

It follows from this equation that  $D/f$  can have a negative coefficient only if the same is true of at least one of the quotients  $D'/\phi$  or  $D''/\psi$ . Exactly similar conclusions hold for  $E/f$  and  $F/f$ . We state this result in the theorem:

**THEOREM 6.2.** *The presence of a negative coefficient in any of the quotients  $D/f, E/f, F/f$  of an RC parallel connected factorable network  $\Gamma$  implies the presence of a negative coefficient in the corresponding quotient  $D'/\phi, E'/\phi, F'/\phi$  respectively of the component network  $\Gamma'$  or in the corresponding quotient  $D''/\psi, E''/\psi, F''/\psi$  respectively of the component network  $\Gamma''$ .*

**7. RC series-parallel networks.** When a series-parallel network is simplified by series removal of an admittance at an external node, the remaining network is reduced by one internal node. When the network is decomposed into two parallel networks, either one of them is a  $\Pi$  network or the number of internal nodes in the decomposed networks is less than in the composite network. Since at least one of these methods of simplification is possible at each stage, the reduction in the number of internal nodes of a finite series-parallel network means that eventually a stage will be reached where the reduced networks are all simple  $T$ -networks or  $\Pi$ -networks. A simple calculation of  $D', E', F'$  for these networks shows that there is no possibility for the appearance of any negative coefficients in  $D'/f, E'/f, F'/f$ . These remarks in conjunction with Theorem 5.2 and Theorem 6.2 lead directly to the following new realizability criterion for RC series-parallel networks.

**THEOREM 7.1.** *For an RC series-parallel factorable network with common factor  $f(s)$ , the ratios  $D/f$ ,  $E/f$  and  $F/f$  are polynomials with non-negative coefficients.*

It is possible that some coefficients in these ratios may be zero. Thus the network  $\Gamma_0$  for which  $\Delta_0$  is given in §4 is an RC series-parallel factorable network with  $f(s) = s + 1$  for which  $D/f = 2s^2 + 2$ . We now explore some consequences of Theorem 7.1. We first prove the following two theorems:

**THEOREM 7.2.** *Let  $\Gamma$  be an RC series-parallel network with admittance functions  $B^*/G^*$ ,  $C^*/G^*$ ,  $D^*/G^*$  written so that  $B^*$ ,  $C^*$ ,  $D^*$ ,  $G^*$  have no common factor. If  $G_1$  is the greatest common divisor of  $G^*$  and  $B^*C^* - D^{*2}$ , then  $D^*G^*/G_1$ ,  $(B^* - D^*)G^*/G_1$  and  $(C^* - D^*)G^*/G_1$  are polynomials with non-negative coefficients.*

**THEOREM 7.3.** *Let  $\Gamma$  be an RC series-parallel network with impedance functions  $B^+/\Delta^+$ ,  $C^+/\Delta^+$ ,  $D^+/\Delta^+$ , written so that  $B^+$ ,  $C^+$ ,  $D^+$ ,  $\Delta^+$  have no common factor. If  $\Delta_1$  is the greatest common divisor of  $\Delta^+$  and  $B^+C^+ - D^{+2}$ , then  $D^+\Delta^+/\Delta_1$ ,  $(B^+ - D^+)\Delta^+/\Delta_1$  and  $(C^+ - D^+)\Delta^+/\Delta_1$  are polynomials with non-negative coefficients.*

The proof follows. Let  $\Gamma$  be an RC series-parallel network for which the network admittance functions  $Y_{11}$ ,  $Y_{12}$ ,  $Y_{22}$  are specified. According to (2.7), this information determines the ratios of  $B$ ,  $C$ ,  $D$  to  $G$ . Let

$$B = fB^*, \quad C = fC^*, \quad D = fD^*, \quad G = fG^*, \tag{7.1}$$

where  $B^*$ ,  $C^*$ ,  $D^*$ ,  $G^*$  have no common factor and are thus determined by the admittance functions up to a multiplicative constant. Suppose that

$$B^*C^* - D^{*2} = G_1H, \quad G^* = G_1G_2, \tag{7.2}$$

where\*  $G_1$  is the greatest common factor of  $B^*C^* - D^{*2}$  and  $G^*$  so that  $G_2$  and  $H$  are relatively prime. Now, from (2.5), (7.1) and (7.2),

$$BC - D^2 = f^2(B^*C^* - D^{*2}) = f^2G_1H = G\Delta = fG_1G_2\Delta,$$

so that, since  $G_2$ ,  $H$  are relatively prime and  $fH = G_2\Delta$ , it follows that  $\Delta = \phi H$  and consequently  $f = \phi G_2$ , where  $\phi$  is arbitrary. It follows from (7.1) that the most general network functions consistent with the given admittance functions are

$$B = \phi G_2 B^*, \quad C = \phi G_2 C^*, \quad D = \phi G_2 D^*, \quad G = \phi G_1 G_2^2, \quad \Delta = \phi H. \tag{7.3}$$

In a similar manner, if the impedance functions  $Z_{11}$ ,  $Z_{12}$ ,  $Z_{22}$  are specified for  $\Gamma$ , from (2.6), they determine functions  $B^+$ ,  $C^+$ ,  $D^+$ ,  $\Delta^+$  without common factor. Proceeding as above with the roles of  $\Delta$  and  $G$  interchanged, it follows that the most general network functions consistent with the given impedance functions are

$$B = \phi \Delta_2 B^+, \quad C = \phi \Delta_2 C^+, \quad D = \phi \Delta_2 D^+, \quad G = \phi H, \quad \Delta = \phi \Delta_1 \Delta_2^2. \tag{7.4}$$

In these equations  $\phi$  is arbitrary and

$$B^+C^+ - D^{+2} = \Delta_1 H, \quad \Delta^+ = \Delta_1 \Delta_2,$$

where  $\Delta_1$  is the greatest common factor of  $B^+C^+ - D^{+2}$  and  $\Delta^+$ . In both (7.3) and (7.4),

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\*An equivalent definition is that  $G_1$  contains all poles of the  $Y_{ij}$  where the residue condition is satisfied with equality sign;  $G_2$  contains all other poles.

the greatest common factor of all the networks functions is  $\phi$ . As a result of Theorem 7.1, this proves that  $D^*G_2$ ,  $(B^* - D^*)G_2$ ,  $(C^* - D^*)G_2$  and  $D^*\Delta_2$ ,  $(B^* - D^*)\Delta_2$ ,  $(C^* - D^*)\Delta_2$  are polynomials with non-negative coefficients. This completes the proof of the theorems.

The preceding discussion may also be used to establish the following theorem:

**THEOREM 7.4.** *Let  $\Gamma$  be an RC series-parallel network with admittance functions  $B^*/G^*$ ,  $C^*/G^*$ ,  $D^*/G^*$  and impedance functions  $B^*/\Delta^*$ ,  $C^*/\Delta^*$ ,  $D^*/\Delta^*$  written so that  $B^*$ ,  $C^*$ ,  $D^*$ ,  $G^*$ ,  $\Delta^*$  have no common factor. Then all the coefficients of  $D^*$ ,  $B^* - D^*$  and  $C^* - D^*$  must be non-negative.*

The proof follows. For any network  $\Gamma$ , with given admittance (or impedance) functions, the network functions may always be written in the form of (7.3) or, alternatively, of (7.4). Consequently when both the admittance and impedance functions are written in simplest form but subject to the condition that corresponding functions of each kind have the same numerator, we obtain, from (2.6) and (2.7),

$$Y_{11} = \frac{G_2 B^*}{G_1 G_2^2}, \quad Y_{22} = \frac{G_2 C^*}{G_1 G_2^2}, \quad -Y_{12} = \frac{G_2 D^*}{G_1 G_2^2},$$

$$Z_{11} = \frac{G_2 C^*}{H}, \quad Z_{22} = \frac{G_2 B^*}{H}, \quad Z_{12} = \frac{G_2 D^*}{H}.$$

Since  $B^*$ ,  $C^*$ ,  $D^*$ ,  $G^*$  are relatively prime, the only common factor of the  $Y_{ij}$  is  $G_2$ , but  $G_2$  is relatively prime to  $H$ . Hence no further simultaneous reduction in both the  $Y_{ij}$ ,  $Z_{ij}$  is possible. If  $\Gamma$  is an RC series-parallel network, the common numerator  $G_2 D^*$  of  $-Y_{12}$ ,  $Z_{12}$  as well as the differences between the numerators of  $Y_{11}$ ,  $Y_{22}$  and the numerator of  $Y_{12}$  must have non-negative coefficients, in accordance with Theorem 7.2. This proves the theorem.

When the five network functions of  $\Gamma$  have all common factors deleted as in Theorem 7.4, the resulting functions  $B^*$ ,  $C^*$ ,  $D^*$ ,  $G^*$ ,  $\Delta^*$  may still have common factors when taken in groups of four. Let  $B^*$ ,  $C^*$ ,  $D^*$ ,  $G^*$  and  $B^*$ ,  $C^*$ ,  $D^*$ ,  $\Delta^*$  have the greatest common factors  $\mu$  and  $\nu$ , respectively. Since all five functions are relatively prime, it follows that the pairs  $\mu, \nu$ ;  $\mu, \Delta^*$  and  $\nu, G^*$  are each relatively prime. Then we may write

$$B^* = \mu\nu B^+, \quad C^* = \mu\nu C^+, \quad D^* = \mu\nu D^+, \quad G^* = \mu G_0, \quad \Delta^* = \nu \Delta_0.$$

From (2.5) and the preceding equations,

$$\mu^2 \nu^2 (B^+ C^+ - D^{+2}) = \mu\nu G_0 \nu_0$$

Hence  $\mu$  divides  $G_0$  and  $\nu$  divides  $\Delta_0$  so that

$$G^* = \mu^2 G^+, \quad \Delta^* = \nu^2 \Delta^+, \quad B^+ C^+ - D^{+2} = G^+ \Delta^+$$

Theorem 7.4 asserts that the coefficients of  $\mu\nu D^+$  are all non-negative. However this need not be true of the reduced numerator of  $-Y_{12} = \nu D^+ / \mu G^+$  when all three admittance functions  $Y_{ij}$  are written in simplest form with common denominator  $\mu G^+$ , nor of the reduced numerator of  $Z_{12} = \mu D^+ / \nu \Delta^+$  when all three impedance functions  $Z_{ij}$  are written in simplest form with common denominator  $\nu \Delta^+$ . Thus in the illustrative examples of §4, which all refer to RC series-parallel networks, the reduced numerator of  $Z_{12}$  has a negative coefficient in example 1 and the reduced numerator of  $-Y_{12}$  has



a negative coefficient in example 3. Example 2 illustrates a situation not discussed above in which  $C^*$ ,  $D^*$ ,  $G_1$ ,  $H$  but not  $B^*$  all have a common factor. When this factor is removed, the common numerator of  $-Y_{12}$  and  $Z_{12}$  has a negative coefficient.

**8. Some realizability questions.** We now consider some aspects of the synthesis problem. For a set of admittance functions to be realized by an  $RC$  series-parallel network, the previously known residue and coefficient conditions must be satisfied. In addition, the new realizability condition of the preceding section (for example, Theorem 7.2), which is independent of the earlier ones, must be obeyed. But no general realization of admittance functions obeying these conditions exists. Most series-parallel synthesis procedures do little more than describe the necessary extraction of series and parallel components without giving any theoretical basis for when and how this will be possible. The synthesis technique of Ozaki [5] lays such a foundation but is applicable primarily to a restricted class of symmetric  $RC$  two-ports. A simple application of Ozaki's method extends the synthesis possibilities as follows:

**THEOREM 8.1.** *Let  $Y_{11}$ ,  $Y_{22}$  be two  $RC$  admittances and let  $-Y_{12} = D^*/G^*$  be the quotient of two polynomials without common factors, where all the coefficients of  $D^*$  are non-negative,  $Y_{12}(0) = 0$  if  $Y_{11}(0)Y_{22}(0) = 0$ , and each pole of  $Y_{12}$  (possibly, including  $\infty$ ) is also a pole of both  $Y_{11}$  and  $Y_{22}$ . Then an  $RC$  series-parallel network exists which realizes  $Y_{11}$ ,  $KY_{12}$ ,  $Y_{22}$  for all sufficiently small positive values of  $K$ .*

To prove this result, we first decompose the  $Y_{ij}$  in any manner into a sum

$$Y_{ij} = \sum_k {}^{(k)}Y_{ij},$$

such that all the numerators of the  ${}^{(k)}Y_{12}$  are Hurwitz polynomials and each pole of  ${}^{(k)}Y_{12}$  is also a pole of  ${}^{(k)}Y_{11}$  and  ${}^{(k)}Y_{22}$ . This decomposition is always possible. Thus one possibility, usually not the best, is the following: If

$$-Y_{12} = \frac{D^*}{G^*} = \frac{\sum_{h=0}^m D_h s^h}{G^*},$$

let

$$-{}^{(k)}Y_{12} = D_k s^k / G^*, \quad {}^{(k)}Y_{11} = Y_{11} / (m + 1), \quad {}^{(k)}Y_{22} = Y_{22} / (m + 1),$$

$$k = 0, 1, 2, \dots, m.$$

Having chosen the  ${}^{(k)}Y_{ij}$ , expand each  ${}^{(k)}Y_{ij}$  so as to exhibit its residues:

$${}^{(k)}Y_{11} = {}^{(k)}a_0 + {}^{(k)}a_\infty s + \sum_{l=1}^{n_k} \frac{{}^{(k)}a_l s}{s + {}^{(k)}\delta_l},$$

$${}^{(k)}Y_{12} = {}^{(k)}b_0 + {}^{(k)}b_\infty s + \sum_{l=1}^{n_k} \frac{{}^{(k)}b_l s}{s + {}^{(k)}\delta_l},$$

$${}^{(k)}Y_{22} = {}^{(k)}c_0 + {}^{(k)}c_\infty s + \sum_{l=1}^{n_k} \frac{{}^{(k)}c_l s}{s + {}^{(k)}\delta_l}.$$

Choose  $K_0$  so small so that, for each value of  $k$ ,

$${}^{(k)}a_0 \geq K_{0(k)} b_0, \quad {}^{(k)}c_0 \geq K_{0(k)} b_0, \quad {}^{(k)}a_\infty \geq K_{0(k)} b_\infty, \quad {}^{(k)}c_\infty \geq K_{0(k)} b_\infty,$$

$${}^{(k)}a_l \geq K_0 |{}^{(k)}b_l|, \quad {}^{(k)}c_l \geq K_0 |{}^{(k)}b_l|, \quad l = 1, 2, \dots, n_k$$

This is possible since, according to the hypothesis of the theorem, if any of the left members of the inequalities are zero, the corresponding right members are also. Now write

$$\frac{1}{K_0} {}^{(k)}Y = {}^{(k)}b_0 + {}^{(k)}b_\infty s + \sum_{l=1}^{n_k} \frac{|{}^{(k)}b_l| s}{s + {}^{(k)}\delta_l},$$

$${}^{(k)}Y_1 = {}^{(k)}Y_{11} - {}^{(k)}Y, \quad {}^{(k)}Y_2 = {}^{(k)}Y_{22} - {}^{(k)}Y,$$

where, according to the preceding inequalities  ${}^{(k)}Y_1, {}^{(k)}Y_2$  are  $RC$  admittances. Now use the method of Ozaki to synthesize the admittance functions

$$Y'_{11} = Y'_{22} = {}^{(k)}Y, \quad Y'_{12} = K_0 {}^{(k)}Y_{12},$$

and modify the resulting network by adding shunt admittances  ${}^{(k)}Y_1$  between nodes 1 and 0 and  ${}^{(k)}Y_2$  between nodes 2 and 0 to form an  $RC$  series-parallel network  $\Gamma_k$ . Then  $\Gamma_k$  realizes the admittance functions  ${}^{(k)}Y_{11}, {}^{(k)}Y_{22}, K_0 {}^{(k)}Y_{12}$ . The parallel connection  $\Gamma$  of the  $\Gamma_k$  then realizes the admittance functions  $Y_{11}, Y_{22}, K_0 Y_{12}$ .

To realize  $Y_{11}, Y_{22}, KY_{12}$ , where  $K \leq K_0$ , we proceed as follows\*: Change the admittance level in  $\Gamma$  by the factor  $K/K_0$  to produce  $\Gamma'$  whose admittance functions are  $KY_{11}/K_0, KY_{22}/K_0, KY_{12}$ . Modify  $\Gamma'$  by introducing two new branches joining nodes 0 and 1 and nodes 0 and 2 respectively, whose admittances are  $(1 - K/K_0)Y_{11}$  and  $(1 - K/K_0)Y_{22}$  respectively. Then the admittance functions of the modified network are  $Y_{11}, Y_{22}, KY_{12}$ . This completes the proof of Theorem 8.1.

In this synthesis, there is no theoretical determination of the maximum value  $K_0$  which can be realized. A more serious defect in the theory is the lack of a complete description of all the possible  $D^*$ 's which are realizable by series-parallel networks by some method. The difficulties associated with this problem arise from the following circumstances. Series-parallel networks exist whose admittance functions  $Y_{ij}$ , specified as in (7.1), are such that the reduced numerator  $D^*$  of  $-Y_{12}$  has negative coefficients (even if, as demanded by Theorem 7.2, the coefficients of  $D^*G_2$  are non-negative). Example 3 of §4 is such a network. For admittance functions of this kind, there is no general series-parallel synthesis technique and consequently we cannot decide a priori whether such functions can or cannot be realized by series-parallel networks.

Actually, not all admittance functions which obey the residue and coefficient conditions and also the new conditions of §7 (such as Theorem 7.2) can be synthesized by series-parallel networks. That is, all the preceding conditions are necessary but not sufficient for  $RC$  series-parallel realizability. We now give two illustrations of this fact, omitting the relevant proofs which will appear elsewhere.

*Example 1.*

$$Y_{11} = a_0 + a_\infty s + \frac{as}{s + 1}, \quad Y_{22} = c_0 + c_\infty s + \frac{cs}{s + 1}, \quad -Y_{12} = b_0 + b_\infty s - \frac{bs}{s + 1}$$

where all the letters are positive. It may be shown that all such  $Y_{ij}$  which satisfy the residue and coefficient conditions may be synthesized *if and only if* the numerator of  $-Y_{12}$  has non-negative coefficients. This last condition is equivalent to

$$0 \leq b \leq b_0 + b_\infty.$$

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\*An alternative method is to replace  $K_0$  by  $K$  throughout the preceding synthesis.

However, if the pole  $s = -1$  is not compact, Theorem 7.2 requires that  $-(s+1)Y_{12}$  have non-negative coefficients or, in equivalent terms, that

$$0 < b \leq b_0 + b_\infty + \text{Min}(b_0, b_\infty).$$

Thus values of  $b$  in the range

$$b_0 + b_\infty < b \leq b_0 + b_\infty + \text{Min}(b_0, b_\infty),$$

generate admittance functions  $Y_{ij}$  (having a non-compact pole at  $s = -1$ ) which satisfy all the known necessary conditions but cannot be synthesized by a series-parallel network.

*Example 2.*

$$-Y_{12} = \frac{b_0 s^3 + b_3}{(s + \gamma_1)(s + \gamma_2)}, \quad Y_{11} = \frac{b_0 s^3 + a_1 s^2 + a_2 s + b_3}{(s + \gamma_1)(s + \gamma_2)}, \quad Y_{22} = \frac{b_0 s^3 + c_1 s^2 + c_2 s + b_3}{(s + \gamma_1)(s + \gamma_2)}$$

where all the letters are positive and  $a_1, a_2, c_1, c_2$  are chosen so that  $Y_{11}, Y_{22}$  are *RC* admittances, all the residue conditions are satisfied with equality signs and the coefficient conditions are obeyed. Since  $b_0 > 0, b_3 > 0$ , Theorem 7.2 is also true. Nevertheless, it can be proved that these  $Y_{ij}$  cannot be realized by any *RC* series-parallel network.

The preceding discussion proves that Theorem 7.2, taken in conjunction with the previously known residue and coefficient conditions still does not constitute a set of sufficient conditions for series-parallel synthesis. These examples suggest an additional problem. Do *RC* admittance functions exist, whose synthesis demands a more general structure than the series-parallel structure? To date, no counter examples are known to the ten year old conjecture of Darlington [2, p. 295] that every realizable set of *RC*  $Y_{ij}$  may be synthesized by a series-parallel network. But neither has the validity of the Darlington conjecture been established. A proof of the truth or falsity of this conjecture may depend upon the new necessary property of series-parallel networks.

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