Series Solutions of Unsteady Boundary-Layer Flows over a Stretching Flat Plate

By Shijun Liao

An analytic technique, namely, the homotopy analysis method, is applied to give series solution of the unsteady boundary-layer flows over an impermeable stretching plate. Different from all previous perturbation solutions, our series solutions are convergent in the whole time region $0 \leq \tau < +\infty$. To the best of our knowledge, such kind of series solution has never been reported for this problem. Besides, two kinds of new similarity transformations about dimensionless time are proposed. Using these two different similarity transformations, we obtain the same convergent solution valid in the whole time region $0 \leq \tau < +\infty$. Furthermore, it is shown that a nonlinear initial/boundary-value problem can be replaced by an infinite number of linear boundary-value subproblems.

1. Introduction

The investigation of the steady boundary layer flows of an incompressible viscous fluid over a stretching surface has many important applications in engineering, such as the aerodynamic extrusion of plastic sheets, the boundary layer along a liquid film condensation process, the cooling process of metallic plate in a cooling bath, and in the glass and polymer industries. The investigation was made by many researchers, including Sakiadis [1], Crane [2], Banks [3], Banks

Address for correspondence: S. Liao, School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200030, China; e-mail: sjliao@sjtu.edu.cn

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and Zaturska [4], Grubka and Bobba [5], Ali [6] for the impermeable plate, and Erickson et al. [7], Gupta and Gupta [8], Chen and Char [9], Chaudhary et al. [10], Elbashbeshy [11], Magyari and Keller [12] for the permeable plate. The unsteady boundary-layer flows due to an impulsively started flat plate were investigated by some researchers [13–18]. However, the investigations of unsteady boundary-layer flows due to an impulsively stretching surface in a viscous fluid [18–21] is relatively little. Currently, Nazar et al. [21] solved an unsteady boundary-layer flow due to an impulsively stretching surface in a rotating fluid by means of a transformation found by Williams and Rhyne [22] and the so-called Keller-box numerical method, and obtained a first-order perturbation approximation. In general, it is hard to obtain analytic solutions of unsteady boundary-layer flows, which are valid and accurate for *all* time.

Consider the unsteady boundary layer viscous flow due to an impulsively stretching impermeable plate. Let t denote the time, ν the kinematic viscosity coefficient of the fluid, y the distance perpendicular to the plate, x the distance parallel to the plate from the original, (u, v) the velocity components of the fluid in the x and y directions, respectively. When $t < 0$, both of the fluid and the plate are at rest. At $t = 0$, the plate suddenly has the velocity $U = a(x + b)^\kappa$, where $a \neq 0$, $a(1 + \kappa) > 0$, and $b > 0$. The unsteady viscous flow is governed by the boundary-layer equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

subject to the boundary conditions

$$u = a(x+b)^\kappa, \quad v = 0 \quad \text{at} \quad y = 0, \quad t \geq 0, \tag{3}$$

$$u \to 0 \quad \text{as} \quad y \to +\infty, \quad t \geq 0, \tag{4}$$

and the initial conditions

$$t = 0 : u = v = 0 \text{ at points } (x, y) \text{ with } y > 0. \tag{5}$$

Let

$$\tau = a(1+\kappa)(x+b)^{\kappa-1} t \tag{6}$$

denote the dimensionless time, ψ the stream function for all τ, ψ_i the stream function of the initial flow as $\tau \to 0$, and ψ_s the stream function of the steady-state flow as $\tau \to +\infty$, respectively. It is well known that, as $\tau \to +\infty$, the corresponding stream function ψ_s for the steady-state flow is in the form

$$\psi_s = a\sqrt{\frac{\nu}{a(1+\kappa)}}(x+b)^{\frac{\kappa-1}{2}} f_s(\eta_s), \tag{7}$$

where

$$\eta_s = \sqrt{\frac{a(1+\kappa)}{\nu}}(x+b)^{\frac{\kappa-1}{2}} y \tag{8}$$

is the corresponding similarity variable for the steady-state solution, and $f_s(\eta_s)$ is governed by the steady-state boundary-layer equation

$$f_s'''(\eta_s) + \frac{1}{2} f_s(\eta_s) f_s''(\eta_s) - \beta (f_s')^2 = 0, \tag{9}$$

subject to the boundary conditions

$$f_s(0) = 0, \quad f_s'(0) = 1, \quad f_s'(+\infty) = 0, \tag{10}$$

where the prime denotes the differentiation with respect to η_s, and

$$\beta = \frac{\kappa}{1+\kappa}. \tag{11}$$

As shown by Liao and Pop [23], the steady-state solution $f_s(\eta_s)$ has the asymptotic property

$$f_s(\eta_s) \sim \delta_s + A\exp(-\delta_s \eta_s/2), \quad \eta_s \to +\infty, \tag{12}$$

where

$$\delta_s = \lim_{\eta_s \to +\infty} f_s(\eta_s). \tag{13}$$

It is known ([13–21]) that, as $\tau \to 0$, the corresponding stream function ψ_i of the initial flow reads

$$\psi_i = a\sqrt{\frac{\nu\tau}{a(1+\kappa)}}(x+b)^{\frac{\kappa+1}{2}} \hat{f}(\zeta) = a\sqrt{\nu t}(x+b)^{\kappa} \hat{f}(\zeta), \tag{14}$$

where

$$\zeta = \sqrt{\frac{a(1+\kappa)}{\nu\tau}}(x+b)^{\frac{\kappa-1}{2}} y = \frac{\eta_s}{\sqrt{\tau}} = \frac{y}{\sqrt{\nu t}} \tag{15}$$

is the corresponding similarity variable of the solution with small time τ, and $\hat{f}(\zeta)$ is governed by the Rayleigh-type equation

$$\hat{f}'''(\zeta) + \frac{\zeta}{2}\hat{f}''(\zeta) = 0, \tag{16}$$

subject to the boundary conditions

$$\hat{f}(0) = 0, \quad \hat{f}'(0) = 1, \quad \hat{f}'(+\infty) = 0, \tag{17}$$

where the prime denotes the differentiation with respect to ζ. The above Rayleigh-type equation has the exact solution

$$\hat{f}(\zeta) = \zeta\,\mathrm{erfc}\left(\frac{\zeta}{2}\right) + \frac{2}{\sqrt{\pi}}\left[1 - \exp\left(-\frac{\zeta^2}{4}\right)\right], \tag{18}$$

where erfc(z) is the error function defined by

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{+\infty} \exp(-s^2)\, ds.$$

Note that

$$\lim_{\zeta \to 0} \hat{f}''(\zeta) = -\frac{1}{\sqrt{\pi}}. \tag{19}$$

For details, please refer to the above-mentioned authors.

Obviously, the similarity variable ζ of the initial solution (18) (when $\tau \to 0$) is quite different from the similarity variable η_s of the steady-state solution (when $\tau \to +\infty$). Certainly, the initial solution (18) given by the *linear* Rayleigh equation (16) has completely different properties from the steady-state solution $f_s(\eta_s)$ governed by the *nonlinear* equation (9). When $\tau = 0$, we have the exact solution (18). When $\tau \to +\infty$, the series solution for steady-state flows was given by Liao and Pop [23]. However, it is hard to obtain analytic solutions of unsteady boundary-layer flows, which are accurate for *all* time $0 \leq \tau < +\infty$. Perturbation techniques are applied by many researchers to this unsteady problem, but their perturbation results are valid only for small time [16, 18, 21]. To the best of our knowledge, no one has reported analytic solutions for the considered unsteady viscous flows, which are accurate and valid for any possible values of $\beta \in (-1, +\infty)$ and all dimensionless time $0 \leq \tau < +\infty$. For details, please see the above-mentioned references.

Recently, a kind of analytic method, namely the homotopy analysis method [24], was proposed in 1992 to solve highly nonlinear problems, and has been modified step by step [25–29]. Different from perturbation techniques [30], the homotopy analysis method does not depend upon any small or large parameters, and thus is valid for most of nonlinear problems in science and engineering. Besides, it logically contains other nonperturbation techniques such as Lyapunov's small parameter method [31], the δ-expansion method [32], and Adomian's decomposition method [33], as proved by Liao [28]. The so-called "homotopy perturbation technique" [34] proposed in 1999 is only a special case of homotopy analysis method, as pointed out by Liao [35]. Thus, the homotopy analysis method is a kind of unification of the nonperturbation techniques, and therefore is more general, so that it is valid for more of highly nonlinear problems. The homotopy analysis method has been successfully applied to many nonlinear problems [29, 36–46]. It should be emphasized that a few new solutions have been found [47, 48] by means of this analytic method, which were not discovered by other analytic techniques and even by numerical techniques. Recently, using Williams and Rhyne's transformation [22], Liao [49] successfully applied the homotopy analysis method to solve unsteady boundary-layer flows over a flat plate that stretches in a special way $U = a(x + b)$, corresponding to $\kappa = 1$. Following Liao [49], Cheng et al.

[50] obtained series solutions of unsteady mixed convection boundary-layer flows near the stagnation point on a vertical surface in a porous medium, and Xu and Liao [51] gave series solutions of magnetohydrodynamic flows of non-Newtonian fluids caused by an impulsively stretching plate. In this paper, we first generalize Williams and Rhyne's transformation to propose two new transformations, and then apply the homotopy analysis method to solve the unsteady boundary-layer flows over a flat plate that impulsively stretches in a rather general way $U = a(x+b)^\kappa$. Different from all other previous analytic results, our series solutions are valid and accurate in the whole time region $0 \le \tau < +\infty$, shown as follows.

2. Generalization of Williams and Rhyne's transformation

Williams and Rhyne [22] introduced a similarity transformation

$$\hat{\xi}(\tau) = 1 - \exp(-\tau) \tag{20}$$

to solve the above-mentioned unsteady boundary layer flows, where $\hat{\xi}$ is dimensionless time. In this paper, Williams and Rhyne's transformation is generalized by means of the dimensionless time $\xi(\tau)$, satisfying

$$0 \le \xi(\tau) \le 1, \quad \lim_{\tau \to 0} \xi(\tau) = 0, \quad \lim_{\tau \to +\infty} \xi(\tau) = 1 \tag{21}$$

and

$$\lim_{\tau \to 0} \xi'(\tau) = \sigma^2, \quad \lim_{\tau \to +\infty} \xi'(\tau) = 0 \tag{22}$$

for $0 \le \tau < +\infty$, where $\sigma > 0$ is a constant. Obviously, Williams and Rhyne's transformation (20) is a special case of the above definition.

Let ψ denote the stream function. Defining the similarity variables

$$\psi = a\sqrt{\frac{\nu\xi}{a(1+\kappa)}}(x+b)^{\frac{\kappa+1}{2}} f(\eta, \xi), \quad \eta = \sqrt{\frac{a(1+\kappa)}{\nu\xi}}(x+b)^{\frac{\kappa-1}{2}} y, \tag{23}$$

we have

$$u = \frac{\partial \psi}{\partial y} = a(x+b)^\kappa \frac{\partial f(\eta, \xi)}{\partial \eta} \tag{24}$$

and

$$v = -\frac{\partial \psi}{\partial x}$$

$$= -\frac{1}{2}\sqrt{a(1+\kappa)\nu\xi}(x+b)^{(\kappa-1)/2}\left\{\left[1+\frac{(2\beta-1)\tau\xi'(\tau)}{\xi}\right]f(\eta,\xi)\right.$$

$$\left.+(2\beta-1)\eta\left[1-\frac{\tau\xi'(\tau)}{\xi}\right]\frac{\partial f(\eta,\xi)}{\partial\eta}+2(2\beta-1)\tau\xi'(\tau)\frac{\partial f(\eta,\xi)}{\partial\xi}\right\}. \quad (25)$$

Substituting above expressions into Equations (1) and (2), we get

$$\frac{\partial^3 f}{\partial\eta^3}+\frac{1}{2}\eta\xi'(\tau)\frac{\partial^2 f}{\partial\eta^2}-\beta\xi\left(\frac{\partial f}{\partial\eta}\right)^2+\frac{1}{2}[\xi+(2\beta-1)\tau\xi'(\tau)]f\frac{\partial^2 f}{\partial\eta^2}$$

$$=\xi\xi'(\tau)\left\{\frac{\partial^2 f}{\partial\eta\partial\xi}-(2\beta-1)\tau\left[\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2}-\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\partial\xi}\right]\right\}, \quad (26)$$

subject to the boundary conditions

$$f(0,\xi)=0, \quad f'(0,\xi)=1, \quad f'(+\infty,\xi)=0, \quad (27)$$

where the prime in Equations (26) and (27) denotes the differentiation with respect to τ and η, respectively.

The entrainment velocity of the fluid is given by

$$v(x,+\infty,t)=-\frac{1}{2}\sqrt{a(1+\kappa)\nu\xi}(x+b)^{\frac{(\kappa-1)}{2}}\{[1+(2\beta-1)\xi^{-1}\tau\xi'(\tau)]\delta(\xi)$$

$$+2(2\beta-1)\tau\xi'(\tau)\delta'(\xi)\}, \quad (28)$$

where

$$\delta(\xi)=\lim_{\eta\to+\infty}f(\eta,\xi), \quad (29)$$

and the prime denotes the differentiation with respect to ξ or τ. The skin friction on the stretching surface reads

$$f_w(x,\xi)=\rho\nu\left.\frac{\partial u}{\partial y}\right|_{y=0}=a\rho\sqrt{\frac{a(1+\kappa)\nu}{\xi}}(x+b)^{\frac{3\kappa-1}{2}}f''(0,\xi). \quad (30)$$

The corresponding skin friction coefficient is

$$c_f^x(x,\xi)=\frac{f_w(\xi)}{\rho a^2(x+b)^{2\kappa}}=\mathrm{sign}(a)\sqrt{\frac{|1+\kappa|}{Re_x\xi}}f''(0,\xi), \quad (31)$$

where $\mathrm{sign}(a) = a/|a|$, and the local Reynolds number Re_x is defined by

$$Re_x = \frac{|a|(x+b)^{1+\kappa}}{\nu}.$$

Note that there exist an infinite number of the similarity transformations $\xi(\tau)$ satisfying (21) and (22). Here, we consider only two special cases,

$$\xi = 1 - \exp(-\sigma^2\tau)$$

and

$$\xi = \sigma^2\tau/(1+\sigma^2\tau).$$

When $\sigma = 1$, the former becomes the so-called Williams and Rhyne's similarity transformation [22]. When

$$\xi = 1 - \exp(-\sigma^2\tau), \tag{32}$$

corresponding to

$$\xi'(\tau) = \sigma^2(1-\xi), \quad \tau = -\sigma^{-2}\ln(1-\xi), \tag{33}$$

Equation (26) becomes

$$\begin{aligned}
&\frac{\partial^3 f}{\partial\eta^3} + \frac{\sigma^2}{2}\eta(1-\xi)\frac{\partial^2 f}{\partial\eta^2} - \beta\xi\left(\frac{\partial f}{\partial\eta}\right)^2 \\
&\quad + \frac{1}{2}[\xi - (2\beta-1)(1-\xi)\ln(1-\xi)]f\frac{\partial^2 f}{\partial\eta^2} \\
&= \xi(1-\xi)\left\{\sigma^2\frac{\partial^2 f}{\partial\eta\partial\xi} + (2\beta-1)\ln(1-\xi)\left[\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2} - \frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\partial\xi}\right]\right\}.
\end{aligned} \tag{34}$$

When

$$\xi = \frac{\sigma^2\tau}{1+\sigma^2\tau}, \tag{35}$$

corresponding to

$$\xi'(\tau) = \sigma^2(1-\xi)^2, \quad \tau = \frac{\xi}{\sigma^2(1-\xi)}, \tag{36}$$

Equation (26) reads

$$\begin{aligned}
&\frac{\partial^3 f}{\partial\eta^3} + \frac{\sigma^2}{2}\eta(1-\xi)^2\frac{\partial^2 f}{\partial\eta^2} - \beta\xi\left(\frac{\partial f}{\partial\eta}\right)^2 + \frac{1}{2}[\xi + (2\beta-1)\xi(1-\xi)]f\frac{\partial^2 f}{\partial\eta^2} \\
&= \xi(1-\xi)\left\{\sigma^2(1-\xi)\frac{\partial^2 f}{\partial\eta\partial\xi} - (2\beta-1)\xi\left[\frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2} - \frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta\partial\xi}\right]\right\}.
\end{aligned} \tag{37}$$

Note that, different from Equation (34), Equation (37) does not contain the logarithmic function $\ln(1-\xi)$ that tends to infinity as $\xi \to 1$. This is the advantage of the similarity transformation (35) over (32).

When $\xi \to 0$, corresponding to $\tau \to 0$, both of Equations (34) and (37) become the Rayleigh-type equation

$$\frac{\partial^3 f}{\partial \eta^3} + \frac{\sigma^2}{2}\eta \frac{\partial^2 f}{\partial \eta^2} = 0, \tag{38}$$

subject to

$$f(0,0) = 0, \quad \left.\frac{\partial f}{\partial \eta}\right|_{\eta=0,\xi=0} = 1, \quad \left.\frac{\partial f}{\partial \eta}\right|_{\eta=+\infty,\xi=0} = 0, \tag{39}$$

where

$$\eta = \sigma^{-1}\sqrt{\frac{a(1+\kappa)}{\nu\tau}}(x+b)^{\frac{\kappa-1}{2}} y \to \sigma^{-1}\zeta \tag{40}$$

is dependent upon τ. The above equation has the exact solution

$$f(\eta, 0) = \eta \,\mathrm{erfc}\left(\frac{\sigma\eta}{2}\right) + \frac{2}{\sigma\sqrt{\pi}}\left[1 - \exp\left(-\frac{\sigma^2\eta^2}{4}\right)\right], \tag{41}$$

which gives

$$\lim_{\eta\to+\infty} f(\eta, 0) = \frac{2}{\sigma\sqrt{\pi}}, \quad \lim_{\eta\to 0} f''(\eta, 0) = -\frac{\sigma}{\sqrt{\pi}}. \tag{42}$$

When $\xi = 1$, corresponding to $\tau \to +\infty$, both of Equations (34) and (37) become

$$\frac{\partial^3 f}{\partial \eta^3} + \frac{1}{2} f \frac{\partial^2 f}{\partial \eta^2} - \beta\left(\frac{\partial f}{\partial \eta}\right)^2 = 0, \tag{43}$$

subject to

$$f(0,1) = 0, \; \left.\frac{\partial f}{\partial \eta}\right|_{\eta=0,\xi=1} = 1, \quad \left.\frac{\partial f}{\partial \eta}\right|_{\eta=+\infty,\xi=1} = 0, \tag{44}$$

where

$$\eta = \sqrt{\frac{a(1+\kappa)}{\nu}}(x+b)^{\frac{\kappa-1}{2}} y = \eta_s \tag{45}$$

is independent upon τ.

Thus, as ξ varies from 0 to 1, i.e., τ increases from 0 to $+\infty$, the similarity variable η defined by (23) varies from $\sigma^{-1}\zeta$, where ζ defined by (15) is the similarity variable for small τ, to the similarity variable η_s defined by (8) for the steady-state flows. Besides, the stream function ψ varies from ψ_i for the initial impulsive flow to ψ_s describing the steady-state flow.

Note that, according to $a(1+\kappa) > 0$, it holds $a > 0$ when $\kappa > -1$, corresponding to $-1 < \beta \le 1$; and $a < 0$ when $\kappa < -1$, corresponding to $\beta > 1$. In case of $a > 0$ and $-1 < \beta \le 1$, the stretching velocity of the surface is positive. However, in case of $a < 0$ and $\beta > 1$, the stretching velocity is negative, corresponding to the so-called backward boundary layer that also has physical meanings, as suggested by Goldstein [52].

3. Series solution given by the HAM

Using the transformation

$$f(\eta,\xi) = \delta(\xi) + \lambda^{-1} F(z,\xi), \quad z = \lambda\eta, \tag{46}$$

where $\delta(\xi)$ is defined by (29), and $\lambda > 0$ is a constant independent upon τ, ξ and η, the original Equation (26) becomes

$$\begin{aligned}
&\lambda^2 \frac{\partial^3 F}{\partial z^3} + \frac{1}{2} z\xi'(\tau)\frac{\partial^2 F}{\partial z^2} - \beta\xi\left(\frac{\partial F}{\partial z}\right)^2 + \frac{1}{2}[\xi + (2\beta-1)\tau\xi'(\tau)]F\frac{\partial^2 F}{\partial z^2} \\
&\quad + \lambda\left\{\frac{1}{2}[\xi + (2\beta-1)\tau\xi'(\tau)]\delta(\xi) + (2\beta-1)\tau\xi\xi'(\tau)\delta'(\xi)\right\}\frac{\partial^2 F}{\partial z^2} \\
&\quad = \xi\xi'(\tau)\left\{\frac{\partial^2 F}{\partial z\partial\xi} - (2\beta-1)\tau\left[\frac{\partial F}{\partial \xi}\frac{\partial^2 F}{\partial z^2} - \frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial z\partial\xi}\right]\right\}
\end{aligned} \tag{47}$$

subject to the boundary conditions

$$F(0,\xi) = -\lambda\delta(\xi), \quad F(+\infty,\xi) = 0, \quad \left.\frac{\partial F(z,\xi)}{\partial z}\right|_{z=0} = 1,$$
$$\left.\frac{\partial F(z,\xi)}{\partial z}\right|_{z=+\infty} = 0, \tag{48}$$

where the prime denotes the differentiation with respect to τ or ξ.

According to the definitions (29) and (46) and the boundary conditions (48), it is reasonable to assume that $F(z,\xi)$ can be expressed by the following set of base functions

$$\{z^m \exp(-nz) \mid m \ge 0, n \ge 1\} \tag{49}$$

such that

$$F(z,\xi) = \sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty} a_{m,n}(\xi) z^m \exp(-nz), \tag{50}$$

where $a_{m,n}(\xi)$ is dependent upon ξ. It provides us with the so-called *solution expression* (see Liao [28, 29]) of the considered problem. Let $F_0(z,\xi)$ and

$\delta_0(\xi)$ denote the guess approximations of $F(z, \xi)$ and $\delta(\xi)$, respectively. According to the boundary conditions (48) and the *solution expression* (50), it is straightforward to choose the auxiliary linear operator

$$\mathcal{L}[\phi(z,\xi;q)] = \frac{\partial^3\phi(z,\xi;q)}{\partial z^3} - \frac{\partial\phi(z,\xi;q)}{\partial z}, \tag{51}$$

which has the property

$$\mathcal{L}\left[C_1(\xi)e^{-z} + C_2(\xi) + C_3(\xi)e^{z}\right] = 0 \tag{52}$$

for any functions $C_1(\xi)$, $C_2(\xi)$, and $C_3(\xi)$ of ξ. Note that the above auxiliary linear operator is *independent* upon the dimensionless time ξ, although the problem is *unsteady*. Besides, it has not very close relationship with the original governing equation (47). For simplicity, we define from Equation (47) the nonlinear operator

$$\begin{aligned}
&\mathcal{N}[\phi(z,\xi;q), \Delta(\xi;q)] \\
&= \lambda^2\frac{\partial^3\phi}{\partial z^3} + \frac{1}{2}z\xi'(\tau)\frac{\partial^2\phi}{\partial z^2} - \beta\xi\left(\frac{\partial\phi}{\partial z}\right)^2 + \frac{1}{2}[\xi + (2\beta-1)\tau\xi'(\tau)]\phi\frac{\partial^2\phi}{\partial z^2} \\
&\quad + \lambda\left\{\frac{1}{2}[\xi + (2\beta-1)\tau\xi'(\tau)]\Delta(\xi;q) + (2\beta-1)\tau\xi\xi'(\tau)\frac{\partial\Delta(\xi;q)}{\partial\xi}\right\}\frac{\partial^2\phi}{\partial z^2} \\
&\quad - \xi\xi'(\tau)\left\{\frac{\partial^2\phi}{\partial z\partial\xi} - (2\beta-1)\tau\left[\frac{\partial\phi}{\partial\xi}\frac{\partial^2\phi}{\partial z^2} - \frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial z\partial\xi}\right]\right\},
\end{aligned} \tag{53}$$

where $\phi(z, \xi; q)$ and $\Delta(\xi; q)$ are mappings of $F(z, \xi)$ and $\delta(\xi)$, respectively.

Let $\hbar \neq 0$ denote a nonzero auxiliary parameter. Using above definitions, we construct the so-called zeroth-order deformation equation (see Liao [28, 29])

$$(1-q)\mathcal{L}[\phi(z,\xi;q) - F_0(z,\xi)] = \hbar q\mathcal{N}[\phi(z,\xi;q), \Delta(\xi;q)], \tag{54}$$

subject to the boundary conditions

$$\phi(+\infty,\xi;q) = 0, \quad \left.\frac{\partial\phi(z,\xi;q)}{\partial z}\right|_{z=0} = 1, \quad \left.\frac{\partial\phi(z,\xi;q)}{\partial z}\right|_{z=+\infty} = 0 \tag{55}$$

and

$$\Delta(\xi;q) + \lambda^{-1}\phi(0,\xi;q) = 0, \tag{56}$$

where $q \in [0, 1]$ is an embedding parameter. Obviously, when $q = 0$ and $q = 1$, we have

$$\phi(z,\xi;0) = F_0(z,\xi), \quad \Delta(\xi;0) = \delta_0(\xi) \tag{57}$$

and

$$\phi(z,\xi;1) = F(z,\xi), \quad \Delta(\xi;1) = \delta(\xi), \tag{58}$$

respectively, where $\delta_0(\xi)$ is the initial guess of $\delta(\xi)$. Thus, as q increases from 0 to 1, $\phi(z, \xi; q)$ varies from the guess $F_0(z, \xi)$ to the solution $F(z, \xi)$ of (47) and (48), so does $\Delta(\xi; q)$ from $\delta_0(\xi)$ to $\delta(\xi)$. Assume that the auxiliary parameter $\hbar$ is so properly chosen that the Taylor series of $\phi(z, \xi; q)$ and $\Delta(\xi; q)$ expanded with respect to q, i.e.,

$$\phi(z, \xi; q) = \phi(z, \xi; 0) + \sum_{n=1}^{+\infty} F_n(z, \xi) q^n, \tag{59}$$

$$\Delta(\xi; q) = \Delta(\xi; 0) + \sum_{n=1}^{+\infty} \delta_n(\xi) q^n, \tag{60}$$

converge at $q = 1$, where

$$F_n(z, \xi) = \frac{1}{n!} \left. \frac{\partial^n \phi(z, \xi; q)}{\partial q^n} \right|_{q=0}, \quad \delta_n(\xi) = \frac{1}{n!} \left. \frac{\partial^n \Delta(\xi; q)}{\partial q^n} \right|_{q=0}. \tag{61}$$

Then, we have from (57) and (58) that

$$F(z, \xi) = F_0(z, \xi) + \sum_{n=1}^{+\infty} F_n(z, \xi) \tag{62}$$

and

$$\delta(\xi) = \delta_0(\xi) + \sum_{n=1}^{+\infty} \delta_n(\xi). \tag{63}$$

The governing equation and boundary conditions of $F_n(z, \xi)$ can be obtained in the following way. Write

$$\vec{F}_n = \{F_0, F_1, F_2, \ldots, F_n\}, \quad \vec{\delta}_n = \{\delta_0, \delta_1, \delta_2, \ldots, \delta_n\}.$$

Differentiating the zeroth-order deformation equations (54)–(56) m times with respect to q, then dividing by $m!$, and finally setting $q = 0$, we have the mth-order deformation equations (see Liao [28, 29])

$$\mathcal{L}[F_m(z, \xi) - \chi_m F_{m-1}(z, \xi)] = \hbar R_m(\vec{F}_{m-1}, \vec{\delta}_{m-1}, z, \xi), \tag{64}$$

subject to the boundary conditions

$$F_m(+\infty, \xi) = 0 \quad \left. \frac{\partial F_m(z, \xi)}{\partial z} \right|_{z=0} = 0, \quad \left. \frac{\partial F_m(z, \xi)}{\partial z} \right|_{z=+\infty} = 0 \tag{65}$$

and

$$\delta_m(\xi) + \lambda^{-1} F_m(0, \xi) = 0, \tag{66}$$

where

$$R_m(\vec{F}_{m-1}, \vec{\delta}_{m-1}, z, \xi)$$

$$= \lambda^2 \frac{\partial^3 F_{m-1}}{\partial z^3} + \frac{1}{2} z \xi'(\tau) \frac{\partial^2 F_{m-1}}{\partial z^2} - \xi \xi'(\tau) \frac{\partial^2 F_{m-1}}{\partial z \partial \xi}$$

$$- \beta \xi \sum_{k=0}^{m-1} \frac{\partial F_{m-1-k}}{\partial z} \frac{\partial F_k}{\partial z} + \frac{1}{2} [\xi + (2\beta - 1)\tau \xi'(\tau)] \sum_{k=0}^{m-1} F_{m-1-k} \frac{\partial^2 F_k}{\partial z^2}$$

$$+ \lambda \sum_{k=0}^{m-1} \left\{ \frac{1}{2} [\xi + (2\beta - 1)\tau \xi'(\tau)] \delta_k(\xi) + (2\beta - 1)\tau \xi \xi'(\tau) \delta_k'(\xi) \right\} \frac{\partial^2 F_{m-1-k}}{\partial z^2}$$

$$+ (2\beta - 1)\tau \xi \xi'(\tau) \sum_{k=0}^{m-1} \left[\frac{\partial F_{m-1-k}}{\partial \xi} \frac{\partial^2 F_k}{\partial z^2} - \frac{\partial F_{m-1-k}}{\partial z} \frac{\partial^2 F_k}{\partial z \partial \xi} \right], \tag{67}$$

and

$$\chi_n = \begin{cases} 1, & n > 1, \\ 0, & n = 1. \end{cases} \tag{68}$$

Let $F_m^*(z, \xi)$ denote a special solution of Equation (64). According to (52), its general solution reads

$$F_m(z, \xi) = F_m^*(z, \xi) + C_1(\xi) \exp(-z) + C_2(\xi) + C_3(\xi) \exp(z),$$

where the coefficients $C_1(\xi)$, $C_2(\xi)$, and $C_3(\xi)$ are determined by the boundary conditions (65), i.e.,

$$C_1(\xi) = \left. \frac{\partial F_m^*}{\partial z} \right|_{z=0}, \quad C_2(\xi) = C_3(\xi) = 0. \tag{69}$$

From (66), we have

$$\delta_m(\xi) = -\lambda^{-1} F_m(0, \xi). \tag{70}$$

In this way, we solve the *linear* boundary-value problem governed by (64)–(66) one after the other in the order $m = 1, 2, 3, \ldots$.

Using the asymptotic property (12) of the steady-state solution (see Liao and Pop [23]), we

$$\lambda = \delta_s / 2, \tag{71}$$

where $\delta_s = \delta(1)$ is defined by (13) and is known from Liao and Pop's work [23] for the corresponding steady-state problem. When $\xi = 0$, Equation (47) reads

$$\lambda^2 \frac{\partial^3 F}{\partial z^3} + \left(\frac{\alpha^2 z}{2} \right) \frac{\partial^2 F}{\partial z^2} = 0, \tag{72}$$

which becomes the standard Rayleigh-type equation in case of $\alpha = \lambda$. So, we choose here

$$\alpha = \lambda = \delta_s/2. \tag{73}$$

Write

$$\gamma(\xi) = \left.\frac{\partial^2 f(\eta,\xi)}{\partial \eta^2}\right|_{\eta=0} = \lambda\left.\frac{\partial^2 F(z,\xi)}{\partial z^2}\right|_{z=0} \tag{74}$$

and

$$\gamma_s = \gamma(1). \tag{75}$$

From Liao and Pop's work [23] for the steady-state flows, both of γ_s given by above expression and δ_s defined by (13) can be regarded to be known. Obviously, these known values of the initial solution and the steady-state solution are useful to choose δ_0 and $F_0(z, \xi)$. Using (42), we choose

$$\delta_0(\xi) = \frac{2}{\sigma\sqrt{\pi}}(1-\xi) + \delta_s\xi \tag{76}$$

as the guess of $\delta(\xi)$. From the boundary conditions (48) and according to the *solution expression* (50), it is straightforward to choose the guess approximation

$$F_0(z,\xi) = [1 - 2\lambda\delta_0(\xi)]\exp(-z) + [\lambda\delta_0(\xi) - 1]\exp(-2z)$$
$$+\,\epsilon\exp(-z)[1-\exp(-z)]^2, \tag{77}$$

where ϵ is a constant. Let $\gamma_0(\xi)$ denote the guess approximation of $\gamma(\xi)$. From (42) and (75), it is obvious to choose

$$\gamma_0(\xi) = -\frac{\sigma}{\sqrt{\pi}}(1-\xi) + \gamma_s\xi \tag{78}$$

as the guess approximation of $\gamma(\xi)$. Enforcing

$$\gamma_0(\xi) = \lambda F_0''(0,\xi),$$

we have

$$\epsilon = \frac{1}{2}\big[3 - 2\lambda\delta_0(\xi) + \lambda^{-1}\gamma_0(\xi)\big]. \tag{79}$$

4. Result analysis

4.1. The convergence of the solution series

According to Liao's proof [28], the series (62) is the solution of Equations (47) and (48), as long as it is convergent. Note that there exists an auxiliary parameter

$\hbar$ in the solution series (62), which can be used to control the convergence of solution series, as shown many times in the previous applications of the homotopy analysis method (see [28–29], [36–39]). Simply speaking, the so-called $\hbar$-curves of $f''(0, \xi)$ are used to find a proper value of $\hbar$, as suggested by Liao [28]. For the considered problem, we first investigate the convergence of the solution series at $\xi = 0$ and $\xi = 1$, and find a proper value of $\hbar$ which ensures that the solution series (62) converges at $\xi = 0$ and $\xi = 1$. Then, we investigate if such a value of $\hbar$ is valid in the whole region $0 \leq \xi \leq 1$. In this way, we can always find a proper value of $\hbar$ that ensures the convergence of the solution series. For details, please refer to Liao [28].

4.2. Results when $\beta = 1/2$

In this special case, the exact solution for the steady-state flow is known, i.e.,

$$F(z, 1) = 1 - \exp(-z), \quad F'(z, 1) = \exp(-z). \tag{80}$$

Two similarity transformations for τ, defined, respectively, by (32) and (35), are used. It is found that, when $\hbar = -1$, our approximations agree well with the exact initial solution $F' = \mathrm{Erfc}(z/2)$ at $\xi = 0$ and also the exact steady-state solution $\exp(-z)$ at $\xi = 1$, as shown in Figure 1 in case of $\xi = 1 - \exp(-\lambda^2\tau)$ and Figure 2 in case of $\xi = \lambda^2\tau/(1 + \lambda^2\tau)$, respectively. Furthermore, it is found that, when $\hbar = -1$, the solution series are convergent in the whole region $\xi \in [0, 1]$, corresponding to the whole time region $0 \leq \tau < +\infty$, as shown in Figure 3. Note that, by means of two *different* similarity transformations (32) and (35), we can obtain the *same* convergent results. For different definitions of ξ, the corresponding expressions of $f''(0, \xi)$ are different, as shown in Figure 3. However, as shown in Figure 4, the two *different* similarity transformations (32) and (35) give the *same* skin friction $C_f^x\sqrt{R_e^x}$ via τ, if expressed by the dimensionless time τ. The velocity profile of our series solutions varies smoothly from the initial solution to the steady-state one, as shown in Figure 5. Note that, our series solution is convergent and accurate for the *whole* dimensionless time $0 \leq \tau < +\infty$. It might be emphasized that such kind of series solutions has never been reported, to the best of the author's knowledge.

Because $f''(0, \xi)$ relates with the skin friction, we give here the 8th-order approximate result

$$\begin{aligned} f''(0, \xi) \\ &= -0.398542 - 0.329020\xi + 2.098202 \times 10^{-2}\xi^2 \\ &\quad - 1.832836 \times 10^{-3}\xi^3 - 6.262510 \times 10^{-5}\xi^4 + 5.965234 \times 10^{-4}\xi^5 \\ &\quad - 8.255860 \times 10^{-3}\xi^6 + 5.423196 \times 10^{-2}\xi^7 - 0.141834\xi^8 \\ &\quad + 0.140349\xi^9 - 3.022535 \times 10^{-2}\xi^{10} - 1.227463 \times 10^{-2}\xi^{11} \end{aligned}$$

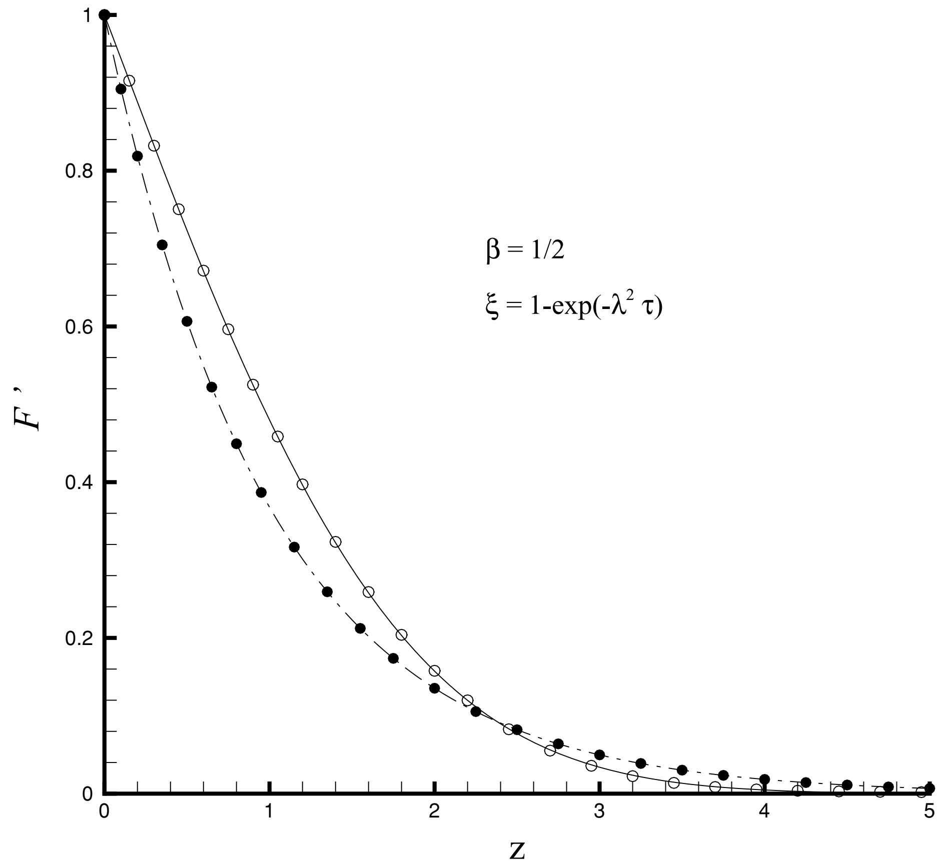

Figure 1. Comparison of $F'(z, \xi)$ with the exact solutions when $\beta = 1/2$, $\lambda = 1/\sqrt{2}$, $\hbar = -1$ and $\xi = 1 - \exp(-\lambda^2\tau)$. Solid line: exact solution $F' = \text{Erfc}(z/2)$ when $\xi = 0$; circle: 8th-order approximation when $\xi = 0$; dashed line: exact solution $F' = \exp(-z)$ when $\xi = 1$; filled circle: 8th-order approximation when $\xi = 1$.

$$- 1.164123 \times 10^{-3}\xi^{12} - 5.381745 \times 10^{-5}\xi^{13} - 1.430504 \times 10^{-6}\xi^{14}$$
$$- 2.257026 \times 10^{-8}\xi^{15} - 1.989010 \times 10^{-10}\xi^{16} - 7.635924 \times 10^{-13}\xi^{17}, \tag{81}$$

where

$$\xi = 1 - \exp(-\tau/2).$$

This expression is accurate and valid in the *whole* time region $\xi \in [0, 1]$, i.e., $0 \le \tau < +\infty$.

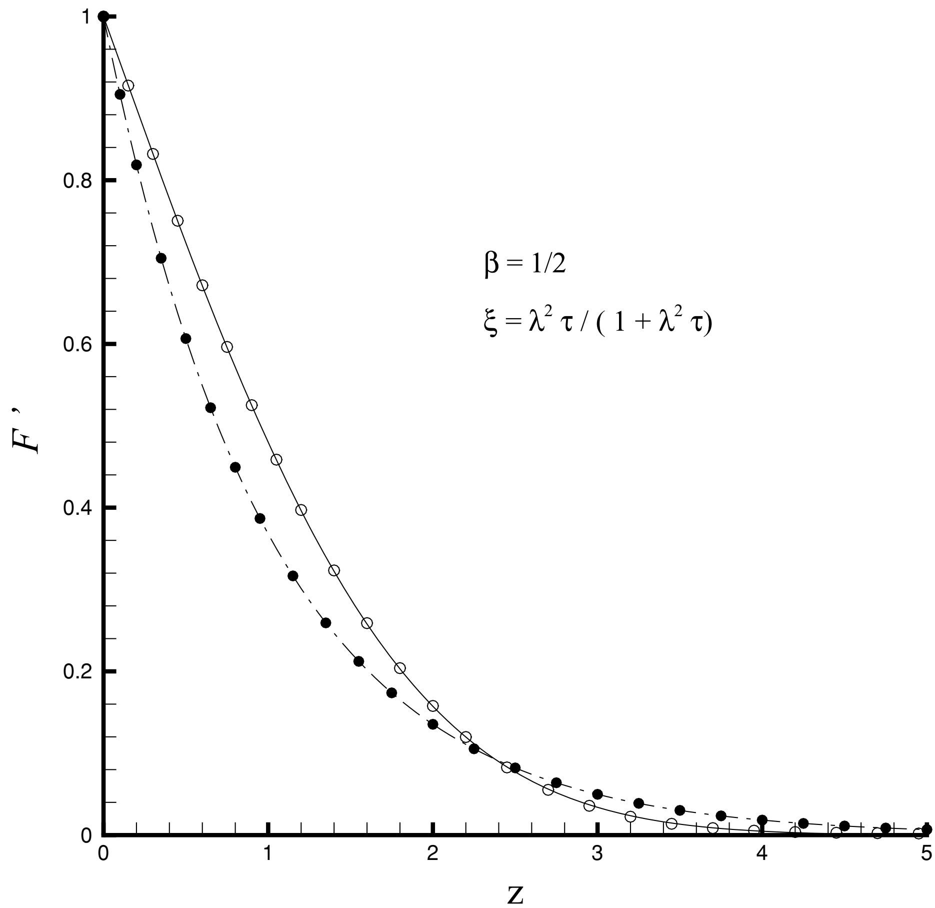

Figure 2. Comparison of $F'(z, \xi)$ with the exact solutions when $\beta = 1/2$, $\lambda = 1/\sqrt{2}$, $\hbar = -1$ and $\xi = \lambda^2\tau/(1 + \lambda^2\tau)$. Solid line: exact solution $F' = \text{Erfc}(z/2)$ when $\xi = 0$; circle: 8th-order approximation when $\xi = 0$; dashed line: exact solution $F' = \exp(-z)$ when $\xi = 1$; filled circle: 8th-order approximation when $\xi = 1$.

4.3. Results when $\beta \neq 1/2$

In the similar way, we obtain the convergent series results for $\beta \neq 1/2$, such as $\beta = 1$, $\beta = 0$, $\beta = -0.3$, and so on. The evolution of the skin frictions and the horizontal velocity profiles are as shown in Figures 6–8. In all considered cases, the velocity profile evolves smoothly, and the skin frictions, given, respectively, by two different similarity transformations (32) and (35), match completely. Note that, when $\beta \leq 0$, the solution series are divergent in case of $\hbar = -1$. However, convergent solutions are obtained by means of a proper value of

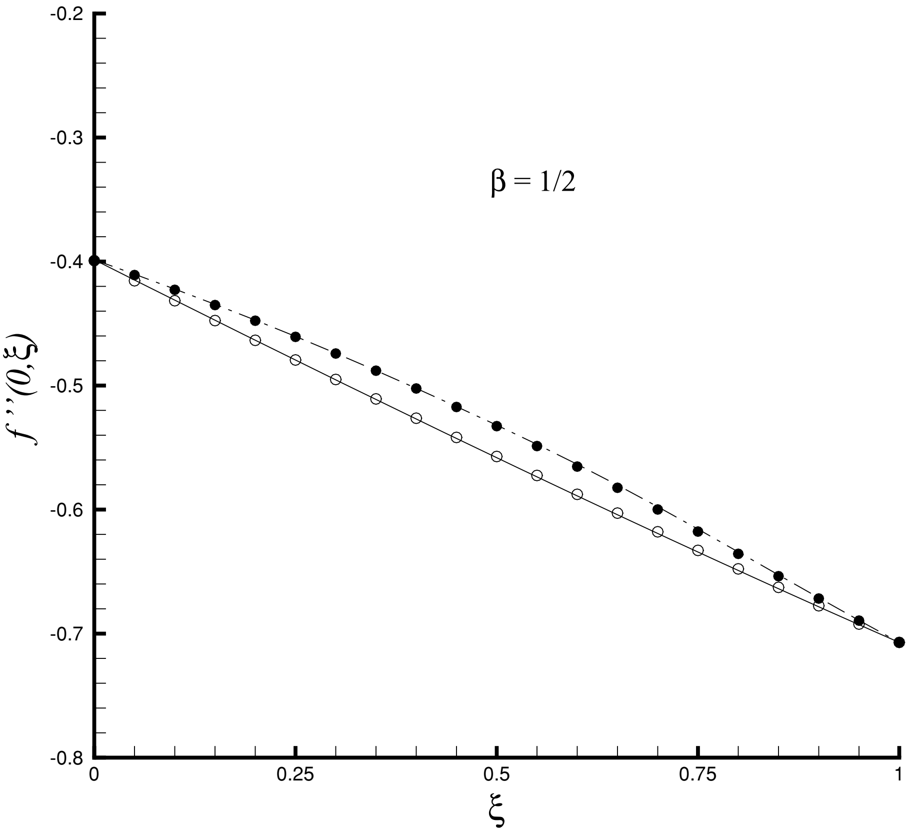

Figure 3. Curve of $f''(0, \xi)$ via ξ when $\beta = 0.5, \lambda = 1/\sqrt{2}$ and $\hbar = -1$. Solid line: 5th-order approximation when $\xi = 1 - \exp(-\lambda^2\tau)$; dashed line: 5th-order approximation when $\xi = \lambda^2\tau/(1 + \lambda^2\tau)$; circle: 8th-order approximation when $\xi = 1 - \exp(-\lambda^2\tau)$; filled circle: 8th-order approximation when $\xi = \lambda^2\tau/(1 + \lambda^2\tau)$.

$-1 < \hbar < 0$. This indicates once again that the auxiliary parameter $\hbar$ is indeed rather important to ensure that all solution series converge. In fact, it is the auxiliary parameter $\hbar$ which provides us a simple way to control and adjust the convergence region and rate of the solution series, as pointed out by Liao [28].

In case of $\xi = \lambda^2\tau/(1 + \lambda^2\tau)$, the logarithmic function $\ln(1 - \xi)$ does not appear in the solution expression, and thus less CPU time is used to employ symbolic computation to get results at the same order of approximations. So, this similarity transformation is more efficient and better than Williams and Rhyne's transformation [22] defined by (20) that brings the logarithmic function $\ln(1 - \xi)$ in the solution expression.

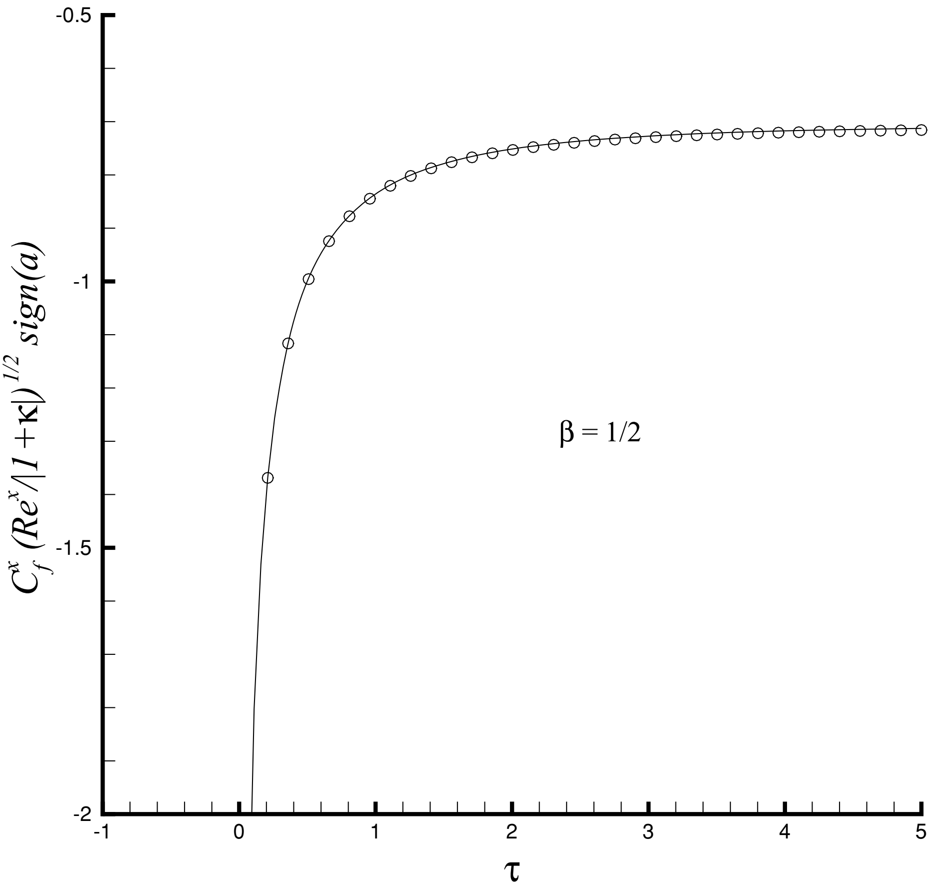

Figure 4. Curve of $C_f^x\sqrt{R_e^x/|1+\kappa|}\text{sign}(a)$ via τ when $\beta = 0.5, \lambda = 1/\sqrt{2}$ and $\hbar = -1$. Solid line: 5th-order approximation when $\xi = 1 - \exp(-\lambda^2\tau)$; circle: 5th-order approximation when $\xi = \lambda^2\tau/(1 + \lambda^2\tau)$.

5. Conclusions and discussions

In this paper, we employ the homotopy analysis method to give the series solution of the unsteady boundary-layer flows over an impermeable stretching plate. Different from previous perturbation solutions, our series solutions are convergent and valid in the whole time region $0 \le \tau < +\infty$. To the best of our knowledge, such kind of series solution has *never* been reported for this problem. It illustrates that the homotopy analysis method is valid for unsteady nonlinear problems, and thus can be applied to other more complicated unsteady nonlinear problems.

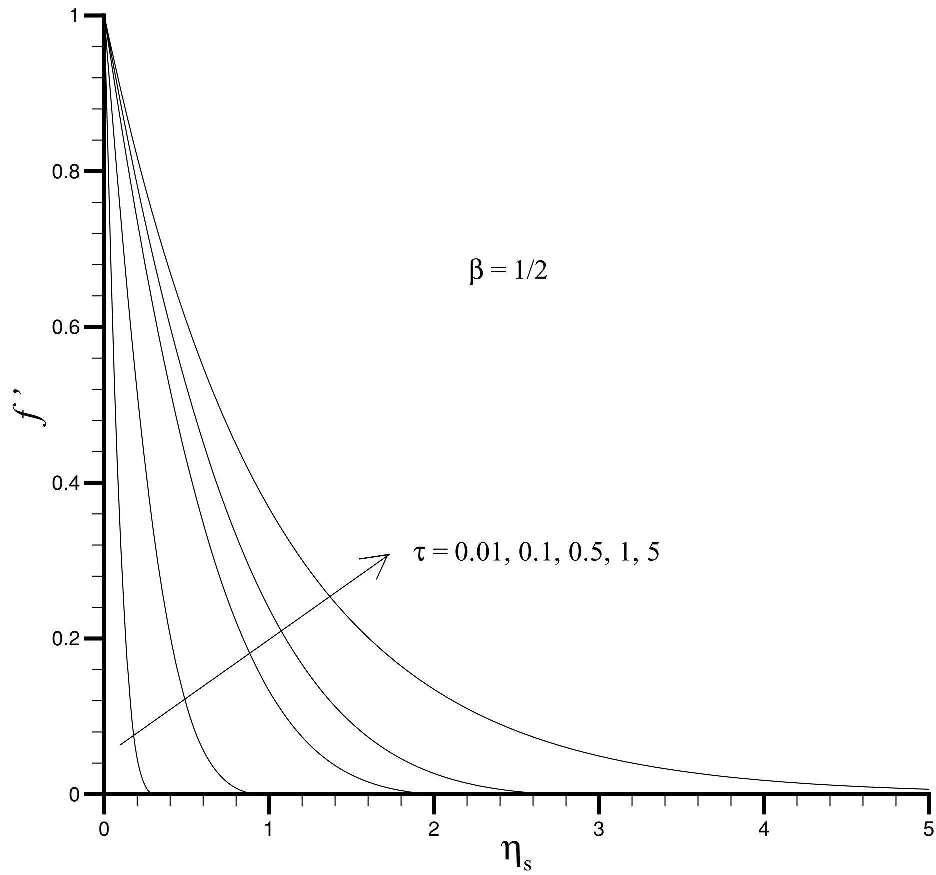

Figure 5. Velocity profile $f'(\eta_s/\sqrt{\xi}, \tau)$ via η_s when $\beta = 0.5, \lambda = 1/\sqrt{2}, \hbar = -1, \tau = 0.01, 0.1, 0.5, 1,$ and 5.

Besides, we first generalize Williams and Rhyne's transformation (20) by means of (21) and (22), and then propose two kinds of new transformations (32) and (35). These two different similarity transformations give the same convergent results in the whole time region $0 \leq \tau < +\infty$. It verifies once again Liao's [28] conclusion: even if a nonlinear problem has a unique solution, its solution may have an infinite number of different expressions. The similarity transformation defined by (21) and (22) is more general, and thus can be applied to other more complicated unsteady nonlinear problems. For the considered problem, the similarity transformation (35) is more efficient and thus better than Williams and Rhyne's transformation (20). However, it is hard to prove that the solution series given by the similarity transformation (35) is the best. So, further investigations are necessary.

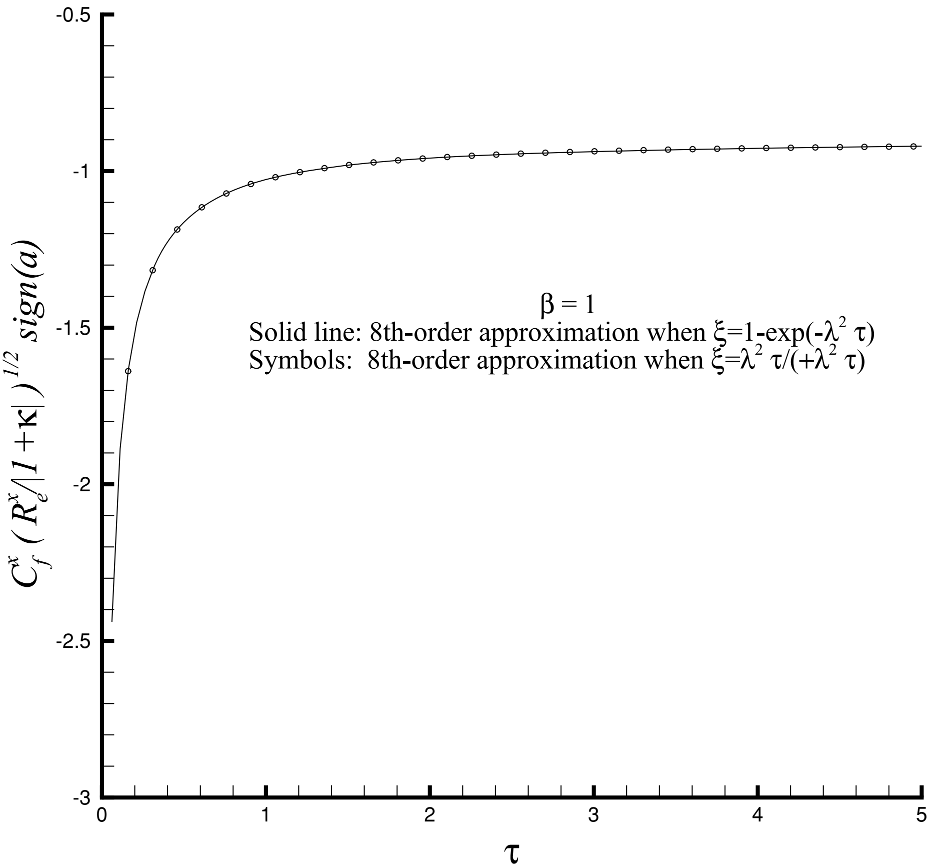

Figure 6. Curve of $C_f^x \sqrt{R_e^x/|1+\kappa|}\mathrm{sign}(a)$ via τ when $\beta = 1$, $\lambda = 1.28083$ and $\hbar = -1$. Solid line: 8th-order approximation when $\xi = 1 - \exp(-\lambda^2\tau)$; circle: 8th-order approximation when $\xi = \lambda^2\tau/(1 + \lambda^2\tau)$.

It should be pointed out that the high-order deformation equations (64)–(66) are linear *boundary-value* problems, because the linear operator $\mathcal{L}$ does *not* contain the time variable ξ at all. Thus, by means of the homotopy analysis method, a nonlinear, combined initial/boundary-value problem can be replaced by an infinite number of linear boundary-value subproblems. The same point of view was given by Liao et al. [40]. This mathematical *fact* is rather interesting. It seems that the time variable ξ is not as important as the spatial variables. However, it is unknown whether this mathematical fact might imply some important physical meanings or not.

The steady-state boundary layer equations (9) and (10) have multiple solutions. There are two branches of *exponentially* decaying solution [47]: the first one exists in the range of $-1 < \beta < +\infty$, the second one in the range of

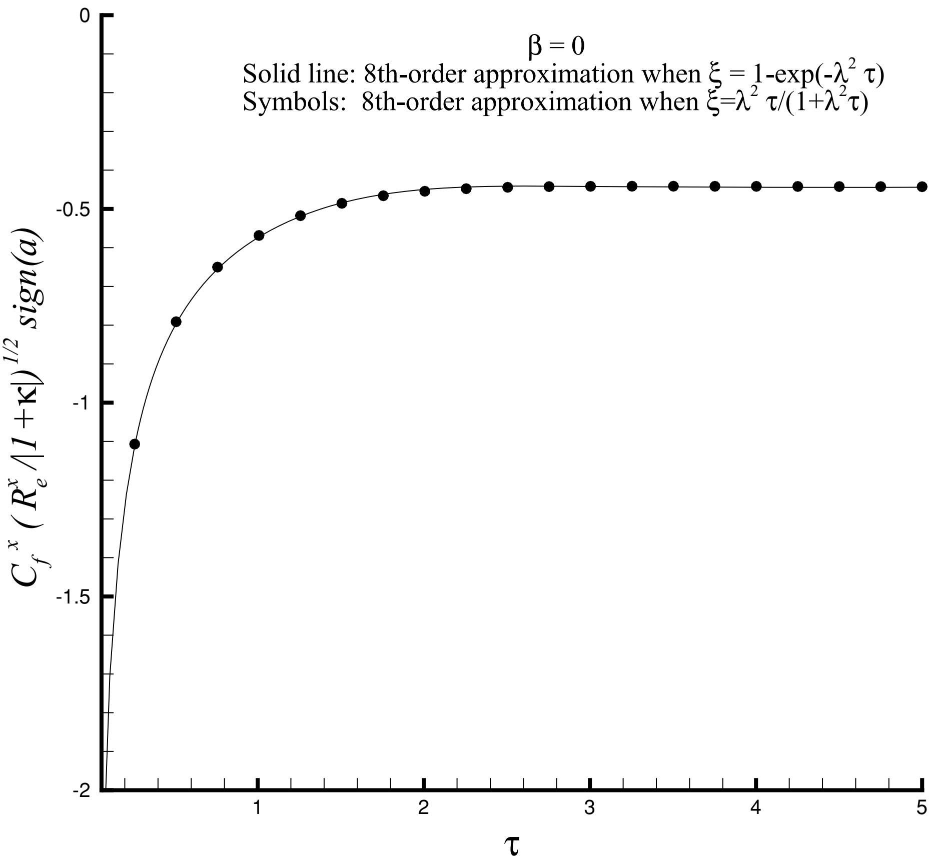

Figure 7. Curve of $C_f^x\sqrt{R_e^x/|1+\kappa|}\mathrm{sign}(a)$ via τ when $\beta = 0$ and $\lambda = 0.808$. Solid line: 8th-order approximation when $\xi = 1 - \exp(-\lambda^2\tau)$ and $\hbar = -1/2$; circle: 8th-order approximation when $\xi = \lambda^2\tau/(1 + \lambda^2\tau)$ and $\hbar = -3/4$.

$1/2 < \beta < +\infty$. For large β, it is difficult to distinct the two branches of solutions. For example, the difference of $f_s''(0)$ of two branches of solutions is only 0.013% for $\beta = 5$ and 0.00077% for $\beta = 10$, as pointed out by Liao [47]. This is the reason why the second branch of solution in the range of $1 < \beta < +\infty$ has not been found even by numerical techniques. So, it is valuable to investigate the *unsteady* solution corresponding to each branch of the two exponentially decaying steady solutions. Besides, Liao and Magyari [48] currently found that Equations (9) and (10) have an *infinite* number of *algebraically* decaying solutions in the range of $-1 < \beta < 0$. It is valuable and challenging to study the *unsteady* solutions corresponding to each of these infinite number of steady solutions, their stability and relationships of each other by means of the homotopy analysis method.

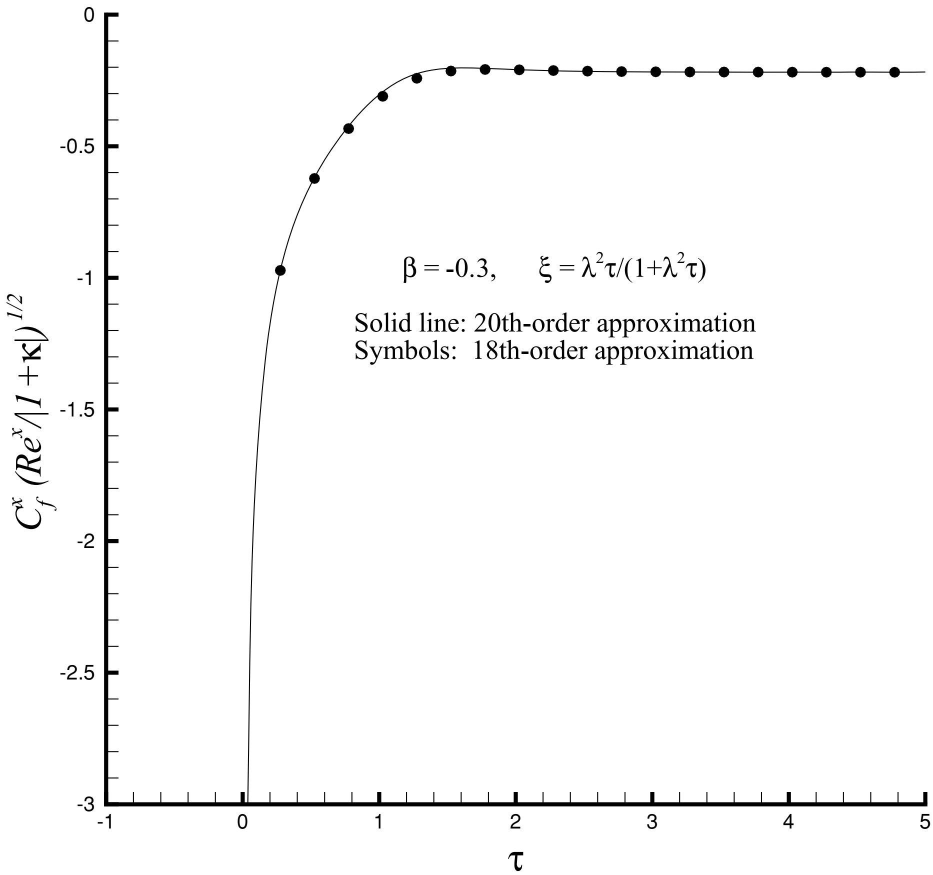

Figure 8. Curve of $C_f^x\sqrt{R_e^x/|1+\kappa|}\text{sign}(a)$ via τ when $\beta = -0.3$, $\lambda = 0.904$, $\xi = \lambda^2\tau/(1+\lambda^2\tau)$ and $\hbar = -1/4$. Solid line: 20th-order approximation; Symbols: 18th-order approximation.

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SHANGHAI JIAO TONG UNIVERSITY

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