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SERIES TRANSFORM SOLUTIONS FOR VOIGT TRANSIENTS*

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Abstract. Equations for plane and spherical waves in a Voigt medium were investigated to find methods of solution by means of Laplace transforms for transient waves. One type of solution was found in the form of products of infinite series in both the s-plane and the t-plane.

The plane Voigt wave equation has been previously solved for particle velocity, stress and strain for a unit impulse forcing function. However, solutions for the displacement due to a unit impulse, and for velocity, stress, strain and displacement for a unit step and a decay exponential involve operational forms for which no transform pairs have been published. A method of solution utilizes a series expansion of the multipliers in the transform plane to give a product of two infinite series which may be inverted term by term. These are further resolved as single series with polynomial coefficients for purposes of computation. Convolution methods may also be applied, but series expansion methods were used because of convenient recursions involved for machine computation.

The same method of solution was applied to Voigt spherical waves for unit impulse, unit step and decay exponential (finite source) forcing functions for displacement, particle velocity, strain and radial stress. Appropriate recursion formulas make them adaptable to computer evaluation. Oscillations occur for a spherical wave whereas for a plane wave they do not.

List of Symbols

a_i	= constants	m	= summation index
b_m	= constants from Heaviside expansion	n	= summation index
c	= elastic velocity of sound	P	= notation for plane wave
	= $(E/\rho)^{1/2}$ for plane wave	P_0	= pressure
	= $((\lambda + 2\mu)/\rho)^{1/2}$ for spherical wave	P'	= pressure multiplier for $\delta(t)$
c_i	= constants	P''	= pressure multiplier for $1(t)$
D_{-n}	= cylinder function of negative integral order	q	= $s^{1/2}$
		r	= radial length
e	= exponential	r_0	= radius of cavity
E	= Young's modulus of elasticity	R	= $\omega_0(r - r_0)/c$, dimensionless distance
k	= summation index	s	= transform variable

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S = notation for spherical wave
t = time
T = dimensionless time = $\omega_0 t$
u = displacement
v = particle velocity
x = linear distance
X = $\omega_0 x/c$, dimensionless distance
ϵ = strain
λ = Lame's constant—elastic
λ' = viscoelastic modulus
μ = shear modulus—elastic
ρ = density
σ = stress
ω_0 = transition frequency = $(\lambda + \mu)/(\lambda' + \mu')$
Σ = summation

Introduction. The Voigt viscoelastic model is among the many physical analogs which have been investigated in an attempt to find valid mathematical representations of wave parameters in natural materials. Solutions for a plane Voigt wave in the literature have been given by Collins [1] and Hanin [2] in which they solved for the particle velocity in response to a unit impulse.

Lee [3] solved the spherical Voigt equation for stress for a continuous harmonic forcing function. However, in solving for response to transient loading he introduced a "constant loss factor" into the transform which changed the problem from a Voigt model to one similar to a solid friction model.

Clark and Rupert [4] and [5] obtained stress and strain by Collins' method and the displacement by numerical integration. The method employed by Collins was to normalize and shift the Laplace transform solution for the Voigt wave equation, expand the positive part of a resulting exponential in a series, and obtain a term by term inversion expressed as negative order parabolic cylinder functions multiplied by a factorial, an exponential and a power term.

Operational solutions for the displacement for a unit impulse and for forcing functions such as a unit step or a decay exponential require that the transform solution also contain an appropriate multiplier. No transform pairs for the resulting expressions exist in published tables.

The transformed multipliers may, however, be expressed as Heaviside expansions, or by Taylor series expansion about $s = \infty$ yielding transformed solutions in the s-plane composed of the product of two infinite series. These may be inverted term by term into the time plane, expressed as a product of two series in the time variable and by proper arrangement utilized for computer evaluation. Some of the double series may be expressed as Cauchy products, which clarifies the mathematical operations and assists in computer programming.

Plane waves. The basic wave equation for a plane Voigt wave may be derived in different ways [6] and in terms of alternative parameters:

$$\left(1 + \frac{1}{\omega_0}\frac{\partial}{\partial t}\right)\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \tag{1}$$

where z represents either displacement, velocity, stress or strain.

Where

ω_0 = viscoelastic or transition frequency factor,
c = bar velocity = E/ρ,
x = distance,
t = time.

Collins [1], however, derived the operational form of the equation directly for particle velocity utilizing the force acceleration relation to obtain:

$$\bar{v} = \frac{P'}{\rho c} \frac{\exp\left[-xs/c(s/\omega_0 + 1)^{1/2}\right]}{(s/\omega_0 + 1)^{1/2}} \tag{2}$$

where the bar indicates a Laplace transform.

The notation of Collins and van der Pol is used herein for the symbol connecting a function to its transform:

$$v(x,\ t) \fallingdotseq \bar{v}(x,\ s). \tag{3}$$

The transform expression is then "normalized" by substituting $\omega_0 s$ for ω_0, or multiplying s by ω_0. This is equivalent to dividing the time and the velocity function by ω_0, thus

$$\frac{1}{\omega_0} v(x,\ t/\omega_0) \fallingdotseq \frac{P'}{\rho c} \frac{\exp\left[-x\omega_0 s/c(s + 1)^{1/2}\right]}{(s + 1)^{1/2}}. \tag{4}$$

The shift theorem is then used to place the transform solution in a form which can be separated, expanded and then inverted by use of tabulated transform pairs, i.e.,

$$v(x,\ t/\omega_0)e^t \fallingdotseq \frac{P'\omega_0}{\rho c} \frac{\exp\left[-X(s - 1)/s^{1/2}\right]}{s^{1/2}} \tag{5}$$

where $X = (\omega_0 x)/c$. The second half of the exponential is then expanded (a Taylor's series expansion about $s = \infty$):

$$\exp\left(-X/s^{1/2}\right) = \sum_{n=0}^{\infty} \frac{X^n}{n!} \frac{1}{s^{n/2}} \tag{6}$$

which gives

$$v(x,\ t/\omega_0)e^t \fallingdotseq \frac{P'\omega_0}{\rho c} \sum_{n=0}^{\infty} \frac{X^n}{n!} \frac{\exp\left(-x(s)^{1/2}\right)}{s^{n/2+1/2}}. \tag{7}$$

By use of formula (9), p. 246 of [7], Eq. (7) may be inverted term by term to give

$$v(x,\ t/\omega_0)e^t = \frac{P'\omega_0}{\rho c} \sum_{n=0}^{\infty} \frac{X^n 2^{n/2} t^{n/2-1/2}}{n!\,(\pi)^{1/2}} \exp\left(-X^2/8t\right) D_{-n}\left(\frac{X}{(2t)^{1/2}}\right) \tag{8}$$

or, rearranging and substituting $\omega_0 t$ for t and letting $\omega_0 t = T$,

$$v(x,\ t) = \frac{P'\omega_0}{\rho c(\pi)^{1/2}} \exp\left(-T - X^2/8T\right) \sum_{n=0}^{\infty} \frac{X^n}{n!} 2^{n/2} T^{(n-1)/2} D_{-n}\left(\frac{X}{(2T)^{1/2}}\right) \tag{9}$$

Similarly, $\sigma = E(1 + 1/\omega_0 \partial/\partial t)\epsilon$,

$$\sigma(x,\ t/\omega_0)e^t \fallingdotseq -P'\omega_0 \exp\left[-X(s - 1)/s^{1/2}\right] \tag{10}$$

or

$$\sigma(x,\ t/\omega_0)e^t \fallingdotseq -P'\omega_0 \sum_{n=0}^{\infty} \frac{X^n}{n!} \frac{\exp\left(-X(s)^{1/2}\right)}{s^{n/2}} \tag{11}$$

whose solution is

$$\sigma(x,\ t) = \frac{-P'\omega_0}{\pi^{1/2}} \sum_{n=0}^{\infty} \frac{X^n}{n!} 2^{(n-1)/2} T^{n/2-1} D_{-n+1}\left(\frac{X}{(2T)^{1/2}}\right). \tag{12}$$

The displacement is the integral of the velocity with respect to time, which is obtained by multiplying the unmodified velocity transform by $1/s$. This operation is valid only for zero displacement at $t = 0$. Hence

$$u(x, t) \doteqdot \frac{P'}{\rho c} \frac{\exp [-xs/c(s/\omega_0 + 1)^{1/2}]}{s(s/\omega_0 + 1)^{1/2}}. \tag{13}$$

Normalizing and shifting gives

$$\frac{1}{\omega_0} u(x, t/\omega_0)e^t \doteqdot \frac{P'}{\rho c} \frac{\exp [-X(s - 1)/s^{1/2}]}{\omega_0(s - 1)s^{1/2}}. \tag{14}$$

In this case $1/(s - 1)$ may be expanded and the final operational solution is a double series, i.e.,

$$\frac{1}{s - 1} = \frac{1}{s} \sum_{m=0}^{\infty} \frac{1}{s^m} \tag{15}$$

which yields

$$u(x, t/\omega_0)e^t \doteqdot \frac{P'}{\rho c} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{X^n}{n!} \frac{\exp (-xs^{1/2})}{s^{n/2+m+3/2}}. \tag{16}$$

This may be inverted term by term to give

$$u(x, t) = \frac{P'}{\rho c \pi^{1/2}} \exp (-T - X^2/8T) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{X^n}{n!} 2^{n/2+m+1} T^{n/2+m+1/2} D_{-n-2m-2}(X/(2T)^{1/2}). \tag{17}$$

The double series is readily amenable to evaluation by computer methods for small values of X and T. Equation (14) could also be solved by convolution or by integration of the velocity equation with respect to time. Of these methods, the series expressions appear to be most tractable for computation purposes.

The expression for strain is found by differentiating equation (13) with respect to x, which gives

$$\epsilon(s, x) = \frac{P'}{\rho c^2} \frac{\exp [-xs/c(s/\omega_0 + 1)^{1/2}]}{(s/\omega_0 + 1)} \tag{18}$$

which yields

$$\frac{1}{\omega_0} \epsilon\left(x, \frac{t}{\omega_0}\right)e^t \doteqdot \frac{P'}{\rho c^2} \sum_{n=0}^{\infty} \frac{X^n}{n!} \frac{\exp (-X(s)^{1/2})}{s^{n/2+1}}. \tag{19}$$

Equation (19) may be inverted term by term to give

$$\epsilon(x, t) = \frac{P'}{\rho c^2 (n)^{1/2}} \exp (-T - X^2/8T) \sum_{n=0}^{\infty} \frac{X^n}{n!} 2^{(n+1)/2} T^{n/2} D_{-n-1}(X/(2T)^{1/2}). \tag{20}$$

The solutions for an exponential decay input can be obtained by multiplying the transform for each particular parameter before normalizing and shifting by $P_0/(s + \beta)$, which is the transform of $P(t) = P_0 \exp (-\beta t)$. This is illustrated by the derivation of the velocity equation:

$$v(x, t) \doteqdot \frac{P_0}{c} \cdot \frac{1}{(s + \beta)} \cdot \exp \left(-\frac{x}{c} \frac{s}{(s/\omega_0 + 1)^{1/2}}\right). \tag{21}$$

Upon shifting and normalizing this becomes

$$\frac{1}{\omega_0} v\left(x, \frac{t}{\omega_0}\right) e^t \doteqdot \frac{P_0}{\rho c} \frac{1}{\omega_0\left[s - \left(1 - \frac{\beta}{\omega_0}\right)\right]} \frac{\exp\left[-x(s-1)/s^{1/2}\right]}{s^{1/2}}. \tag{22}$$

Letting

$$[s - (1 - \beta/\omega_0)] = (s - \alpha) \tag{23}$$

and expanding the positive exponential as previously done, and letting

$$\frac{1}{s - \alpha} = \frac{1}{s} \sum_{m=0}^{\infty} \left(\frac{\alpha}{s}\right)^m \tag{24}$$

the transform then becomes

$$\frac{1}{\omega_0} v\left(x, \frac{t}{\omega_0}\right) e^t \doteqdot \frac{P_0}{\rho c \omega_0} \cdot \frac{1}{s} \sum_{m=0}^{\infty} \left(\frac{\alpha}{s}\right)^m \sum_{n=0}^{\infty} \frac{X^n}{n!} \frac{\exp\left(-X(s)^{1/2}\right)}{s^{n/2+1/2}}. \tag{25}$$

This may be rewritten

$$\frac{1}{\omega_0} v\left(x, \frac{t}{\omega_0}\right) e^t \doteqdot \frac{P_0}{\rho c \omega_0} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{X^n}{n!} \alpha^m \frac{\exp\left(-X(s)^{1/2}\right)}{s^{n/2+m+1/2}}. \tag{26}$$

The inverse is found term by term to be

$$v(x, t) = \frac{P_0}{\rho c \pi^{1/2}} \exp\left(-T - X^2/8T\right) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{X^n}{n!} \alpha^m 2^{n/2+m+1} T^{n/2+m+1/2} D_{-n-2m-2}(X/(2T)^{1/2}) \tag{27}$$

which likewise can be evaluated by computer methods.

The other parameters are determined in a like manner each resulting in a double series. The values for a unit step input can be readily determined by letting $\beta = 0$ which makes $\alpha = 1$.

Spherical wave. Collins [1] presented a Laplace transform solution for particle velocity in terms of the s-variable for a spherical Voigt wave with a unit impulse, $\delta(t)$, input, but did not invert it.

The spherical Voigt wave equation may be developed in more than one way [4], but is probably in its most tractable form when expressed in terms of the displacement potential, ϕ:

$$\left(1 + \frac{1}{\omega_0} \frac{\partial}{\partial t}\right) \frac{\partial^2 r\phi}{\partial r^2} = \frac{1}{c^2} \frac{\partial^2 r\phi}{\partial t^2}. \tag{28}$$

The boundary conditions are:

$$\begin{array}{llll} t < 0 & P(t) = 0 & \phi = 0 & r = r_0 \\ t \geq 0 & P(t) = \sigma_r(t) & & r = r_0 \\ & & \phi = 0 & r = \infty \end{array}$$

and at $r = r_0$

$$\left[(\lambda + 2\mu) + (\lambda' + 2\mu') \frac{\partial}{\partial t}\right] \frac{\partial^2 \phi}{\partial r^2} + \frac{2(\lambda + \lambda' \,\partial/\partial t)}{r} \frac{\partial \phi}{\partial r} = -\sigma_r(t). \tag{29}$$

No experimental values are available for the viscoelastic moduli, hence, for purposes of computation it is assumed that $\lambda = \mu$ and $\lambda' = \mu'$, although the following methods are applicable without this assumption.

Equation (29) then reduces to

$$\mu\left(1 + \frac{1}{\omega_0}\frac{\partial}{\partial t}\right)\left(3\,\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}\right) = -\sigma_r(t). \tag{30}$$

For a Dirac delta forcing function, $P(t) = P'\,\delta(t)$, the operational solution of Eq. (28) becomes

$$r\phi(r,\,t) \doteqdot -\frac{P'r_0 \exp\left[-s(r - r_0)/c(s/\omega_0 + 1)^{1/2}\right]}{r\mu s(s/\omega_0 + 1)\left[\dfrac{3s^2}{c^2(s/\omega_0 + 1)} + \dfrac{4s}{r_0c(s/\omega_0 + 1)^{1/2}} + \dfrac{4}{r_0^2}\right]}. \tag{31}$$

However, interest is centered on parameters such as displacement, particle velocity, stress and strain. The displacement is

$$u = \partial\phi/\partial r \tag{32}$$

or

$$u(r,\,t) \doteqdot \frac{P'r_0 \exp\left[-s(r - r_0)/c(s/\omega_0 + 1)^{1/2}\right]}{\mu(s/\omega_0 + 1)[B]}\left[\frac{1}{r^2} + \frac{s}{rc(s/\omega_0 + 1)^{1/2}}\right] \tag{33}$$

where

$$[B] = \left[\frac{3s^2}{c^2(s/\omega_0 + 1)} + \frac{4s}{r_0c(s/\omega_0 + 1)^{1/2}} + \frac{4}{r_0^2}\right]. \tag{34}$$

Let $s = \omega_0 s$ (normalize) and then $s = (s - 1)$(shift), and Eq. (33) becomes

$$\frac{1}{\omega_0}u(r,\,t/\omega_0)e^t \doteqdot \frac{P'r_0 \exp\left(-Rs^{1/2} + R/s^{1/2}\right)}{\mu\omega_0^2[A]}\left[\frac{1}{r^2} + \frac{\omega_0(s - 1)}{rcs^{1/2}}\right] \tag{35}$$

where

$$[A] = \left[s^2 + \frac{4c}{3r_0\omega_0}s^{3/2} + \left(\frac{4c^2}{3r_0^2\omega_0^2} - 2\right)s - \frac{4c}{3r_0\omega_0}s^{1/2} + 1\right] \tag{36}$$

and

$$R = \omega_0(r - r_0)/c. \tag{37}$$

The fraction involving the quadratic in s in the denominator may be expanded as follows:

$$1/[A] = \frac{1}{s^2}\sum_{m=0}^{\infty} b_m\,\frac{1}{s^{m/2}} \tag{38}$$

where the coefficients b_m are functions of the coefficients of the quadratic. The region of convergence is $1 < \operatorname{Re} s \leq \infty$.

The expansion may be accomplished either as a Taylor series about $s = \infty$ or by long division. The latter method provides a recursion for determining the values of the coefficients for computation purposes. The positive exponential may likewise be expanded to yield:

$$\exp R/s^{1/2} = \sum_{n=0}^{\infty}\frac{R^n}{n!}\frac{1}{(s)^{n/2}}. \tag{39}$$

Substituting in (35) and separating into appropriate terms gives the operational solution in terms of double infinite series:

$$u(r, t/\omega_0)e^t \doteqdot \frac{P'r_0c^2}{\mu\omega_0}\left\{\frac{1}{r^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{R^n}{n!}\,b_m\,\frac{\exp(-Rs^{1/2})}{s^{n/2+m/2+2}}\right.$$
$$\left.\cdot\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{R^n}{n!}\,b_m\,\frac{\exp(-Rs^{1/2})}{s^{n/2+m/2+3/2}} - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{R^n}{n!}\,b_m\,\frac{\exp(-Rs^{1/2})}{s^{n/2+m/2+5/2}}\right]\right\}. \quad (40)$$

Equation (40) may then be inverted term by term [7] to yield the solution in the time plane.

The velocity for a unit impulse may be found by differentiating the displacement with respect to t, or multiplying (33) by s. The strain is determined by differentiating the displacement Eq. (34) with respect to r, and the stress by substituting into Eq. (30). All of these lead to solutions consisting of double series similar to Eq. (40), which may then be inverted, and evaluated by computer methods.

For example, inversion of (40) gives

$$u(r, t) = \frac{P'r_0c^2}{\mu\omega_0}\exp(-T - R^2/8T)\left\{\frac{1}{r^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{R^n}{n!}\,b_m 2^{n/2+m/2+5/2}T^{n/2+m/2+2}D_{-n-m-5}\left(\frac{R}{(2T)^{1/2}}\right)\right.$$
$$+\frac{\omega_0}{rc}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{R^n}{n!}\,b_m 2^{n/2+m/2+2}T^{n/2+m/2+3/2}D_{-n-m-4}\left(\frac{R}{(2T)^{1/2}}\right)\right.$$
$$\left.\left.-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{R^n}{n!}\,b_m 2^{n/2+m/2+3}T^{n/2+m/2+5/2}D_{-n-m-6}\left(\frac{R}{(2T)^{1/2}}\right)\right]\right\} \quad (41)$$

where

$T = \omega_0 t.$

D_{-n-m-s} = cylinder functions of negative integral order.

The expansion of fractions has been shown to be valid for both rational [8] and fractional exponents [9] of the denominator terms of the fraction (see below). One limitation on the use for computing of the expansion is that the series involved converge slowly for large values of distance, time and ω_0. Other methods of obtaining a function which could be inverted were investigated. None offered a ready means of solution and consequently no other tractable method of solving for and evaluating the desired parameters was found, except convolution. The latter led to evaluation of integrals by numerical methods and was beyond the scope of the current investigation.

In every term in the inverted equations, there are essentially five components, an exponential, a power-factorial term, 2 to an exponential value, T to an exponential value, and a parabolic cylinder function of negative integral order. The exponential decreases rapidly with increasing values of T, and the cylinder functions also decrease with an increase in order or an increase in value of the argument. The power-factorial term increases rapidly until $n = R$ and then it decreases. Cylinder functions are calculated by means of an appropriate recursion formula [1].

Both 2 and T to exponential increase without bound as n increases. The behavior of each of these functions must be considered in programming inasmuch as some of the numbers may become very large, in excess of 10^{100}, and some very small, less than 10^{-100}. In some cases the order of multiplication becomes important so that two small or two large numbers are not multiplied in succession.

For computation of the double series, it is more convenient to find the multipliers of each successive cylinder function. These are found to be increasing truncated series of power-factorial terms. That is, a double series may be expressed as a Cauchy product:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_m \frac{R^n}{b!} 2^{(n+m+z)/2} T^{(n+m+z-1)/2} D_{-n-m-z}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{b_k R^{n-k}}{(n-k)!} \right) 2^{(n+z)/2} T^{(n+z-1)/2} D_{-n-z} . \tag{42}$$

The infinite series all converge rapidly for relatively small values of R and T, but not so rapidly for larger values. For purposes of computation specific values are chosen for r, r_0, ω_0, and c, which was not necessary in the case of plane waves.

A rational function of the following type may be expanded [8] in an infinite series, and has a zero of order n at ∞, i.e.,

$$y(s) = \frac{1}{a_0 s^n + a_1 s^{n-1} + \cdots a_n} \tag{43}$$

may be expanded as

$$y(s) = \frac{1}{a_0 s^n} - \frac{a_1}{a_0^2} \frac{1}{s^{n+1}} + \frac{a_1 - a_0 a_2}{a_0^3} \frac{1}{s^{n+2}} + \cdots \tag{44}$$

and inverting term by term yields

$$y(t) = c_0 \frac{t^{n-1}}{(n-1)!} + c_1 \frac{t^n}{n!} + c_2 \frac{t^{n+1}}{(n+1)!} + \cdots . \tag{45}$$

However, the individual coefficients of the terms in the series rapidly become very cumbersome to develop and to use in computation. For digital computer calculations, on the other hand, a simple sequential loop procedure can be employed to determine the successive coefficients.

Let Eq. (44) be written in the following infinite series form:

$$y(s) = \frac{c_0}{s^n} + \frac{c_1}{s^{n+1}} + \frac{c_2}{s^{n+2}} + \cdots \frac{c_n}{s^{2n}} + \cdots \tag{46}$$

which may be inverted term by term. The same process may be carried out for irrational fractions [9]:

$$y(q) = \frac{1}{a_0 q^n + a_1 q^{n-1} + \cdots a_n} \tag{47}$$

where $q = s^{1/2}$. This may likewise be expanded in the form

$$y(q) = \frac{c_0}{q^n} + \frac{c_1}{q^{n-1}} + \frac{c_2}{q^{n-2}} + \cdots \tag{48}$$

and may also be inverted term by term to give a solution in the time plane.

Thus, the inversion of the double series resolves itself into the inversion of a single series with a polynomial coefficient, each term of the series having a valid inversion. The inverted terms are similarly expressed and are in a convenient form for computation.

The operational solutions for unit step and decay exponentials are obtained in a similar manner. In these cases the operational expression is multiplied into the quadratic in s before it is expanded.

It is notable that there is an oscillation of all parameters for the spherical wave, while the plane Voigt wave does not exhibit oscillations [1]. The behavior is somewhat similar to an elastic wave, but with greater damping and a difference in wave shape. For a unit impulse and a decay exponential forcing function the oscillations are about the zero axis. For a unit step, however, the oscillations are about a curve which is parallel to the zero axis. In an elastic material, a wave caused by a unit step function also oscillates about a line parallel to the zero axis.

For larger travel distances the wave spreads out and becomes somewhat more symmetrical. However, very small disturbances are indicated before the arrival of the main wave, which disturbances are characteristic of a "diffusion model."

Solutions for unit step and exponential inputs for spherical waves are found by utilizing the methods employed for a plane wave and those developed for the spherical wave for a $\delta(t)$ input. Convolution solutions are also feasible, but are not given in this paper.

Expressions for velocity, displacement, stress and strain for three different forcing functions are given in the following tabulation for plane and spherical waves.

Values for particle velocity, displacement, stress and strain were calculated and curves drawn for the three input functions, but only two types are included here. The particle velocity for the decay exponential (Figs. 1–3) illustrates several of the char-

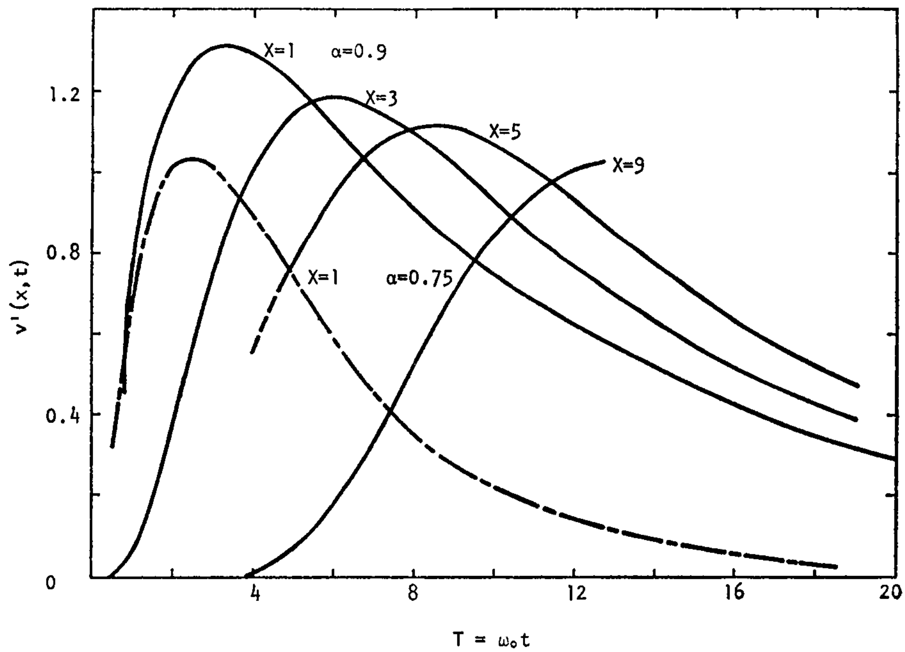

Fig. 1. Normalized velocity $v'(x,t) = v(x,t) \div \frac{P_o}{\rho c\sqrt{\pi}}$ for $P_o e^{-\beta t}$ forcing function for plane Voigt wave.

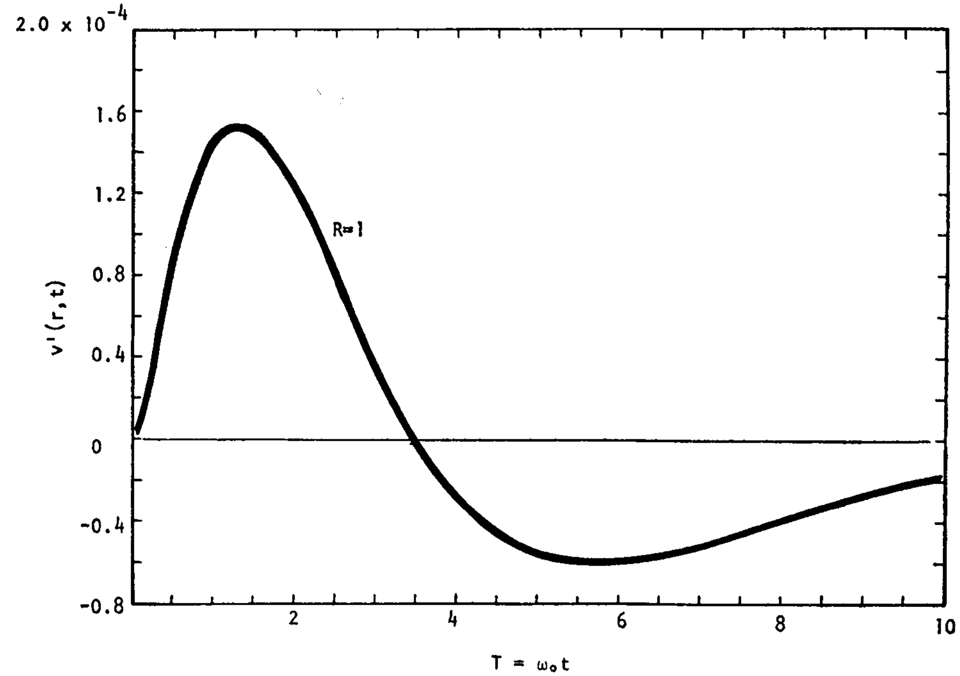

Fig. 2. Normalized velocity $v'(r,t) = v(r,t) \div \frac{P_0 r_0 c^2}{3\mu\omega_0\sqrt{\pi}}$ for $P(t) = P_0 e^{-\beta t}$ for a spherical Voigt wave, $r_0 = 50$ ft., $\omega_0 = 600$ and $c = 20{,}000$ ft/sec.

acteristics of the Voigt wave. The plane wave (Fig. 1) does not oscillate, whereas the spherical wave (Figs. 2 and 3) does. A change in ω_0 alters only the amplitude of the plane wave, but not its rate of decay, whereas for a spherical wave both are affected by the value of ω_0 . Acceleration and strain may be inferred from the particle velocity curves.

For spherical pulses the primary differences between a Voigt and an elastic wave is the early arrival time and the symmetrical shape of the former. The Voigt wave velocity depends upon the frequencies of the input function as well as the value of ω_0 . That is, for very high frequencies the Voigt model behaves as a rigid element. For lower frequencies and larger values of ω_0 it behaves more nearly as an elastic element. Upper limits were found to exist for values of t, r and ω_0 which could be employed in calculations, as well as lower limits for values of t.

Summation Symbols

Plane wave—single summation:

$$\sum\nolimits_{\nu}^{(s,P)} = \sum_{n=0}^{\infty} \frac{X^n}{n!} \frac{\exp(-Xs^{1/2})}{s^{n/2+\nu}},$$

$$\sum\nolimits_{s}^{(t,P)} = \sum_{n=0}^{\infty} \frac{X^n}{n!} 2^{(n+s)/2} T^{(n+s-1)/2} D_{-n-s}(X/(2T)^{1/2}) \cdot \mathrm{EXP}.$$

Plane wave—double summation:

$$\sum \sum_y^{(s,P)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_m \frac{X^n}{n!} \frac{\exp(-Xs^{1/2})}{s^{m+n/2+y}}$$

$$\sum \sum_z^{(t,P)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_m \frac{X^n}{n!} 2^{m+(n+z)/2} T^{m+(n+z-1)/2} D_{-n-2m-z}(X/(2T)^{1/2}) \cdot \mathrm{EXP}.$$

Spherical wave:

$$\sum \sum_y^{(s,S)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_m \frac{R^n}{n!} \frac{\exp(-Rs^{1/2})}{s^{(m+n)/2+y}},$$

$$\sum \sum_z^{(t,S)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_m \frac{R^n}{n!} 2^{(m+n+z)/2} T^{(m+n+z-1)/2} D_{-n-m-z}(R/(2T)^{1/2}) \cdot \mathrm{EXP}.$$

$(s, P) = s$ variable, plane wave, $(s, S) = s$ variable, spherical wave,

$(t, P) = t$ variable, plane wave, $(t, S) = t$ variable, spherical wave.

On the left side of the above equations the first letter in the superscript represents the s-plane or the t-plane, and the second a plane or spherical wave. The subscripts y and z are, respectively, the numerical values of the transform variable exponent other than the summation indices, and the subscript of the Weber function other than the summation indices. In all cases $z = 2y - 1$, EXP $= \exp(-T - X^2/8T)$ for the plane wave, and EXP $= \exp(-T - R^2/8T)$ for the spherical wave.

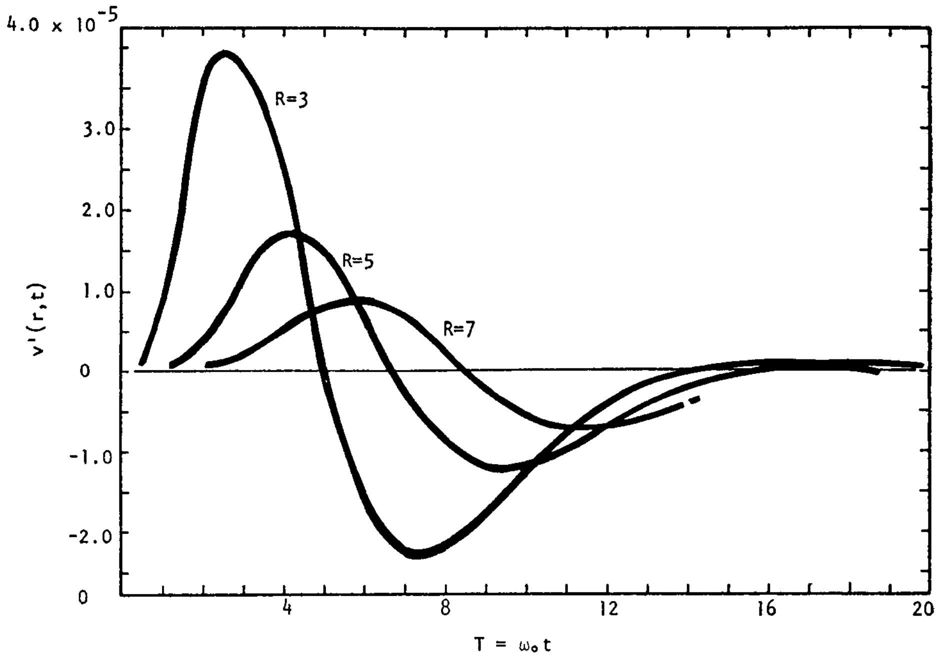

Fig. 3. Normalized particle velocity $v'(r,t) = v(r,t) \div \frac{P_o r_o c^2}{3\mu\omega_o\sqrt{\pi}}$ for $P(t) = P_o e^{-\beta t}$ for spherical Voigt wave, $r_o = 50$ ft., $\omega_o = 600$ and $c = 20{,}000$ ft/sec.

SUMMARY OF SOLUTIONS

Plane Wave: $P(t) = P'\,\delta(t)$

$$v(x, t) = \frac{P'\omega_0}{\rho c\pi^{1/2}} \sum_0^{(t,P)}, \qquad u(x, t) = \frac{P'}{\rho c\pi^{1/2}} \sum \sum_2^{(t,P)},$$

$$\epsilon(x, t) = -\frac{P'\omega_0}{\rho c^2\pi^{1/2}} \sum_1^{(t,P)}, \qquad \sigma(x, t) = -P'\omega_0 \sum_{-1}^{(t,P)}.$$

Plane Wave: $P(t) = P''1(t)$

$$v(x, t) = \frac{P''}{\rho c\pi^{1/2}} \sum \sum_2^{(t,P)}, \qquad u(x, t) = \frac{P''}{\rho c\omega_0\pi^{1/2}} \sum \sum_2^{(t,P)},$$

$$\epsilon(x, t) = -\frac{P''}{\rho c^2\pi^{1/2}} \sum \sum_3^{(t,P)}, \qquad \sigma(x, t) = -P'' \sum \sum_1^{(t,P)}.$$

Plane Wave: $P(t) = P_0 e^{-\beta t}$

$$v(x, t) = \frac{P_0}{\rho c\pi^{1/2}} \sum \sum_2^{(t,P)}, \qquad u(x, t) = \frac{P_0}{\rho c\omega_0\pi^{1/2}} \sum \sum_4^{(t,P)},$$

$$\epsilon(x, t) = -\frac{P_0}{\rho c^2\pi^{1/2}} \sum \sum_3^{(t,P)}, \qquad \sigma(x, t) = -\frac{P_0}{\pi^{1/2}} \sum \sum_1^{(t,P)}.$$

Spherical Wave: $P(t) = P'\,\delta(t)$

$$u(r, t) = \frac{P'r_0c^2}{3\mu\omega_0\pi^{1/2}}\left[\frac{1}{r^2} \sum \sum_3^{(t,S)} + \frac{\omega_0}{rc} \sum \sum_2^{(t,S)} - \frac{\omega_0}{rc} \sum \sum_4^{(t,S)}\right],$$

$$v(r, t) = \frac{P'r_0c^2}{3\mu\pi^{1/2}}\left\{\frac{1}{r^2}\left[\sum \sum_1^{(t,S)} - \sum \sum_3^{(t,S)}\right] + \frac{\omega_0}{rc}\left[\sum \sum_0^{(t,S)} - 2 \sum \sum_2 + \sum \sum_4^{(t,S)}\right]\right\},$$

$$\epsilon(r, t) = -\frac{P'r_0c^2}{3\mu\omega_0\pi^{1/2}}\left\{\frac{2}{r^3} \sum \sum_3^{(t,S)} + \frac{2\omega_0}{r^2c}\left[\sum \sum_2^{(t,S)} - \sum \sum_4^{(t,S)}\right] + \frac{\omega_0^2}{rc^2}\left[\sum \sum_1^{(t,S)} - 2 \sum \sum_3^{(t,S)} + \sum \sum_5^{(t,S)}\right]\right\},$$

$$\sigma(r, t) = -\frac{P'r_0c^2}{3\omega_0\pi^{1/2}}\left\{\frac{4}{r^3} \sum \sum_1^{(t,S)} + \frac{4\omega_0}{r^2c}\left[\sum \sum_0^{(t,S)} - \sum \sum_2^{(t,S)}\right] + \frac{3\omega_0^2}{rc^2}\left[\sum \sum_{-1}^{(t,S)} - 2 \sum \sum_1^{(t,S)} + \sum \sum_3^{(t,S)}\right]\right\},$$

Spherical Wave: $P(t) = P''1(t)$

$$u(r, t) = \frac{P''r_0c^2}{3\mu\omega_0\pi^{1/2}}\left\{\frac{1}{r^2} \sum \sum_5^{(t,S)} + \frac{\omega_0}{rc}\left[\sum \sum_4^{(t,S)} - \sum \sum_6^{(t,S)}\right]\right\},$$

$$v(r, t) = \frac{P''r_0c^2}{3\mu\omega_0\pi^{1/2}}\left\{\frac{1}{r^2} \sum \sum_3^{(t,S)} + \frac{\omega_0}{rc}\left[\sum \sum_2^{(t,S)} - \sum \sum_4^{(t,S)}\right]\right\},$$

$$\epsilon(r, t) = -\frac{P''r_0c^2}{3\mu\omega_0^2\pi^{1/2}}\left\{\frac{2}{r^3}\sum\sum_5^{(t,S)} + \frac{2\omega_0}{r^2c}\left[\sum\sum_4^{(t,S)} - \sum\sum_6^{(t,S)}\right]\right.$$
$$\left. + \frac{\omega_0^2}{rc^2}\left[\sum\sum_3^{(t,S)} - 2\sum\sum_5^{(t,S)} + \sum\sum_7^{(t,S)}\right]\right\},$$

$$\sigma(r, t) = -\frac{P''r_0c^2}{3\omega_0^2\pi^{1/2}}\left\{\frac{4}{r^3}\sum\sum_3^{(t,S)} + \frac{4\omega_0}{r^2c}\left[\sum\sum_2^{(t,S)} - \sum\sum_4^{(t,S)}\right]\right.$$
$$\left. + \frac{3\omega_0^2}{rc^2}\left[\sum\sum_1^{(t,S)} - 2\sum\sum_3^{(t,S)} + \sum\sum_5^{(t,S)}\right]\right\}.$$

Spherical Wave: $P(t) = P_0e^{-\beta t}$

$$u(r, t) = \frac{P_0r_0c^2}{3\mu\omega_0^2\pi^{1/2}}\left\{\frac{1}{r^2}\sum\sum_5^{(t,S)} + \frac{\omega_0}{rc}\left[\sum\sum_4^{(t,S)} - \sum\sum_6^{(t,S)}\right]\right\},$$

$$v(r, t) = \frac{P_0r_0c^2}{3\mu\omega_0\pi^{1/2}}\left\{\frac{1}{r^2}\left[\sum\sum_3^{(t,S)} - \sum\sum_5^{(t,S)}\right]\right.$$
$$\left. + \frac{\omega_0}{rc}\left[\sum\sum_2^{(t,S)} - 2\sum\sum_4^{(t,S)} + \sum\sum_6^{(t,S)}\right]\right\},$$

$$\epsilon(r, t) = -\frac{P_0r_0c^2}{3\mu\omega_0^2\pi^{1/2}}\left\{\frac{2}{r^3}\sum\sum_5^{(t,S)} + \frac{2\omega_0}{r^2c}\left[\sum\sum_4^{(t,S)} - \sum\sum_6^{(t,S)}\right]\right.$$
$$\left. + \frac{\omega_0^2}{rc^2}\left[\sum\sum_3^{(t,S)} - 2\sum\sum_5^{(t,S)} + \sum\sum_7^{(t,S)}\right]\right\},$$

$$\sigma(r, t) = -\frac{P_0r_0c^2}{3\omega_0^2\pi^{1/2}}\left\{\frac{4}{r^3}\sum\sum_3^{(t,S)} + \frac{4\omega_0}{r^2c}\left[\sum\sum_2^{(t,S)} - \sum\sum_4^{(t,S)}\right]\right.$$
$$\left. + \frac{3\omega_0^2}{rc^2}\left[\sum\sum_1^{(t,S)} - 2\sum\sum_3^{(t,S)} + 2\sum\sum_5^{(t,S)}\right]\right\}.$$

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