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SERRIN-TYPE REGULARITY CRITERION FOR THE
NAVIER-STOKES EQUATIONS INVOLVING ONE VELOCITY
AND ONE VORTICITY COMPONENT

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Abstract. We consider the Cauchy problem for the three-dimensional Navier-Stokes equations, and provide an optimal regularity criterion in terms of u_3 and ω_3 , which are the third components of the velocity and vorticity, respectively. This gives an affirmative answer to an open problem in the paper by P. Penel, M. Pokorný (2004).

Keywords: regularity criterion; Navier-Stokes equation

MSC 2010: 35B65, 35Q30, 76D03

1. INTRODUCTION

In this paper, we consider the Cauchy problem for the three-dimensional (3D) Navier-Stokes equations

$$(1.1) \quad \begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla\pi = \mathbf{0}, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the fluid velocity field, π is a scalar pressure, $\nu > 0$ is the kinematic viscosity and is assumed to be 1 in the rest of the paper, \mathbf{u}_0 is the prescribed initial data satisfying $\nabla \cdot \mathbf{u}_0 = 0$ in the distributional sense, and $\mathbf{u} \cdot \nabla = \sum_{i=1}^3 u_i \partial_i$ with $\partial_i = \partial/\partial x_i$.

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It is well-known that (1.1) possesses a global weak solution

$$\mathbf{u} \in L^\infty(0, \infty; L^2(\mathbb{R}^3)) \cap L^2(0, \infty; \bar{H}^1(\mathbb{R}^3))$$

for each set of initial data \mathbf{u}_0 of finite energy, see Leray [8] and Hopf [5]. However, whether or not such a weak solution is regular and unique is still a challenging open problem. Consequently, various criteria ensuring the regularity of the solutions are proposed. The classical Prodi-Serrin conditions (see [4], [9], [12], [13]) state that if

$$(1.2) \quad \mathbf{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 \leq q \leq \infty,$$

then the solution is smooth on $(0, T)$.

Later on, da Veiga in [1] showed a Serrin-type regularity criterion involving the velocity gradient (or vorticity)

$$(1.3) \quad \nabla \mathbf{u} \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} \leq q \leq \infty.$$

Due to the divergence-free condition, refinements of (1.2) and (1.3) attract many authors' attention, interested readers are referred to [3], [2], [6], [7], [11], [10], [14], [17], [18], [19], [20] and references therein.

In particular, Penel and Pokorný in [10], Theorem 1 (b) and Remark 1, discovered the following regularity criterion:

$$(1.4) \quad \begin{aligned} u_3 \in L^p(0, T; L^q(\mathbb{R}^3)); \quad \partial_2 u_1, \partial_1 u_2 \in L^r(0, T; L^s(\mathbb{R}^3)), \\ \frac{2}{p} + \frac{3}{q} = 1, \quad 3 < q \leq \infty; \quad \frac{2}{r} + \frac{3}{s} = 2, \quad 2 \leq s \leq 3; \end{aligned}$$

and remarked that it is an interesting open problem whether we could make assumptions on $\omega_3 = \partial_1 u_2 - \partial_2 u_1$ instead of $\partial_2 u_1$ and $\partial_1 u_2$, where ω_3 is the third component of the vorticity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = \nabla \times \mathbf{u}$. In this direction, Zhang et al. in [18], Theorem 1.2, proved the regularity of the solution under the condition

$$(1.5) \quad \begin{aligned} \partial_3 u_3 \in L^p(0, T; L^q(\mathbb{R}^3)); \quad \omega_3 \in L^r(0, T; L^s(\mathbb{R}^3)), \\ \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \leq \infty; \quad \frac{2}{r} + \frac{3}{s} = 2, \quad \frac{3}{2} < s \leq \infty. \end{aligned}$$

Remark 1.1. In [18], Theorem 1.4, the authors also provide a regularity criterion involving u_3 and ω_3 . It reads

$$(1.6) \quad \begin{aligned} u_3 \in L^p(0, T; L^q(\mathbb{R}^3)); \quad \omega_3 \in L^r(0, T; L^s(\mathbb{R}^3)), \\ \frac{2}{p} + \frac{3}{q} = 1, \quad 3 < q \leq \infty; \quad \frac{2}{r} + \frac{3}{s} = 2, \quad \frac{3}{2} < s \leq 3. \end{aligned}$$

However, (1.6) is not right. We thank Dr. X. J. Jia for pointing out that [18], equation (4.1), is not correct, in fact, one more term should be added, and its correct form should be

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}\|_{L^2}^2 + \nu \|\Delta \mathbf{u}\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} (\omega_2 u_3 - \omega_3 u_2) \Delta u_1 \, dx + \int_{\mathbb{R}^3} (\omega_3 u_1 - \omega_1 u_3) \Delta u_2 \, dx \\ & \quad + \int_{\mathbb{R}^3} [(\mathbf{u} \cdot \nabla) u_3] \cdot \Delta u_3 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} \partial_1 |\mathbf{u}|^2 \cdot \Delta u_1 + \partial_2 |\mathbf{u}|^2 \cdot \Delta u_2 \, dx, \end{aligned}$$

where the first three terms can be estimated as in [18], but the last term was neglected and was not treated. Although we can reformulate it as

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^3} (\partial_1 |\mathbf{u}|^2 \cdot \Delta u_1 + \partial_2 |\mathbf{u}|^2 \cdot \Delta u_2) \, dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^3} (\Delta |\mathbf{u}|^2 \cdot \partial_1 u_1 + \Delta |\mathbf{u}|^2 \cdot \partial_2 u_2) \, dx \\ &= \frac{1}{2} \int_{\mathbb{R}^3} \Delta |\mathbf{u}|^2 \cdot \partial_3 u_3 \, dx \\ &= \int_{\mathbb{R}^3} (\mathbf{u} \cdot \Delta \mathbf{u} + |\nabla \mathbf{u}|^2) \cdot \partial_3 u_3 \, dx \\ &= - \int_{\mathbb{R}^3} (\partial_3 \mathbf{u} \cdot \Delta \mathbf{u} + \mathbf{u} \cdot \Delta \partial_3 \mathbf{u} + \nabla \mathbf{u} \cdot \nabla \partial_3 \mathbf{u}) \cdot u_3 \, dx, \end{aligned}$$

we still have no means to bound the term

$$\int_{\mathbb{R}^3} \mathbf{u} \cdot \Delta \partial_3 \mathbf{u} \cdot u_3 \, dx.$$

The motivation of this paper is to give an affirmative answer to the above-mentioned open problem; that is, if

$$(1.7) \quad \begin{aligned} & u_3 \in L^p(0, T; L^q(\mathbb{R}^3)); \quad \omega_3 \in L^r(0, T; L^s(\mathbb{R}^3)), \\ & \frac{2}{p} + \frac{3}{q} = 1, \quad 3 < q \leq \infty; \quad \frac{2}{r} + \frac{3}{s} = 2, \quad \frac{3}{2} < s \leq \infty; \end{aligned}$$

then the solution is smooth on $(0, T)$.

Before stating the precise result, let us recall the weak formulation of (1.1), see [16] for instance.

Definition 1.2. Let $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot \mathbf{u}_0 = 0$, $T > 0$. A measurable \mathbb{R}^3 -valued function \mathbf{u} defined in $[0, T] \times \mathbb{R}^3$ is said to be a weak solution to (1.1) if

- (1) $\mathbf{u} \in L^\infty(0, T; L^2(\mathbb{R}^3) \cap L^2(0, T; H^1(\mathbb{R}^3)))$;
- (2) (1.1)₁ and (1.1)₂ hold in the sense of distributions, i.e.,

$$\int_0^t \int_{\mathbb{R}^3} \mathbf{u} \cdot [\partial_t \Phi + (\mathbf{u} \cdot \nabla) \Phi] dx ds + \int_{\mathbb{R}^3} \mathbf{u}_0 \cdot \Phi(0) dx = \int_0^T \int_{\mathbb{R}^3} \nabla \mathbf{u} : \nabla \Phi dx dt,$$

for each $\Phi \in C_c^\infty([0, T] \times \mathbb{R}^3)$ with $\nabla \cdot \Phi = 0$, where $A : B = \sum_{i,j=1}^3 a_{ij} b_{ij}$ for 3×3 matrices $A = (a_{ij})$, $B = (b_{ij})$, and

$$\int_0^T \int_{\mathbb{R}^3} \mathbf{u} \cdot \nabla \psi dx dt = 0,$$

for each $\psi \in C_c^\infty(\mathbb{R}^3 \times [0, T])$;

- (3) the energy inequality holds, that is,

$$\|\mathbf{u}(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla \mathbf{u}(s)\|_{L^2}^2 ds \leq \|\mathbf{u}_0\|_{L^2}^2, \quad 0 \leq t \leq T.$$

Now, our main result reads as follows.

Theorem 1.3. Let $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot \mathbf{u}_0 = 0$, $T > 0$. Assume that \mathbf{u} is a weak solution to (1.1) in $[0, T]$ with initial data \mathbf{u}_0 . If (1.7) holds, then the solution \mathbf{u} is smooth in $(0, T] \times \mathbb{R}^3$.

2. PROOF OF THEOREM 1.3

In this section, we shall prove Theorem 1.3. For any $\varepsilon \in (0, T)$, due to the fact that $\nabla \mathbf{u} \in L^2(0, T; L^2(\mathbb{R}^3))$, we may find a $\delta \in (0, \varepsilon)$, such that $\nabla \mathbf{u}(\delta) \in L^2(\mathbb{R}^3)$. Taking this $\mathbf{u}(\delta)$ as initial data, there exists an $\tilde{\mathbf{u}} \in C([\delta, \Gamma^*), H^1(\mathbb{R}^3)) \cap L^2(0, \Gamma^*; H^2(\mathbb{R}^3))$, where $[\delta, \Gamma^*)$ is the life span of the unique strong solution, see [16]. Moreover, $\tilde{\mathbf{u}} \in C^\infty(\mathbb{R}^3 \times (\delta, \Gamma^*))$. According to the uniqueness result, $\tilde{\mathbf{u}} = \mathbf{u}$ on $[\delta, \Gamma^*)$. If $\Gamma^* \geq T$, we have already that $\mathbf{u} \in C^\infty(\mathbb{R}^3 \times (0, T))$, due to the arbitrariness of $\varepsilon \in (0, T)$. In the case $\Gamma^* < T$, our strategy is to show that $\|\nabla \mathbf{u}(t)\|_2$ is uniformly bounded for $t \in [\delta, \Gamma^*)$. The standard continuation argument then yields that $[\delta, \Gamma^*)$ cannot be the maximal interval of existence of $\tilde{\mathbf{u}}$, and consequently $\Gamma^* \geq T$. This concludes the proof.

To bound $\|\nabla \mathbf{u}(t)\|_{L^2}$, taking the inner product of (1.1)₁ with $-\Delta \mathbf{u}$ in $L^2(\mathbb{R}^3)$, we obtain as in [18], (3.1)–(3.2) that

$$\begin{aligned}
 (2.1) \quad & \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}\|_{L^2}^2 + \|\Delta \mathbf{u}\|_{L^2}^2 \\
 &= \sum_{i,j,k,l=1}^3 \alpha_{11ijkl} \int_{\mathbb{R}^3} \partial_1 u_1 \partial_i u_j \partial_k u_l \, dx + \sum_{i,j,k,l=1}^3 \alpha_{12ijkl} \int_{\mathbb{R}^3} \partial_1 u_2 \partial_i u_j \partial_k u_l \, dx \\
 &+ \sum_{i,j,k,l=1}^3 \alpha_{21ijkl} \int_{\mathbb{R}^3} \partial_2 u_1 \partial_i u_j \partial_k u_l \, dx + \sum_{i,j,k,l=1}^3 \alpha_{22ijkl} \int_{\mathbb{R}^3} \partial_2 u_2 \partial_i u_j \partial_k u_l \, dx \\
 &\equiv I_1 + I_2 + I_3 + I_4,
 \end{aligned}$$

where $\alpha_{mni jkl}$, $1 \leq m, n \leq 2$, $1 \leq i, j, k, l \leq 3$, are some suitable integers.

To proceed further, we need to represent $\partial_m u_n$, $1 \leq m, n \leq 2$ by u_3 and ω_3 . Denoting by $\Delta_h = \partial_1 \partial_1 + \partial_2 \partial_2$ the horizontal Laplacian, we recall from [18], (1.4), (1.5) that

$$(2.2) \quad \Delta_h u_1 = -\partial_2 \omega_3 - \partial_1 \partial_3 u_3, \quad \Delta_h u_2 = \partial_1 \omega_3 - \partial_2 \partial_3 u_3.$$

Consequently, for $1 \leq k \leq 2$,

$$(2.3) \quad \partial_k u_1 = \frac{\partial_2}{\sqrt{-\Delta_h}} \frac{\partial_k}{\sqrt{-\Delta_h}} + \frac{\partial_2}{\sqrt{-\Delta_h}} \frac{\partial_1}{\sqrt{-\Delta_h}} \partial_3 u_3 = \mathcal{R}_2 \mathcal{R}_k \omega_3 + \mathcal{R}_1 \mathcal{R}_k \partial_3 u_3,$$

$$(2.4) \quad \partial_k u_2 = \mathcal{R}_1 \mathcal{R}_k \omega_3 + \mathcal{R}_2 \mathcal{R}_k \partial_3 u_3,$$

where $\mathcal{R}_k = \partial_k / \sqrt{-\Delta_h}$ is the two-dimensional Riesz transformation, see [15]. With (2.3), we may integrate by parts to dominate I_{11} as

$$\begin{aligned}
 (2.5) \quad I_{11} &= \sum_{i,j,k,l=1}^3 \alpha_{11ijkl} \int_{\mathbb{R}^3} \partial_1 u_1 \cdot \partial_i u_j \cdot \partial_k u_l \, dx \\
 &= \sum_{i,j,k,l=1}^3 \alpha_{11ijkl} \int_{\mathbb{R}^3} (\mathcal{R}_2 \mathcal{R}_1 \omega_3 + \mathcal{R}_1 \mathcal{R}_1 \partial_3 u_3) \cdot \partial_i u_j \cdot \partial_k u_l \, dx \\
 &= \sum_{i,j,k,l=1}^3 \alpha_{11ijkl} \int_{\mathbb{R}^3} \mathcal{R}_2 \mathcal{R}_1 \omega_3 \cdot \partial_i u_j \cdot \partial_k u_l \, dx \\
 &\quad - \sum_{i,j,k,l=1}^3 \alpha_{11ijkl} \int_{\mathbb{R}^3} \mathcal{R}_1 \mathcal{R}_1 u_3 \cdot (\partial_3 \partial_i u_j \cdot \partial_k u_l + \partial_i u_j \cdot \partial_3 \partial_k u_l) \, dx.
 \end{aligned}$$

Observe that the Riesz transformation is bounded in $L^q(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ for $1 < q < \infty$, we have for $f \in L^q(\mathbb{R}^3)$ and $1 \leq j \leq 2$,

$$\begin{aligned}
 (2.6) \quad \|\mathcal{R}_j f\|_{L^q}^q &= \int_{\mathbb{R}^3} |\mathcal{R}_j f|^q dx \\
 &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^2} |\mathcal{R}_j f|^q dx_1 dx_2 \right] dx_3 \\
 &\leq C \int_{\mathbb{R}} \left[\int_{\mathbb{R}^2} |f|^q dx_1 dx_2 \right] dx_3 \\
 &= C \|f\|_{L^q}^q,
 \end{aligned}$$

and thus I_{11} can be further estimated by Hölder and Gagliardo-Nirenberg inequalities,

$$\begin{aligned}
 (2.7) \quad I_{11} &\leq C \|\omega_3\|_{L^s} \|\nabla \mathbf{u}\|_{L^{2s/(s-1)}}^2 + C \|u_3\|_{L^q} \|\nabla \mathbf{u}\|_{L^{2q/(q-2)}} \|\nabla^2 \mathbf{u}\|_{L^2} \\
 &\leq C \|\omega_3\|_{L^s} \|\nabla \mathbf{u}\|_{L^2}^{(2s-3)/s} \|\nabla^2 \mathbf{u}\|_{L^2}^{3/s} + C \|u_3\|_{L^q} \|\nabla \mathbf{u}\|_{L^2}^{(q-3)/q} \|\nabla^2 \mathbf{u}\|_{L^2}^{(q+3)/q} \\
 &\leq C \left(\|\omega_3\|_{L^s}^{2s/(2s-3)} + \|u_3\|_{L^q}^{2q/(q-3)} \right) \|\nabla \mathbf{u}\|_{L^2}^2 + \frac{1}{8} \|\Delta \mathbf{u}\|_{L^2}^2.
 \end{aligned}$$

Substituting (2.7) and similar bounds for I_{12} , I_{21} , I_{22} into (2.1), we conclude by Gronwall inequality that

$$\begin{aligned}
 (2.8) \quad \|\nabla \mathbf{u}(t)\|_{L^2}^2 &\leq \|\nabla \mathbf{u}(\delta)\|_{L^2}^2 \\
 &\quad \times \exp \left[\int_{\delta}^t \left(\|\omega_3\|_{L^s}^{2s/(2s-3)} + \|u_3\|_{L^q}^{2q/(q-3)} \right) ds \right] < \infty, \quad \delta \leq t < \Gamma^*.
 \end{aligned}$$

This bound completes the proof of Theorem 1.3. □

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