Seshadri constants via toric degenerations

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- But it is not enough.

Example

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Then

- $K_X + L_1$ is nef for $k \geq 2$ and very ample for $k \geq 3$,
- but $K_X + L_2$ is not effective for any $k > 0$

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- $\varepsilon(\mathbb{P}^1 \times \mathbb{P}^1; \mathcal{O}(a,b); p) = \min\{a, b\}$ for $a, b > 0, {}^\forall p$,
- For a smooth cubic surface $S \subset \mathbb{P}^3$,

$$\varepsilon(S, \mathcal{O}(1); p) = \begin{cases} 1 & \text{if } p \in \text{line} \\ 3/2 & \text{otherwise.} \end{cases}$$

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holds for very general t (lower semicontinuity).

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- Gromov width (Mcduff-Polterovich), and so on.

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- $\varepsilon(X, L; 1) \geq 1/\dim X$ holds (Ein-Küchle-Lazarsfeld),
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- X : toric, p : torus invariant point (Di rocco).

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By this strategy, we obtain the following results;

Theorem (Hypersurfaces)

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Remark

Note that the upper bound comes from $\varepsilon(X, \mathcal{O}(1); 1) \leq \sqrt[n]{\mathcal{O}(1)^n}$.

Theorem (Fano 3-folds with Picard number 1)

For each family of smooth Fano 3-folds with Picard number 1 (note that there are 17 such families),

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$$\varepsilon(X, -K_X; 1) = \begin{cases} 6/5 & (6) \subset \mathbb{P}(1,1,1,1,3) \\ 4/3 & (4) \subset \mathbb{P}^4 \\ 3/2 & (2) \cap (3) \subset \mathbb{P}^5 \\ 2 & \textit{otherwise} \\ 3 & (2) \subset \mathbb{P}^4 \\ 4 & \mathbb{P}^3 \end{cases}$$

holds, where X is a very general member in the family.

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The polarized toric variety corrsponding to P is

$$(X_P, L_P) := (\operatorname{Proj} \bigoplus_{k \geq 0} V_{kP}, \mathcal{O}(1)),$$

where $V_{kP} := \bigoplus_{u \in kP \cap M} \mathbb{C}x^u \subset \mathbb{C}[M]$.

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For $\sigma \prec P$ and $p \in O_\sigma$, it holds that

$$\varepsilon(X_P, L_P; p) = \min\{\varepsilon(X_\sigma, L_\sigma; p), \varepsilon(X_{P'}, L_{P'}; x_{v'})\},$$

where $\pi : M_{\mathbb{R}} \to M_{\mathbb{R}}/(\mathbb{R}(\sigma - \sigma))$ and
$P' = \pi(P), v' = \pi(\sigma)$.

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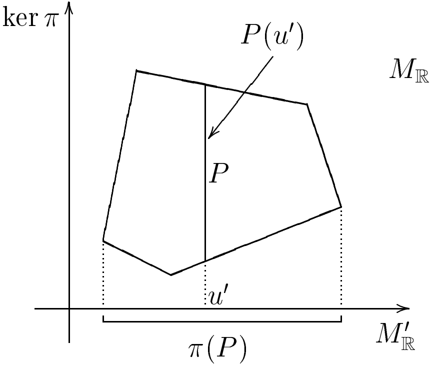

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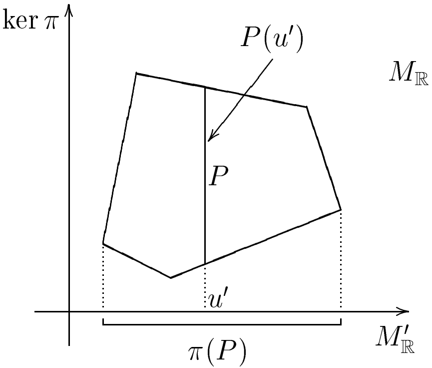

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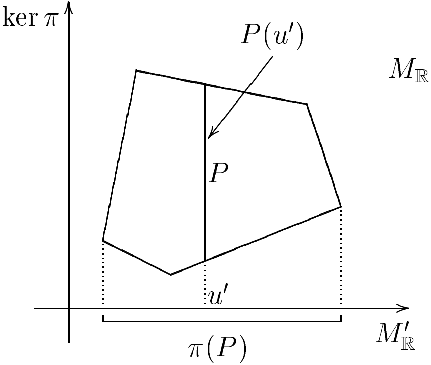
We can construct a rational map $\varphi : X_P \dashrightarrow X_{P(u')}$ such that $X_{\pi(P)} \fallingdotseq$ the general fiber of φ.



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Note that $(X_{P_n}, L_{P_n}) = (\mathbb{P}^n, \mathcal{O}(1))$.

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(2) $P = \mathrm{conv}(e_1, e_2, -e_1 - e_2)$,
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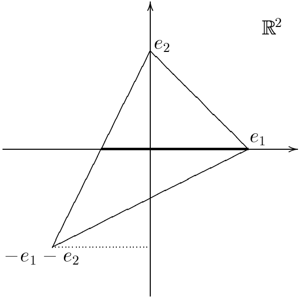
Then we have
$\min\{2, 3/2\} = 3/2 \le \varepsilon(X_P, L_P; 1_P) \le 2.$



Note that X_P is the cubic surface in $\mathbb{P}^3$ defined by $T_0^3 = T_1 T_2 T_3$,

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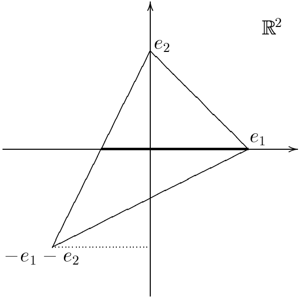

Note that X_P is the cubic surface in $\mathbb{P}^3$ defined by $T_0^3 = T_1T_2T_3$, and $\varepsilon(X_P, L_P; 1_P) = 3/2$ holds.

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In (2) $a_1 = a_2 = 1$, hence
$$\min\left\{\frac{a_1 + a_2 + 1}{a_2 + 1}, \frac{a_2 + a_1}{1}\right\} = \min\left\{\frac{3}{2}, 2\right\} = \frac{3}{2}.$$

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Then

$$\varepsilon(X_t, L_t; 1) \geq \varepsilon(X_P, L_P; 1_P)$$

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In particular, we have

$$\lfloor \sqrt[n]{d} \rfloor \leq \varepsilon(X, \mathcal{O}(1); 1) \leq \sqrt[n]{d}.$$

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The last part follows if we take
$c_n = c, c_{n-1} = c^2, \ldots, c_2 = c^{n-1}$ for $c = \lfloor \sqrt[n]{d} \rfloor$. □

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(2) $\varepsilon(X_{22}^3, \mathcal{O}(1); 1) \geq \min\{3, 8/3, 22/8\} = 8/3$ from $1 \leq 3 \leq 8 \leq 22$.
(3) $\varepsilon(X_{c^n}^n, \mathcal{O}(1); 1) = c$ holds for any $c, n \in \mathbb{N}$.

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$X \subset \mathbb{P}^{n+k}$: very general c.i. of degrees $d_1, \ldots, d_k$.
Then it holds that $\varepsilon(X, \mathcal{O}(1); 1) = d_1/(d_1 - 1)$.

Proof.

We prove only $k = 1$ case, thus we show
$\varepsilon(X, \mathcal{O}(1); 1) = (n+1)/n$ since $d_1 = n + 1$.
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Then we have $p \in C \subset X$ and
$\deg C = (n-1)!(n+1)$, $\mathrm{mult}_p(C) = (n-1)!n$.
Thus
$\varepsilon(X, \mathcal{O}(1); 1) \leq C.\mathcal{O}(1)/\mathrm{mult}_p(C) = (n+1)/n$. □

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$$\varepsilon(X, -K_X; 1) = \begin{cases} 6/5 & (6) \subset \mathbb{P}(1,1,1,1,3) \\ 4/3 & (4) \subset \mathbb{P}^4 \\ 3/2 & (2) \cap (3) \subset \mathbb{P}^5 \\ 2 & \textit{otherwise} \\ 3 & (2) \subset \mathbb{P}^4 \\ 4 & \mathbb{P}^3 \end{cases}$$

holds, where X is a very general member in the family.

Proof.

Ilten, Lewis, and Przyjalkowski showed that such X degenerates to a toric variety. We use it to show $\geq$. $\leq$ is proved by finding a suitable curve $C \subset X$. $\square$

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Anderson gave an interesting partial answer;

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(X, L): polarized var.

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Thus $\varepsilon(X, L; 1) \geq \varepsilon(X_{\Delta(L)}, L_{\Delta(L)}; 1_{\Delta(L)})$ holds in this case.
I proved that $\varepsilon(X, L; 1) \geq \varepsilon(X_{\Delta(L)}, L_{\Delta(L)}; 1_{\Delta(L)})$ holds without the finitely generatedness condition if we define $\varepsilon(X_\Delta, L_\Delta; 1_\Delta)$ for any closed convex set $\Delta \subset \mathbb{R}^n$ suitably.

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Remark

$\varepsilon(X, L; t\overline{m}) = t^{-1} \varepsilon(X, L; \overline{m})$ holds for any $t > 0$.

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Assume that general fibers are red. and irred. and $X_0 = \bigcup_{i=1}^{r} Y_i$: reduced.
Then

$$\varepsilon(X_t, L_t; \varepsilon_1, \dots, \varepsilon_r) \geq 1$$

holds for very general $t \in T$, where $\varepsilon_i = \varepsilon(Y_i, L_0|_{Y_i}; 1)$.

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holds for any $\overline{m} = (m_1, \ldots, m_r) \in (\mathbb{N} \setminus 0)^r$.

Remark

Note that the above theorem is false for $\overline{m} \in (\mathbb{R}_{>0})^r$ in general.

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Since X_d degenerates to $\bigcup_{i=1}^r X_{d_i}$, we have
$\varepsilon(X_d, \mathcal{O}(1); \varepsilon_1, \ldots, \varepsilon_r) \geq 1$ for $\varepsilon_i := \varepsilon(X_{d_i}, \mathcal{O}(1); 1)$.
We take d_i such that $d_i \geq (cm_i)^n$,
where $c = \lfloor \sqrt[n]{d / \sum_{i=1}^r m_i^n} \rfloor$.

Sketch of proof.

Let $d_1, \ldots, d_r \in \mathbb{N} \setminus 0$ such that $\sum d_i = d$.
Since X_d degenerates to $\bigcup_{i=1}^r X_{d_i}$, we have
$\varepsilon(X_d, \mathcal{O}(1); \varepsilon_1, \ldots, \varepsilon_r) \geq 1$ for $\varepsilon_i := \varepsilon(X_{d_i}, \mathcal{O}(1); 1)$.
We take d_i such that $d_i \geq (cm_i)^n$,
where $c = \lfloor \sqrt[n]{d / \sum_{i=1}^r m_i^n} \rfloor$.
Then $\varepsilon_i = \varepsilon(X_{d_i}, \mathcal{O}(1); 1) \geq \lfloor \sqrt[n]{d_i} \rfloor \geq cm_i$.

$\square$