

that \mathfrak{A} not $\vDash A$ and \mathfrak{B} not $\vDash B$. Consider then the ω -model $(\mathfrak{A} + \mathfrak{B})/\mathfrak{D}$. Since it is an ω -model of HAS we have that

$$(\mathfrak{A} + \mathfrak{B})/\mathfrak{D} \vDash (A \vee B).$$

However using Lemma 5.7 we see that it is not possible to find a bar B in $(\mathfrak{A} + \mathfrak{B})/\mathfrak{D}$ such that for all nodes $k \in B$ either A or B is satisfied at k . But then $(A \vee B)$ is not true in $(\mathfrak{A} + \mathfrak{B})/\mathfrak{D}$.

Remark. It should be clear that similar methods could be applied to obtain other common closure properties of intuitionistic systems.

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Set-valued mappings on metric spaces

by

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Abstract. In this paper we consider a mapping F of a complete metric space (X, d) into the class $B(X)$ of nonempty, bounded subsets of X . For A in $B(X)$ we define $FA = \bigcup_{a \in A} Fa$ and for A, B in $B(X)$ we define $\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}$. It is proved that if F maps $B(X)$ into $B(X)$ and satisfies the inequality

$$\delta(Fx, Fy) \leq c \cdot \max\{\delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx), d(x, y)\}$$

for all x, y in X , where $0 \leq c < 1$, then there exists a unique point z in X such that $z \in Fz$ and further $Fz = \{z\}$.

In a paper by Kaulgud and Pai, see [3], they consider mappings F of a metric space (X, d) into either $b(X)$, the class of nonempty, closed and bounded subsets of X , or $\text{Cpt}(X)$, the class of nonempty, compact subsets of X , or 2^X , the class of nonempty, closed subsets of X . The classes $b(X)$ and $\text{Cpt}(X)$ are given the Hausdorff metric D induced by the metric d . With F satisfying various conditions, they prove a number of fixed point theorems for F , a fixed point being defined as a point z in X for which z is in the set Fz . For example, they prove the following theorem in which $d(x, A)$ with x in X and A in $\text{Cpt}(X)$ is defined by

$$d(x, A) = \inf \{d(x, a) : a \in A\}.$$

THEOREM 1. Let F be a mapping of a complete metric space (X, d) into $\text{Cpt}(X)$ satisfying the inequality

$$D(Fx, Fy) \leq a_1 d(x, Fx) + a_2 d(y, Fy) + a_3 d(x, Fy) + a_4 d(y, Fx) + a_5 d(x, y)$$

for all x, y in X , where $a_1, \dots, a_5 \geq 0$ and $a_1 + \dots + a_5 < 1$. Then F has a fixed point in X .

In the following we consider a mapping F of a metric space (X, d) into $B(X)$, the class of all nonempty, bounded subsets of X . We define the function $\delta(A, B)$ with A, B in $B(X)$ by

$$\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}.$$

If the set A consists of a single point a we write

$$\delta(A, B) = \delta(a, B)$$

and if B also consists of single point b we write

$$\delta(A, B) = \delta(a, b) = d(a, b).$$

It follows easily from the definition of δ that

$$\delta(A, B) = \delta(B, A) \geq 0 \quad \text{and} \quad \delta(A, B) \leq \delta(A, C) + \delta(C, B)$$

for A, B, C in $B(X)$.

Further, if A is any nonempty subset of X , we define the set FA by

$$FA = \bigcup_{a \in A} Fa.$$

We now prove the following theorem

THEOREM 2. *Let F be a mapping of a complete metric space (X, d) into $B(X)$ satisfying the inequality*

$$(1) \quad \delta(Fx, Fy) \leq c \max\{\delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx), d(x, y)\}$$

for all x, y in X , where $0 \leq c < 1$. If F also maps $B(X)$ into itself, that is $FA \in B(X)$ for $A \in B(X)$, then F has a unique fixed point z in X and further, $Fz = \{z\}$.

Proof. It follows from inequality (1) that if A, B are any sets in $B(X)$ then

$$\delta(Fa, Fb) \leq c \max\{\delta(a, Fa), \delta(b, Fb), \delta(a, Fb), \delta(b, Fa), d(a, b)\}$$

for all a in A and b in B and so on taking the supremum over a in A and b in B of both sides of this inequality we have

$$(2) \quad \delta(FA, FB) \leq c \max\{\delta(A, FA), \delta(B, FB), \delta(A, FB), \delta(B, FA), \delta(A, B)\}$$

for all A, B in $B(X)$, both sides being finite since we are supposing that F maps $B(X)$ into itself.

Now let x be an arbitrary point in X and define the set $F^n x$ inductively by

$$F^n x = F(F^{n-1} x)$$

for $n = 2, 3, \dots$. Let us suppose that the sequence $\{\delta(F^n x, Fx) : n = 1, 2, \dots\}$ is unbounded. Then there exists some $n > 1$ such that

$$(3) \quad \begin{aligned} \delta(F^n x, Fx) &> \frac{c}{1-c} \delta(x, Fx) \\ &\geq \max\{\delta(F^r x, Fx) : r = 1, 2, \dots, n-1\}. \end{aligned}$$

Note that $n > 1$, since if $n = 1$ we would have

$$\begin{aligned} (1-c)\delta(Fx, Fx) &> c\delta(x, Fx) \\ &\geq c[\delta(x, Fx) - \delta(Fx, Fx)] \end{aligned}$$

which implies that $\delta(Fx, Fx) > c\delta(x, Fx)$ where as inequality (1) implies that $\delta(Fx, Fx) \leq c\delta(x, Fx)$. It follows from inequalities (3) that

$$\begin{aligned} (1-c)\delta(F^n x, Fx) &> c\delta(x, Fx) \\ &\geq c[\delta(x, F^r x) - \delta(F^r x, Fx)] \\ &\geq c[\delta(x, F^r x) - \delta(F^n x, Fx)] \end{aligned}$$

for $r = 1, 2, \dots, n$ and so

$$(4) \quad \delta(F^n x, Fx) > c \max\{\delta(x, F^r x) : r = 1, 2, \dots, n\}.$$

We will now prove that

$$(5) \quad \delta(F^n x, Fx) > c \max\{\delta(F^r x, F^s x) : r, s = 0, 1, 2, \dots, n\}$$

where $F^0 x = x$ and so

$$\delta(F^0 x, F^0 x) = d(x, x) = 0.$$

For if not

$$\begin{aligned} \delta(F^n x, Fx) &\leq c \max\{\delta(F^r x, F^s x) : r, s = 0, 1, 2, \dots, n\} \\ &\leq c \max\{\delta(F^r x, F^s x) : r, s = 1, 2, \dots, n\} \end{aligned}$$

on using inequality (4). We can now apply inequality (2) indefinitely to these terms, since whenever a term of the form $\delta(x, F^r x)$ appears, it can be omitted because of inequality (4). This means that

$$\delta(F^n x, Fx) \leq c^k \max\{\delta(F^r x, F^s x) : r, s = 1, 2, \dots, n\}$$

for $k = 1, 2, \dots$ and on letting k tend to infinity it follows that $\delta(F^n x, Fx) = 0$, giving a contradiction. Inequality (5) is thus proved. However, on using inequality (2), we now have

$$\begin{aligned} \delta(F^n x, Fx) &\leq c \max\{\delta(F^{n-1} x, F^n x), \delta(x, Fx), \delta(F^{n-1}, Fx), \delta(x, F^n x), \delta(F^{n-1} x, x)\} \\ &\leq c \max\{\delta(F^r x, F^s x) : r, s = 0, 1, 2, \dots, n\} \end{aligned}$$

which is impossible because of inequality (4). This contradiction implies that the sequence $\{\delta(F^n x, Fx) : n = 1, 2, \dots\}$ is in fact bounded.

Thus since

$$\delta(F^r x, F^s x) \leq \delta(F^r x, Fx) + \delta(Fx, F^s x);$$

it follows that

$$M = \sup\{\delta(F^r x, F^s x) : r, s = 0, 1, 2, \dots\}$$

is finite. Now, for arbitrary $\epsilon > 0$, choose N such that $c^N M < \epsilon$. It follows that for $m, n \geq N$, inequality (2) can be applied N times to the term $\delta(F^m x, F^n x)$ and so

$$(6) \quad \delta(F^m x, F^n x) \leq c^N M < \epsilon.$$

Choosing a point x_n in $F^n x$ for $n = 1, 2, \dots$, we have

$$d(x_m, x_n) \leq \delta(F^m x, F^n x) < \epsilon$$

for $m, n \geq N$. The sequence $\{x_n: n = 1, 2, \dots\}$ is therefore a Cauchy sequence in the complete metric space X and so has a limit z in X . Further,

$$\begin{aligned} \delta(z, F^n x) &\leq d(z, x_m) + \delta(x_m, F^n x) \\ &\leq d(z, x_m) + \delta(F^m x, F^n x) \\ &< d(z, x_m) + \varepsilon \end{aligned}$$

for $m, n \geq N$. Letting m tend to infinity we have

$$(7) \quad \delta(z, F^n x) \leq \varepsilon,$$

for $n \geq N$ and so

$$(8) \quad \begin{aligned} \delta(F^n x, Fz) &\leq \delta(F^n x, z) + \delta(z, Fz) \\ &\leq \delta(z, Fz) + \varepsilon \end{aligned}$$

for $n \geq N$. On using inequalities (2), (6), (7) and (8), we have for $n > N$

$$\begin{aligned} \delta(F^n x, Fz) &\leq c \max\{\delta(F^{n-1} x, F^n x), \delta(z, Fz), \delta(F^{n-1} x, Fz), \delta(z, F^n x), \delta(F^{n-1} x, z)\} \\ &\leq c \max\{\varepsilon, \delta(z, Fz), \delta(z, Fz) + \varepsilon, \varepsilon, \varepsilon\} \\ &= c[\delta(z, Fz) + \varepsilon]. \end{aligned}$$

It follows that since x_n is in $F^n x$, we have

$$\delta(x_n, Fz) \leq \delta(F^n x, Fz) \leq c[\delta(z, Fz) + \varepsilon]$$

for $n > N$ and on letting n tend to infinity we have

$$[\delta(z, Fz) \leq c\delta(z, Fz)]$$

since ε was arbitrary. It follows that $\delta(z, Fz) = 0$ and so $Fz = \{z\}$.

Now suppose F has a second fixed point w in X , so that w is in Fw . Then on using inequality (1) we have

$$\delta(Fw, Fw) \leq c\delta(w, Fw) \leq c\delta(Fw, Fw).$$

It follows that $\delta(Fw, Fw) = 0$ and so Fw only contains the single point w . Then

$$\begin{aligned} d(z, w) &= \delta(Fz, Fw) \\ &\leq c \max\{\delta(z, Fz), \delta(w, Fw), \delta(z, Fw), \delta(w, Fz), d(z, w)\} \\ &= cd(z, w). \end{aligned}$$

The uniqueness of z now follows. This completes the proof of the theorem.

We now note that although the mapping F in the theorem has a unique fixed point z it is possible for the point z to be contained in other sets Fx . To see this let x be the closed interval $[0, 1]$ with the usual metric. Define the function F by putting

$$Fx = \begin{cases} \{0\} & \text{for } x = 0, \\ [0, \frac{1}{2}x] & \text{for } x \neq 0. \end{cases}$$

Inequality (1) is satisfied with $c = \frac{1}{2}$ and the fixed point 0 is contained in every set in the range of F .

We finally prove two corollaries to Theorem 2. First of all we have

COROLLARY 1. Let F be a mapping of a complete metric space (X, d) into $B(X)$ satisfying the inequality

$$\delta(Fx, Fy) \leq a_1 \delta(x, Fx) + a_2 \delta(y, Fy) + a_3 \delta(x, Fy) + a_4 \delta(y, Fx) + a_5 d(x, y)$$

for all x, y in X , where $a_1, \dots, a_5 \geq 0$ and $a_1 + \dots + a_5 < 1$. If F also maps $B(X)$ into itself, then F has a unique fixed point z in X and further, $Fz = \{z\}$.

Proof. We have

$$\begin{aligned} \delta(Fx, Fy) &\leq a_1 \delta(x, Fx) + a_2 \delta(y, Fy) + a_3 \delta(x, Fy) + a_4 \delta(y, Fx) + a_5 d(x, y) \\ &\leq c \max\{\delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx), d(x, y)\} \end{aligned}$$

where $c = a_1 + \dots + a_5$. The result follows immediately from the theorem.

COROLLARY 2. Let T be a mapping of a complete metric space (X, d) into itself satisfying the inequality

$$d(Tx, Ty) \leq c \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\}$$

for all x, y in X , where $0 \leq c < 1$. Then T has a unique fixed point z .

Proof. Define a mapping F of X into $B(X)$ by putting $Fx = \{Tx\}$ for all x in X . It follows that F satisfies inequality (1). Further, by noting the condition that F maps bounded sets into bounded sets was only used to prove inequality (2) and because we required the sets $\{F^n x: n = 1, 2, \dots\}$ to be bounded, this condition is not needed in this corollary since $F^n x$ is now a set always consisting of a single point. In such a case inequality (1) can always be used instead of inequality (2) throughout the proof of the corollary. Thus, there exists a unique point z in X with $Fz = \{z\} = \{Tz\}$. The result now follows.

The result of this corollary was given in [2].

It should also be noted that it follows from Theorem 1 that the condition M be F -orbitally complete in Theorem 3 of Ćirić [1] can be replaced by the condition M be complete and it follows from Corollary 2 that the condition M be T -orbitally complete in Theorem 1 of [1] can also be replaced by the condition M be complete.

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