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Set-Valued Solutions to the Cauchy Problem for Hyperbolic Systems of Partial Differential Inclusions

J.-P. Aubin
H. Frankowska

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IIASA
International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 71521 □ Telex: 079 137 iiasa a □ Telefax: +43 2236 71313

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Jean-Pierre Aubin & Hélène Frankowska

CEREMADE, Université de Paris-Dauphine
&
IIASA, International Institute for Applied Systems Analysis

FOREWORD

We prove the existence of global set-valued solutions to the Cauchy problem for partial differential equations and inclusions, with either single-valued or set-valued initial conditions.

The method is based on the equivalence between this problem and problem of finding viability tubes of the associated characteristic system of ordinary differential equations or differential inclusions.

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Abstract

We prove the existence of global set-valued solutions to the Cauchy problem for partial differential equations and inclusions, with either single-valued or set-valued initial conditions.

The method is based on the equivalence between this problem and problem of finding viability tubes of the associated characteristic system of ordinary differential equations or differential inclusions.

Résumé

On démontre l'existence de solutions multivoques globales du problème de Cauchy pour les systèmes hyperboliques du premier ordre d'équations ou d'inclusions aux dérivées partielles, pour des conditions initiales univoques ou multivoques.

La méthode est basée sur l'équivalence entre ce problème et celui de l'existence de tubes de viabilité pour le système caractéristique d'équations différentielles ordinaires ou d'inclusions différentielles.

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1 Introduction

It is well known from the method of characteristics that first-order systems of hyperbolic partial differential equations may and do possess set-valued solutions, the set-valued character of a given solution providing an explanation for shocks.

One can use the differential calculus of set-valued maps for looking for global set-valued solutions to such hyperbolic systems of both partial differential equations and inclusions.

We shall prove the existence of a largest set-valued solutions with closed graph, which is unique (among closed graph single or set-valued solutions) whenever the characteristic system enjoys the uniqueness property.

The method we use is based on the equivalence between solutions $u(t,x) = (u_1(t,x),\ldots,u_m(t,x))$ to the system of partial differential equations

$$\forall\, j = 1,\ldots,m,\;\; 0 \;=\; \frac{\partial u_j(t,x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial u_j(t,x)}{\partial x_i} f_i(t,x,u(t,x)) - g_j(t,x,u(t,x)) \tag{1}$$

and *bilateral viable tubes*[1] $P(\cdot)$ *under* the characteristic system

$$\begin{cases} i) & x'(t) \;=\; f(t,x(t),y(t)) \\ ii) & y'(t) \;=\; g(t,x(t),y(t)) \end{cases} \tag{2}$$

The link between (single-valued or set-valued) solutions to (1) and tubes bilaterally viable under the characteristic system (2) is given by the relation

$$\forall\, t \geq 0,\;\; P(t) \;=\; \mathrm{Graph}(u(t,\cdot)) \;\subset\; X \times Y$$

[1] We recall that a solution $t \mapsto (x(t),y(t)) \in X \times Y$ to (2) defined on $[0,+\infty[$ is *viable* in the tube P if

$$\forall\, t \geq 0,\;\; (x(t),y(t)) \;\in\; P(t)$$

A tube P is *bilaterally viable* under the system (2) if, for all $t_0 \geq 0$ and $(x_{t_0}, y_{t_0}) \in P(t_0)$, there exists at least one solution $(x(\cdot),y(\cdot))$ to the differential system (2) satisfying $(x,y)(t_0) = (x_{t_0}, y_{t_0})$ which is viable in the tube P.

Therefore, the existence of solution to the Cauchy problem for (1) satisfying the initial condition

$$\forall\, x \in X, \;\; u(0,x) \;=\; u_0(x)$$

is equivalent to the existence of a tube bilaterally viable under the characteristic system (2) satisfying the initial condition

$$P(0) \;=\; K := \mathrm{Graph}(u_0)$$

Our objectives are twofold:

- to prove the equivalence between Cauchy problems for hyperbolic systems of partial differential equations and initial value problems for viable tubes of ordinary differential equations on one hand,
- to prove the existence of the largest tube bilaterally viable on the other hand and to characterize it.

This equivalence allows also to transfer other properties of viable tubes to corresponding properties of solutions to partial differential systems.

There are obvious advantages in doing so. First, dealing with graphs of solutions, we do not have to worry about the univocity issue: the viable tube provides the graph of a solution, single-valued or set-valued. We can tackle for instance the question of the existence of a largest solution as well as the existence of minimal solutions containing a given function.

The other advantage is that we can treat in the same way not only systems of partial differential equations, but also *partial differential inclusions*, since the results about viable tubes are still valid for ordinary differential inclusions

$$\begin{cases} i) & x'(t) \;\in\; F(t,x(t),y(t)) \\ ii) & y'(t) \;\in\; G(t,x(t),y(t)) \end{cases} \tag{3}$$

First-order systems of partial differential inclusions arise naturally in control theory (see [7,9,8]).

For instance, we shall prove a *stability theorem*: the *graphical upper limit*[2] of a sequence of solutions U_n is still a solution and that in the time independent case, the *graphical upper limit* of the solutions $U(t,\cdot)$ when $t \to \infty$ is a solution to the stationary problem.

We shall provide an explicit formula in the decomposable (set-valued) case from which we derive useful estimates. They are applied later on to prove the existence of single-valued Lipschitz contingent solution to the Cauchy problem for systems of partial differential inclusions

$$\forall\, t,x \in X, \;\; 0 \;\in\; \frac{du}{dt}(t,x) + \frac{du}{dx}(t,x)\cdot f(t,x,u(t,x)) - G(t,x,u(t,x))$$

on a small time interval by using fixed point arguments.

[2]The graph of the *graphical upper limit* $U^\sharp := \mathrm{Lim}^\sharp_{n\to\infty} U_n$ of a sequence of set-valued maps $U_n : X \leadsto Y$ is by definition the graph of the upper limit of the graphs of the maps U_n.

2 Cauchy Problem for Viability Tubes

The differential calculus for single-valued maps, including inverse function theorems, can be extended to set-valued maps.

We recall that the contingent derivative $DU(x,y)$ of a set-valued map $U : X \rightsquigarrow Y$ at $(x,y) \in \text{Graph}(U)$ is defined by

$$\text{Graph}(DU(x,y)) \;:=\; T_{\text{Graph}(U)}(x,y)$$

where

$$T_K(z) \;:=\; \{v \in X \mid \liminf_{h \to 0+} d(z+hv;K)/h = 0\}$$

denotes the *contingent cone* to a subset K at $z \in K$.

When $U = u$ is single-valued, we set $Du(x) := Du(x,u(x))$. See [5, Chapters 4, 5] for more details on contingent cones and differential calculus of set-valued maps.

We say that a set-valued map $P : t \in [0,+\infty[\rightsquigarrow P(t) \subset X$ is a *tube*, and that a tube is closed if its graph is closed.

We shall say that a set-valued map F is a *Marchaud map* if it is nontrivial, upper semicontinuous, has compact convex images and linear growth.

In finite dimensional spaces, this amounts to saying that

$$\begin{cases} i) & \text{the graph and the domain of } F \text{ are closed} \\ ii) & \text{the values of } F \text{ are convex} \\ iii) & \text{the growth of F is linear} \end{cases}$$

We consider a Marchaud map $F : [0,+\infty[\times X \rightsquigarrow X$ and the differential inclusion

$$x'(t) \;\in\; F(t,x(t)) \tag{4}$$

Definition 2.1 *A tube P is* viable *under F (or enjoys the* viability property*) if and only if, for all $t_0 \geq 0$ and $x_0 \in P(t_0)$, there exists at least one solution $x(\cdot)$ to the differential inclusion (4) starting at x_0 at time t_0 which is viable in the tube P.*

It is said to be backward viable *under F if for every $t_0 \in [0,+\infty[$, $x_0 \in P(t_0)$, there exists at least one solution $x(\cdot)$ to the differential inclusion (4) on the interval $[0,t_0]$ starting at $P(0)$, viable in the tube P on $[0,t_0]$ and such that $x(t_0) = x_0$.*

It is said to be bilaterally viable *under F if it is both viable and backward viable, i.e., if and only if $\forall\, t_0 \in [0,+\infty[, \forall\, x_0 \in P(t_0)$, there exists at least one viable solution starting at $P(0)$ and passing through x_0 at time t_0 (in the sense that $x(t_0) = x_0$).*

A tube $P : [0,+\infty[\rightsquigarrow X$ is called a viability tube *of a set-valued map $F : [0,+\infty[\times X \rightsquigarrow X$ if*

$$\forall\, t \in [0,+\infty[,\ \forall\, x \in P(t),\ \ F(t,x) \cap DP(t,x)(1) \neq \emptyset$$

a backward viability tube *if*

$$\forall\, t \in [0, +\infty[,\ \forall\, x \in P(t),\ \ F(t,x) \cap -DP(t,x)(-1) \neq \emptyset$$

and a bilateral viability tube *if it is both a viability tube and a backward viability tube.*

We say that a tube P is invariant under F *(or enjoys the* invariance property*) if and only if for all t_0 and $x_0 \in P(t_0)$,* all the solutions *to differential inclusion (4) starting at x_0 at time t_0 are viable in the tube P.*

It is called an invariance tube *if*

$$\forall\, t \in [0, +\infty[,\ \forall\, x \in P(t),\ \ F(t,x)\ \subset\ DP(t,x)(1)$$

For Marchaud maps, we recall some properties of the viability tubes (see Theorem 11.1.3 of VIABILITY THEORY, [2, Aubin]): *If $F : [0, +\infty[\times X \rightsquigarrow X$ is a Marchaud map, then a tube is viable under F if and only if it is a viability tube.*

Let us consider a sequence of set-valued maps $F_n : X \rightsquigarrow Y$. The set-valued map $F^\sharp := \mathrm{Lim}^\sharp{}_{n\to\infty} F_n$ from X to Y defined by

$$\mathrm{Graph}(\mathrm{Lim}^\sharp{}_{n\to\infty} F_n) := \mathrm{Limsup}_{n\to\infty} \mathrm{Graph}(F_n)$$

is called the *(graphical) upper limit* of the set-valued maps F_n.

We derive the following characterization of bilateral viability:

Proposition 2.2 *Assume that $F : [0, +\infty[\times X \rightsquigarrow X$ is a Marchaud map. Then a closed tube P is backward viable under F if and only if it is a backward viability tube.*

As a consequence, P is bilaterally viable under F if and only it is a bilateral viability tube.

Furthermore, if P_n is a sequence of closed tubes bilaterally viable under F, then so is its graphical upper limit P.

Consequently, any closed tube Q contained in a tube P bilaterally viable under F and satisfying $Q(0) = P(0)$ is actually contained in a minimal tube $\widehat{Q} \subset P$ bilaterally viable under F and satisfying $\widehat{Q}(0) = P(0)$, called a viability envelope *of Q.*

Proof — Let $P(\cdot)$ be a backward viability tube and x_0 belong to $P(t_0)$. First, consider the tube $\check{P}_{t_0}(s) := P(t_0 - s)$ defined by

$$\check{P}_{t_0}(s)\ :=\ \begin{cases} P(t_0 - s) & \text{if}\ \ s\ \in\ [0, t_0] \\ K & \text{if}\ \ s\ \geq\ t_0 \end{cases}$$

We observe that

$$D\check{P}_{t_0}(s,x)(\lambda)\ =\ DP(t_0 - s, x)(-\lambda)$$

because one can check easily that

$$(\lambda, u) \in T_{\mathrm{Graph}(\check{P}_{t_0})}(s, x)$$

if and only if

$$(-\lambda, u) \in T_{\mathrm{Graph}(P)}(t_0 - s, x)$$

Second, we consider the set-valued map $\check{G}_{t_0}$ defined by

$$\check{G}_{t_0}(s,x) := \begin{cases} \{-1\} \times -F(t_0 - s, x) & \text{if } s \in]0, t_0] \\ [-1, 0] \times \overline{\mathrm{co}}(\{0\} \cup -F(0,x)) & \text{if } s = t_0 \\ \{0\} \times \overline{\mathrm{co}}(\{0\} \cup -F(0,x)) & \text{if } s > t_0 \end{cases}$$

It is a Marchaud map since F is assumed to be a Marchaud map. Then, we observe that P is a backward viability tube if and only if the graph of $\check{P}_{t_0}$ is a viability domain of $\check{G}_{t_0}$.

Therefore, Theorem 3.3.5 of VIABILITY THEORY, [2, Aubin] implies that this is equivalent to say that the graph of $\check{P}_{t_0}$ is viable under $\check{G}_{t_0}$.

This means that for every $t_0 \in [0, +\infty[$, $x_0 \in P(t_0)$, there exists a solution $z(\cdot)$ to the backward differential inclusion $z'(t) \in -F(t_0 - t, z(t))$ starting at x_0 at time 0 and viable in the tube $t \rightsquigarrow P(t_0 - t)$ for all $t \in [0, t_0]$. By setting $x(t) := z(t_0 - t)$ when $t \in [0, t_0]$, we infer that $x(\cdot)$ is a solution to the differential inclusion $x' \in F(t, x)$ starting at $x(0) = z(t_0) \in P(0)$ and satisfying $x(t_0) = x_0$.

We show next that the upper graphical limit $P^\sharp$ of a sequence of tubes P_n bilaterally viable under F is still bilaterally viable under F.

Let x belong to $P^\sharp(t)$. This means that t is the limit of a subsequence $t_{n'}$ and that x is the limit of a subsequence $x_{n'} \in P_{n'}(t_{n'})$. Since the tubes P_n are bilaterally viable under F, there exist solutions $y_{n'}(\cdot)$ to differential inclusion (4) starting at $P_{n'}(0)$, satisfying $y_{n'}(t_{n'}) = x_{n'}$ and viable in $P_{n'}$. Theorem 3.5.2 of VIABILITY THEORY, [2, Aubin] implies that these solutions remain in a compact subset of $\mathcal{C}(0, +\infty; X)$. Hence a subsequence (again denoted) $y_{n'}(\cdot)$ converges uniformly on compact intervals to a solution $y(\cdot)$ to differential inclusion (4) starting at $P^\sharp(0)$ and satisfying $x(t) = x$. Since $y_{n'}(t)$ belongs to $P_{n'}(t)$ for all n', we deduce that $y(t)$ does belong to $P^\sharp(t)$ for all $t \geq 0$.

When the sequence P_n is decreasing, we know that its upper limit is equal to the intersection of the P_n : $P^\sharp(t) = \bigcap_{n \geq 0} P_n(t)$.

Therefore, by Zorn's Lemma for the inclusion order on the family of closed tubes bilaterally viable under F and satisfying $Q(0) = P(0)$, we deduce that any closed tube Q starting at $P(0)$ is contained in a minimal closed tube bilaterally viable and starting at $P(0)$. □

For Lipschitz maps, we recall a characterization of the invariant tubes. Theorem 11.6.2 of VIABILITY THEORY, [2, Aubin] states that *whenever* $F : [0, +\infty[\times X \to X$ *is upper semicontinuous and Lipschitz with respect to* x[3], *then a closed tube* $t \rightsquigarrow P(t) \subset X$ *is invariant under* F *if and only if it is an invariance tube.*

Let us single out the following property :

Proposition 2.3 *Assume that* P *is a closed tube invariant under a set-valued map* F. *Then, if for some* $s > 0$, $x_s \notin P(s)$, *then for every solution* $x(\cdot)$ *to differential inclusion (4) satisfying* $x(s) = x_s$ *and for every* $t \in [0, s]$, $x(t) \notin P(t)$.

Proof — If not, there would exist a solution $x(\cdot)$ and a time $t_0 \in [0, s[$ such that $x(t_0) \in P(t_0)$ and $y(s) = x_s$. This solution is viable in the tube P since all the solutions starting from $x(t_0) \in P(t_0)$ are viable, because the tube is assumed to be invariant. Therefore $x(s)$ belongs to $P(s)$, a contradiction. □

We now provide examples of tubes invariant under a set-valued map F.

Let us denote by $\mathcal{S}_F(s, K) \subset \mathcal{C}(s, +\infty; X)$ the subset of solutions to differential inclusion (4) starting from K at time $s \geq 0$.

The **reachable tube** $R_K(\cdot)$ of F starting at K defined by

$$R_K(t) \; := \; \{x(t)\}_{x(\cdot)\in\mathcal{S}_F(0,K)}$$

is obviously closed whenever F is Marchaud (see VIABILITY THEORY, [2, Aubin]).

Theorem 2.4 *The* **reachable tube** $R_K(\cdot)$ *is invariant under* F *and "minimal" in the sense that there is no other tube* P *invariant under* F *starting at* K *and strictly contained in* $R_K(\cdot)$.

It is also backward viable under F *and is the largest closed bilateral viability tube starting at* R_K.

If $K^\sharp := \mathrm{Limsup}_{n\to\infty} K_n$ *denotes the upper limit of a sequence of closed subsets* $K_n \subset X$, *then the graphical upper limit of the reachable tubes starting at* K_n *is a bilateral tube starting at* $K^\sharp$ *and thus*

$$\mathrm{Lim}^\sharp{}_{n\to\infty} R_{K_n} \; \subset \; R_{K^\sharp}$$

[3]in the sense that for some positive constant λ

$$F(t, x) \; \subset \; F(t, y) + \lambda \|x - y\| B$$

(B is a unit ball)

Equality holds true if the set-valued maps $F(t,\cdot)$ are λ-Lipschitz for every $t \geq 0$.

Proof— The reachable tube $R_K(\cdot)$ is obviously invariant and backward viable under F: Indeed, if $x_0 \in R_K(t_0)$, there exists by definition a solution $x(\cdot)$ to the differential inclusion (4) starting from K at time 0 and passing through x_0 at t_0. Furthermore, every solution $y(\cdot)$ to differential inclusion (4) starting at x_0 at time t_0, concatenated to $x(\cdot)$ restricted to the interval $[0, t_0]$ being a solution to our differential inclusion starting at K, $R_K(\cdot)$ is invariant.

Let us consider a closed tube $P \subset R_K$ invariant under F starting at K. We claim that it is equal to the reachable tube. Otherwise, there would exist $x_s \in R_K(s)$ such that $x_s \notin P(s)$. Since the reachable tube is backward viable, there exists a solution $x(\cdot)$ to the differential inclusion (4) starting from $x(0) \in K$ such that $x(s) = x_s$. But starting from $x(0)$, the solution is viable in the tube P since it is invariant under F and satisfies $P(0) = K$. Therefore $x(s)$ belongs to $P(s)$, a contradiction.

Let now P be any closed bilateral viability tube starting from K at time 0 and let us check that it is contained in the reachable tube. For that purpose, take $x_0 \in P(t_0)$. By Proposition 2.2, we know that there exists at least one viable solution starting at $P(0) = K$ and passing through x_0 at time t_0. Hence $x_0 \in R_K(t_0)$.

By Proposition 2.2, we know that the graphical upper limit is a bilateral viability tube. Proposition 7.1.4 of SET-VALUED ANALYSIS, [5, Aubin & Frankowska] implies that

$$\left(\mathrm{Lim}^{\sharp}_{n\to\infty} R_{K_n}\right)(0) \;=\; \mathrm{Limsup}_{n\to\infty}(R_{K_n}(0)) \;=\; K^{\sharp}$$

we infer that it is a bilateral tube starting at $K^{\sharp}$, and thus, contained in $R_{K^{\sharp}}$.

Conversely, let us choose $x_{t_0} \in R_{K^{\sharp}}(t_0)$. Then there exist a solution $x(\cdot)$ to (4) starting from some $x(0) \in K^{\sharp}$ and satisfying $x(t_0) = x_{t_0}$ and a subsequence (again denoted by) $x_n \in K_n$ converging to $x(0)$. By the Filippov

Theorem[4], there exist solutions $x_n(\cdot)$ to (4) starting at x_n such that

$$\begin{cases} \|x(t) - x_n(t)\| \\ \leq e^{\lambda t}\left(\|x(0) - x_n\| + \int_0^t e^{-\lambda s} d(x'(s), F(s, x_n(s)))ds\right) \\ \leq e^{\lambda t}\left(\|x(0) - x_n\| + \lambda \int_0^t e^{-\lambda s}\|x(s) - x_n(s)\|ds\right) \end{cases}$$

We thus derive from Gronwall's Lemma that $x_n(t_0) \in R_{K_n}(t_0)$ converges to $x(t_0) = x_{t_0}$. □

Proposition 2.5 *Let $c > 0$ be the growth constant of $F : [0, +\infty[\times X \rightsquigarrow X$:*

$$\forall\, t \geq 0,\ x \in X,\ \|F(t, x)\| \leq c(\|x\| + 1)$$

Then the tube $Q_K(\cdot)$ defined by

$$\forall\, t \geq 0,\ Q_K(t) := K + (\|K\| + 1)(e^{ct} - 1)B$$

is invariant under F and satisfies $Q_K(0) = K$.

Proof — Indeed, we know that every solution $x(\cdot)$ to differential inclusion (4) satisfies

$$\forall\, t \geq t_0,\ \|x'(t)\| \leq c(\|x(t_0)\| + 1)e^{c(t-t_0)}$$

so that

$$\|x(t) - x(t_0)\| \leq \int_{t_0}^t \|x'(\tau)\|d\tau = (\|x(t_0)\| + 1)(e^{c(t-t_0)} - 1)$$

Therefore, every solution $x(\cdot)$ to differential inclusion (4) starting from the closed subset $Q_K(t_0)$ at time t_0 satisfies

$$x(t) \in Q_K(t_0) + (\|Q_K(t_0)\| + 1)e^{c(t-t_0)}B$$

Since $Q_K(t_0) := K + (\|K\| + 1)(e^{ct_0} - 1)B$, we infer from these two inclusions that $x(t)$ remains in the tube $Q_K(t)$ for $t \geq t_0$. □

[4]Filippov's Theorem (see [3, Theorem 2.4.1] for instance), yields an estimate on any finite interval $[0, T]$: If for every $t \geq 0$ $F(t, \cdot)$ is λ-Lipschitz with nonempty closed values, and if an absolutely continuous function $y(\cdot)$ and an initial state x_0 are given, then there exists a solution $x(\cdot)$ to the differential inclusion (4) defined on $[0, T]$, starting at x_0 and satisfying the estimate

$$\|x(t) - y(t)\| \leq e^{\lambda t}\left(\|x_0 - y(0)\| + \int_0^t d(y'(s), F(y(s)))e^{-\lambda s}ds\right) \quad (5)$$

3 Contingent Solutions

Consider two finite dimensional vector-spaces X and Y and two set-valued maps $F : [0, +\infty[\times X \times Y \rightsquigarrow X$, $G : [0, +\infty[\times X \times Y \rightsquigarrow Y$. Let $DU(t, x, y)$ denote the *contingent derivative* of U at a point (t, x, y) of the graph of U.

Definition 3.1 *We shall say that a closed set-valued map $U : [0, +\infty[\times X \rightsquigarrow Y$ satisfying*

$$\forall\, y \in U(t, x), \;\; 0 \;\in\; DU(t, x, y)(1, F(t, x, y)) - G(t, x, y) \tag{6}$$

is a forward (contingent) set-valued solution *to the partial differential inclusion (6).*

It is said to be a backward (contingent) set-valued solution *to (6) if it satisfies*

$$\forall\, y \in U(t, x), \;\; 0 \;\in\; DU(t, x, y)(-1, -F(t, x, y)) + G(t, x, y) \tag{7}$$

and a (contingent) set-valued solution *to (6) if it is both a forward and a backward solution.*

Naturally, whenever the contingent derivatives $DU(t, x, y)$ are even, then forward and a backward solutions do coincide.

When $U = u : [0, +\infty[\times X \mapsto Y$ is a single-valued map with closed graph, the partial contingent differential inclusion (6) becomes

$$\forall\, (t, x), \;\; 0 \;\in\; Du(t, x)(1, F(t, x, u(x))) - G(t, x, u(x)) \tag{8}$$

Let the initial condition $U_0 : X \rightsquigarrow Y$, a single- or set-valued map be given.

Theorem 3.2 *Let us define the set-valued map $U_\propto : (t, x) \in [0, +\infty[\times X \rightsquigarrow U_\propto(t, x) \in Y$ by the method of characteristics: $y \in U_\propto(t, x)$ if there exists a solution $(x(\cdot), y(\cdot))$ to the system of differential inclusions (3) starting in the graph of U_0 and such that $x(t) = x$ and $y(t) = y$.*

Assume that $F : [0, +\infty[\times X \times Y \rightsquigarrow X$ and $G : [0, +\infty[\times X \times Y \rightsquigarrow Y$ are Marchaud maps. Then, for any initial condition $U_0 : X \rightsquigarrow Y$, $U_\propto$ is the largest set-valued solution to (6) satisfying the initial condition

$$\forall\, x \in X, \;\; U_\propto(0, x) \;=\; U_0(x)$$

It is the unique solution with closed graph whenever the characteristic system has the uniqueness property[5]*.*

[5]This happens whenever F and G enjoy a monotonicity property of the form: there exists a real constant c such that for every $t \geq 0$, for every pair x_i, y_i, $u_i \in F(t, x_i, y_i)$ and $v_i \in G(t, x_i, y_i)$ $(i = 1,\ 2)$, we have

$$\langle u_1 - u_2, x_1 - x_2 \rangle + \langle v_1 - v_2, y_1 - y_2 \rangle \leq c \left(\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2 \right)$$

Furthermore, we can associate with any selection $V(t,x) \subset U_\propto(t,x)$ satisfying $V(0,x) = U_0(x)$ a minimal solution $\widehat{V} \subset U_\propto$ to (6) satisfying the same initial condition and containing V.

Proof of Theorem 3.2 — By Theorem 2.4, the reachable tube $R_K(\cdot) : \mathbf{R}_+ \rightsquigarrow X \times Y$ starting at $K := \text{Graph}(U_0)$ at time $t = 0$ for the system of differential inclusions (3) is the largest closed bilateral viability tube of the system of differential inclusion (3). The map $U_\propto(\cdot,\cdot) : \mathbf{R}_+ \times X \rightsquigarrow Y$ defined by the method of characteristics is equal to

$$U_\propto(t,x) \ := \ \{y \in Y \mid (x,y) \in R_K(t)\}$$

Then $U_\propto(0,\cdot) = U_0(\cdot)$ and $\text{Graph}(U_\propto) = \text{Graph}(R_K)$. Since $R_K(\cdot)$ is a viability tube, its graph is a viability domain of the set-valued map $\{1\} \times F(t,x,y) \times G(t,x,y)$. This amounts to saying that

$$\forall\, y \in U_\propto(t,x), \ \ (\{1\} \times F(t,x,y) \times G(t,x,y)) \cap T_{\text{Graph}(U_\propto)}(t,x,y) \neq \emptyset$$

Since $T_{\text{Graph}(U_\propto)}(t,x,y) \ = \ \text{Graph}(DU_\propto(t,x,y))$, the above relation means that

$$\forall\, y \in U(t,x), \ \ 0 \ \in \ DU_\propto(t,x,y)(1, F(t,x,y)) - G(t,x,y)$$

In the same way, to say that R_K is a backward viability tube amounts to saying that

$$\forall\, y \in U(t,x), \ \ 0 \ \in \ DU_\propto(t,x,y)(-1, -F(t,x,y)) + G(t,x,y)$$

Let us consider any closed selection V of the solution $U_\propto$ to the Cauchy problem for (6) with which we associate a closed tube Q defined by $Q(t) := \text{Graph}(V(t,\cdot))$. Then there exists a minimal bilateral viability tube containing the closed tube Q, with which we associate a minimal set-valued solution to (6) containing this selection V. □

Let us derive the corollaries in the case of hyperbolic systems of partial differential equations.

Corollary 3.3 *Assume that $f : [0,+\infty[\times X \times Y \mapsto X$, $g : [0,+\infty[\times X \times Y \mapsto Y$ are continuous maps with linear growth. Then, for any initial condition $u_0 : X \mapsto Y$, there exists a largest set-valued solution $(t,x) \rightsquigarrow U_\propto(t,x)$ to*

$$\forall\, y \in U(t,x), \ \ 0 \ \in \ DU(t,x,y)(1, f(t,x,y)) - g(t,x,y) \qquad (9)$$

satisfying the initial condition

$$\forall\, x \in X, \ \ U_\propto(0,x) \ = \ \{u_0(x)\}$$

It is the unique solution with closed graph whenever the characteristic system has the uniqueness property[6].

[6]This is the case when the functions f and g are Lipschitz with respect to (x,y), or when they enjoy monotonicity properties.

Dealing with set-valued initial conditions is justified for instance to study the case when disturbances $U_{0n}(x) = u_0(x) + \frac{1}{n}B$ of the initial condition $u_0(x)$ are involved. This approximation procedure makes sense since we obtain the following stability result with respect to the initial conditions:

Theorem 3.4 (Stability) *Assume that $F : [0,+\infty[\times X \times Y \rightsquigarrow X$, $G : [0,+\infty[\times X \times Y \rightsquigarrow Y$ are Marchaud maps. Consider a sequence of initial conditions $U_{0n} : X \rightsquigarrow Y$ and denote by $U_n(t,\cdot)$ the largest set-valued solution to (6) satisfying the initial condition*

$$\forall\, x \in X, \;\; U_n(0,x) \;=\; U_{0n}(x)$$

Let $U_0^\sharp := \mathrm{Lim}^\sharp_{n\to\infty} U_{0n}$ denote the graphical upper limit of the initial conditions. Then the graphical upper limit $U^\sharp(t,\cdot) := \mathrm{Lim}^\sharp_{n\to\infty} U_n(t,\cdot)$ of the solutions is a solution to (6) satisfying the initial condition

$$\forall\, x \in X, \;\; U^\sharp(0,x) \;=\; U_0^\sharp(x)$$

so that $U^\sharp(t,\cdot) \subset U_\propto(t,\cdot)$ for every $t \geq 0$.

If we assume furthermore that the set-valued maps $F(t,\cdot,\cdot)$ and $G(t,\cdot,\cdot)$ are λ-Lipschitz, then $U^\sharp(t,\cdot) = U_\propto(t,\cdot)$.

The proof follows from the second statement of Theorem 2.4.

We also derive the following asymptotic result:

Theorem 3.5 *Consider two time independent set-valued maps $F : X \times Y \rightsquigarrow X$ and $G : X \times Y \rightsquigarrow Y$ and a forward set-valued solution $(t,x) \rightsquigarrow U(t,x)$ to*

$$\forall\, y \in U(t,x), \;\; 0 \;\in\; DU(t,x,y)(1,F(x,y)) - G(x,y)$$

Then the graphical upper limit $U_\infty(\cdot) := \mathrm{Lim}^\sharp_{t\to\infty} U(t,\cdot)$ is a closed solution to the stationary problem

$$\forall\, y \in U_\infty(x), \;\; 0 \;\in\; DU(x,y)(F(x,y)) - G(x,y) \;\; \square$$

Proof of Theorem 3.4 — It follows from the fact that the upper limit when $t \to \infty$ of the values $P(t)$ of a viability tube is a viability domain (see Theorem 11.3.1 of VIABILITY THEORY, [2, Aubin]), because, in terms of graphs, this means that the graph of the graphical upper limit of the set-valued maps $U(t,\cdot)$ is a viability domain of the system of differential inclusions (3)

$$\begin{cases} i) & x'(t) \;\in\; F(x(t),y(t)) \\ ii) & y'(t) \;\in\; G(x(t),y(t)) \end{cases}$$

i.e., a solution $U_\infty(\cdot)$ to the stationary problem. $\square$

We also deduce the following characterization of the solution $U_\propto$:

Theorem 3.6 *Let us assume that the maps $F(t,\cdot,\cdot) : X \times Y \rightsquigarrow X$ and $G(t,\cdot,\cdot) : X \times Y \rightsquigarrow Y$ are λ-Lipschitz maps with compact values. Then, for any initial condition $U_0 : X \rightsquigarrow Y$, $(t,x) \rightsquigarrow U_\propto(t,x)$ is also the solution to*

$$\forall\, y \in U(t,x),\;\; G(t,x,y) \;\subset \bigcap_{u\in F(t,x,y)} DU(t,x,y)(1,u) \tag{10}$$

satisfying the initial condition

$$\forall\, x \in X,\;\; U_\propto(0,x) \;=\; U_0(x)$$

It is minimal in the sense that any closed set-valued map U contained in $U_\propto$, satisfying (10) and the same initial condition is equal to $U_\propto$.

Proof — We know that the reachable tube $R_{\mathrm{Graph}(U_0)}(\cdot)$ is an invariance tube thanks to Proposition 2.3, the smallest of the invariance tubes starting at $\mathrm{Graph}(U_0)$. We have defined $U_\propto$ as the set-valued map the graph of which is equal to $R_{\mathrm{Graph}(U_0)}(\cdot)$. By Invariance Theorem 11.6.2 of VIABILITY THEORY, [2, Aubin], this graph is an invariance tube. This means that

$$\forall\, y \in U_\propto(t,x),\;\; (\{1\} \times F(t,x,y) \times G(t,x,y)) \;\subset\; T_{\mathrm{Graph}(U_\propto)}(t,x,y)$$

Since $T_{\mathrm{Graph}(U_\propto)}(t,x,y) \;=\; \mathrm{Graph}(DU_\propto(t,x,y))$, this is equivalent to say that $U_\propto$ satisfies property (10). □

Remark — When the maps F and G are both Marchaud and λ-Lipschitz with respect to x, y, we deduce that $U_\propto$ satisfies

$$\forall\, y \in U(t,x),\;\; \begin{cases} G(t,x,y) \;\subset\; \bigcap_{u\in F(t,x,y)} DU(t,x,y)(1,u) \\ \\ 0 \;\in\; DU(t,x,y)(-1,-F(t,x,y)) + G(t,x,y) \end{cases} \tag{11}$$

4 Decomposable Case

We shall consider first the decomposable case for which we have explicit formulas, that we next use to solve the general problem of finding a contingent solution to the problem

$$\forall\, t, x \in X,\;\; 0 \;\in\; Du(t,x)(1, f(t,x,u(t,x))) - G(t,x,u(t,x))$$

If $u : X \mapsto Y$, we set

$$\|u\|_\infty \;:=\; \sup_{x\in X} \|u(x)\| \;\;\&\;\; \|u\|_\Lambda \;:=\; \sup_{x\neq y} \frac{\|u(x)-u(y)\|}{\|x-y\|}$$

When G is Lipschitz with nonempty closed images, we denote by $\|G\|_\Lambda$ its Lipschitz constant, the smallest of the constants l satisfying

$$\forall z_1, z_2, \;\; G(z_1) \subset G(z_2) + l\,\|z_1 - z_2\|\, B$$

where B is the unit ball. For time-dependent set-valued maps $G(t,\cdot)$ which are uniformly Lipschitz, we still set $\|G\|_\Lambda \;:=\; \sup_{t\geq 0} G(t,\cdot)$ to denote the common Lipschitz constant.

Let $\Phi : \mathbf{R}_+ \times X \rightsquigarrow X$ and $\Psi : \mathbf{R}_+ \times X \rightsquigarrow Y$ be set-valued maps.

Consider the *decomposable* system of hyperbolic partial differential inclusions

$$\forall\,(t,x,y) \in \mathrm{Graph}(U), \;\; 0 \in DU(t,x,y)(1,\Phi(t,x)) - \Psi(t,x) \qquad (12)$$

and its associated characteristic system of differential inclusions

$$\begin{cases} x'(t) \;\in\; \Phi(t,x(t)) \\ \\ y'(t) \;\in\; \Psi(t,x(t)) \end{cases} \qquad (13)$$

We denote by $\mathcal{S}^t_{-\Phi}(x)$ the set of solutions $x(\cdot)$ to the differential inclusion $x'(s) \in -\Phi(t-s,x(s))$ on $[0,t]$ starting at x.

Define the set-valued map $U_\propto : \mathbf{R}_+ \times X \rightsquigarrow Y$ by[7]

$$U_\propto(t,x) \;:=\; \left\{u_t + \int_0^t \Psi(s,x(t-s))ds\right\}_{u_t \in U_0(x(t)),\; x(\cdot)\in \mathcal{S}^t_{-\Phi}(x)} \qquad (14)$$

We set

$$e_1^\alpha(t) \;:=\; \frac{e^{\alpha t}-1}{\alpha} \;\;\&\;\; e_2^\alpha(t) \;:=\; \frac{e^{\alpha t}-\alpha t-1}{\alpha^2}$$

Theorem 4.1 *Assume that $\Phi : \mathbf{R}_+ \times X \rightsquigarrow X$ and $\Psi : \mathbf{R}_+ \times X \rightsquigarrow Y$ are Marchaud maps and that U_0 is closed with linear growth. Then the set-valued map $U_\propto : \mathbf{R}_+ \times X \rightsquigarrow Y$ defined by (14) is the solution defined by the method of characteristics, and is thus the largest solution to (12) satisfying the initial condition*

$$\forall\, x \in X, \;\; U_\propto(0,x) \;=\; U_0(x)$$

[7]By definition of the integral of a set-valued map (see Chapter 8 of SET-VALUED ANALYSIS, [5, Aubin & Frankowska] for instance), this means that for every $y \in U_\propto(t,x)$, there exist a solution $x(\cdot) \in \mathcal{S}^t_{-\Phi}(x,\cdot)$ to the differential inclusion $x'(s) \in -\Phi(t-s,x(s))$ starting at x, $u_t \in U_0(x(t))$ and $z(s) \in \Psi(s,x(t-s))$ such that

$$y \;:=\; u_t + \int_0^t z(s)ds \;\in\; U_\propto(t,x)$$

If there exist positive constants α, δ, β_0, γ_0, β, γ such that

$$\|\Phi(t,x)\| \leq \delta + \alpha\|x\|, \quad \|U_0(x)\| \leq \beta_0 + \gamma_0\|x\| \quad \& \quad \|\Psi(t,x)\| \leq \beta + \gamma\|x\|$$

then

$$\|U_\propto(t,x)\| \leq \beta_0 + \beta t + \gamma_0\|x\|e^{\alpha t} + (\gamma_0\delta + \gamma\|x\|)e_1^\alpha(t) + \gamma\delta e_2^\alpha(t) \tag{15}$$

Moreover, if U_0, Φ, Ψ are λ-Lipschitz with respect to x, then the maps $U_\propto(t,\cdot) : X \rightsquigarrow Y$ are also Lipschitz (with nonempty values):

$$U_\propto(t,x_1) \subset U_\propto(t,x_2) + \left(\|U_0\|_\Lambda e^{\|\Phi\|_\Lambda t} + \|\Psi\|_\Lambda e_1^{\|\Phi\|_\Lambda}(t)\right)\|x_1 - x_2\| B$$

We recall that $U_\propto(t,\cdot)$ being the solution defined by the method of characteristics, it is both a forward and backward solution to (12): This means that it satisfies (12) and

$$\forall\,(t,x,y) \in \mathrm{Graph}(U), \quad 0 \in DU(t,x,y)(-1,-\Phi(t,x)) + \Psi(t,x) \tag{16}$$

Formula (14) shows also under mere inspection that the graph of $U_\propto(t,\cdot)$ is convex (respectively $U_\propto(t,\cdot)$ is a closed convex process) whenever the graphs of the set-valued maps U_0, $\Phi(t,\cdot)$ and $\Psi(t,\cdot)$ are convex (respectively $\Phi(t,\cdot)$ and $\Psi(t,\cdot)$ are closed convex processes).

Proof

1. — We prove first that the map $U_\propto$ is the largest solution to inclusion (12), i.e., that the tube $\mathrm{Graph}(U_\propto(t,\cdot))$ is the reachable map $R_{\mathrm{Graph}(U_0)}(t)$.

Indeed, a pair (x,y) belongs to $R_{\mathrm{Graph}(U_0)}(t)$ if and only if there exist solutions $(z(\cdot),y(\cdot))$ to the characteristic system (13) starting from the graph of U_0 and satisfying $(z(t),y(t)) = (x,y)$. This solution can be written in the form

$$\begin{cases} z(t) = w_t + \int_0^t \Phi(s,z(s))ds \\ y(t) = u_t + \int_0^t \Psi(s,z(s))ds \end{cases}$$

where $u_t \in U_0(w_t)$. By setting $x(s) := z(t-s)$, we observe that it is a solution $x(\cdot) \in \mathcal{S}^t_{-\Phi}(x)$ to the differential inclusion $x'(s) \in -\Phi(t-s,x(s))$ starting at x and such that $w_t = x(t)$ and that

$$w_t = x(t) \quad \& \quad y = y(t) = u_t + \int_0^t \Psi(s,x(t-s)ds$$

Hence this solution $U_\propto$ coincides with the largest solution.

2. — Estimate (15) is obvious since any solution $x(\cdot) \in \mathcal{S}^t_{-\Phi}(x)$ satisfies by Gronwall's Lemma the estimate

$$\forall\, t \geq 0, \quad \|x(t)\| \leq \|x(s)\|e^{\alpha(t-s)} + \delta e_1^\alpha(t-s)$$

Therefore,

$$\|U_\alpha(t,x)\| \;\leq\; \beta_0 + \beta t + \gamma_0\|x\|e^{\alpha t} + (\gamma_0\delta + \gamma\|x\|)e_1^\alpha(t) + \gamma\delta e_2^\alpha(t)$$

3. — Assume now that $\Phi(t,\cdot)$ and $\Psi(t,\cdot)$ are Lipschitz, take any pair of elements x_1 and x_2 and choose $y_1 = u_1 + \int_0^t z_1(s)ds \in U_\alpha(t,x_1)$, where

$$x_1(\cdot) \;\in\; \mathcal{S}_{-\Phi}^t(x_1), \;\; z_1(s) \;\in\; \Psi(s, x_1(t-s)) \;\;\&\;\; u_1 \;\in\; U_0(x_1(t))$$

By the Filippov Theorem, there exists a solution $x_2(\cdot) \in \mathcal{S}_{-\Phi}^t(x_2,\cdot)$ such that

$$\forall\, s \in [0,t], \;\; \|x_1(s) - x_2(s)\| \;\leq\; e^{\|\Phi\|_\Lambda s}\|x_1 - x_2\|$$

We denote by $z_2(s)$ the projection of $z_1(s)$ onto the closed convex subset $\Psi(s, x_2(t-s))$, which is measurable thanks to Corollary 8.2.13 of SET-VALUED ANALYSIS, [5, Aubin & Frankowska] and which satisfies

$$\begin{cases} \forall\, s \in [0,t], \; \|z_1(s) - z_2(s)\| \;\leq\; \|\Psi\|_\Lambda \|x_1(t-s) - x_2(t-s)\| \\ \leq\; \|\Psi\|_\Lambda e^{\|\Phi\|_\Lambda (t-s)}\|x_1 - x_2\| \end{cases}$$

Let u_2 denote the projection of u_1 onto the closed convex set $U_0(x_2(t))$. Then $y_2 := u_2 + \int_0^t z_2(s)ds$ belongs to $U_\alpha(x_2)$ and satisfies

$$\begin{cases} \|y_1 - y_2\| \;\leq\; \|U_0\|_\Lambda \|x_1(t) - x_2(t)\| + \int_0^t \|\Psi\|_\Lambda e^{\|\Phi\|_\Lambda (t-s)}\|x_1 - x_2\|ds \\ \leq\; \left(\|U_0\|e^{\|\Phi\|_\Lambda t} + \|\Psi\|_\Lambda e_1^{\|\Phi\|_\Lambda}(t)\right)\|x_1 - x_2\| \quad \square \end{cases}$$

We prove now a comparison result between solutions to two decomposable partial differential inclusions.

When $L \subset X$ and $M \subset X$ are two closed subsets of a metric space, we denote by

$$\Delta(L,M) \;:=\; \sup_{y\in L}\inf_{z\in M} d(y,z) \;=\; \sup_{y\in L} d(y,M)$$

their *semi-Hausdorff distance*[8], and recall that $\Delta(L,M) = 0$ if and only if $L \subset M$. If Φ and Ψ are two set-valued maps, we set

$$\Delta(\Phi,\Psi)_\infty \;=\; \sup_{x\in X}\Delta(\Phi(x),\Psi(x)) \;:=\; \sup_{x\in X}\sup_{y\in\Phi(x)} d(y,\Psi(x))$$

Theorem 4.2 *Consider now two triples $(U_{0,1},\Phi_1,\Psi_1)$ and $(U_{0,2},\Phi_2,\Psi_2)$ of maps and their associated solutions*

$$U_{\alpha i}(t,x) \;:=\; \left\{u_{i,t} + \int_0^t \Psi_i(s, x_i(t-s))ds\right\}_{u_{i,t}\in U_{0,i}(x_i(t)),\; x_i(\cdot)\in\mathcal{S}_{-\Phi_i}^t(x)} \qquad (i = 1,2)$$

[8]The *Hausdorff distance* between L and M is $\max(\Delta(L,M),\Delta(M,L))$, which may be equal to ∞.

If the set-valued maps $U_{0,i}$ are closed and bounded, Φ_i and Ψ_i are Marchaud maps ($i = 1,\ 2$) and $\Phi_2(t,\cdot)$ and $\Psi_2(t,\cdot)$ are λ-Lipschitz, then

$$\left\{\begin{array}{l} \Delta(U_{\propto_1}(t,\cdot), U_{\propto_2}(t,\cdot))_\infty \\ \leq\ \Delta(U_{0,1}, U_{0,2})_\infty + \Delta(\Psi_1,\Psi_2)_\infty t + \Delta(\Phi_1,\Phi_2)_\infty \|\Psi_2\|_\Lambda e_2^{\|\Phi_2\|_\Lambda}(t) \end{array}\right.$$

Proof — Choose $y_1 = u_1 + \int_0^t z_1(s)ds \in U_{\propto 1}(t,x)$ where

$$x_1(\cdot)\ \in\ \mathcal{S}^t_{-\Phi_1}(x),\ u_1 \in U_{0,1}(x_1(t))\ \ \&\ \ z_1(s)\ \in\ \Psi_1(s, x_1(t-s))$$

In order to compare $x_1(\cdot)$ with the solution-set $\mathcal{S}^t_{-\Phi_2}(x)$ via the Filippov Theorem, we use the estimate

$$d(x_1'(s), -\Phi_2(t-s, x_1(s))) \leq \sup_{z\in\Phi_1(t-s,x_1(s))} d(z, \Phi_2(t-s, x_1(s)))) \leq \Delta(\Phi_1,\Phi_2)_\infty$$

Therefore, by Filippov's Theorem, there exists a solution $x_2(\cdot) \in \mathcal{S}^t_{-\Phi_2}(x)$ such that

$$\forall\, s \in [0,t],\ \ \|x_1(s) - x_2(s)\|\ \leq\ \Delta(\Phi_1,\Phi_2)_\infty e_1^{\|\Phi_2\|_\Lambda}(s)$$

As before, we denote by $z_2(s)$ the projection of $z_1(s)$ onto the closed convex set $\Psi_2(s, x_2(s))$, which is measurable and satisfies

$$\left\{\begin{array}{l} \forall\, s \in [0,t],\ \ \|z_1(s) - z_2(s)\|\ \leq\ \Delta(\Psi_1,\Psi_2)_\infty + \|\Psi_2\|_\Lambda \|x_1(t-s) - x_2(t-s)\| \\ \\ \leq\ \Delta(\Psi_1,\Psi_2)_\infty + \|\Psi_2\|_\Lambda \Delta(\Phi_1,\Phi_2)_\infty e_1^{\|\Phi_2\|_\Lambda}(t-s) \end{array}\right.$$

Therefore, denoting by u_2 a projection of u_1 onto the closed set $U_{0,2}(x_2(t))$, the element $y_2 = u_2 + \int_0^t z_2(s)ds$ belongs to $U_{\propto 2}(t,x)$ and satisfies

$$\left\{\begin{array}{l} \|y_1 - y_2\| \\ \\ \leq\ \Delta(U_{0,1}, U_{0,2})_\infty + \Delta(\Psi_1,\Psi_2)_\infty t + \|\Psi_2\|_\Lambda \Delta(\Phi_1,\Phi_2)_\infty e_2^{\|\Phi_2\|_\Lambda}(t)\ \ \square \end{array}\right.$$

When $U_0, \Phi,\ \Psi$ are single-valued, we obtain:

Proposition 4.3 *Assume that u_0, $\varphi(t,\cdot)$ and $\psi(t,\cdot)$ are λ-Lipschitz. Then the map $u_\propto := \Gamma(u_0, \varphi, \psi)$ defined by*

$$u_\propto(t,x)\ = u_0(S^t_{-\varphi}(x)(t)) + \int_0^t \psi\left(s, S^t_{-\varphi}(x)(t-s)\right) ds$$

is the unique (contingent) single-valued solution to

$$0\ \in\ Du(t,x)(1,\varphi(t,x)) - \psi(t,x) \tag{17}$$

and satisfies

$$\|u_\alpha(t,\cdot)\|_\infty \;\le\; \|u_0\|_\infty + \|\psi(t,\cdot)\|_\infty t \tag{18}$$

and

$$\|u_\alpha(t,\cdot)\|_\Lambda \;\le\; \|u_0\|_\Lambda e^{\|\varphi\|_\Lambda t} + \|\psi(t,\cdot)\|_\Lambda e_1^{\|\varphi\|_\Lambda}(t) \tag{19}$$

The map $(u_0,\varphi,\psi) \mapsto \Gamma(u_0,\varphi,\psi)$ *is continuous from* $\mathcal{C}(X,Y) \times \mathcal{C}([0,T] \times X, X) \times \mathcal{C}([0,T] \times X, Y)$ *to* $\mathcal{C}([0,T] \times X, Y)$*:*

$$\left\{ \begin{array}{l} \|\Gamma(u_{0,1},\varphi_1,\psi_1) - \Gamma(u_{0,2},\varphi_2,\psi_2)\|_\infty \\ \le\; \|u_{0,1} - u_{0,2}\|_\infty + \|\psi_1 - \psi_2\|_\infty t + \|\varphi_1 - \varphi_2\|_\infty \|\psi_2\|_\Lambda e_2^{\|\varphi_2\|_\Lambda}(t) \end{array} \right.$$

The result follows from Theorems 4.1 and 4.2.

5 Single-Valued Lipschitz Contingent Solutions

We shall now prove the local existence of a (contingent) single-valued solution to

$$0 \;\in\; Du(t,x)(1, f(t,x,u(t,x))) - G(t,x,u(t,x)) \tag{20}$$

on some interval $[0,T]$ satisfying the initial condition $u(0,x) = u_0(x)$.

Theorem 5.1 *Assume that the maps* $f(t,\cdot) : X \times Y \mapsto X$ *are* λ*-Lipschitz, that* $G(t,\cdot) : X \rightsquigarrow Y$ *are* λ*-Lipschitz with nonempty convex compact values and that*

$$\forall\, t,\ x,\ y,\ \ \|G(t,x,y)\| \le c(1 + \|y\|)$$

Then for any Lipschitz initial condition u_0*, there exist* $T > 0$ *and a bounded Lipschitz (contingent) solution to the partial differential inclusion (20) on the interval* $[0,T]$.

Proof — Since for uniformly Lipschitz single-valued maps $v(t,\cdot)$, the set-valued map $x \rightsquigarrow G(t,x,v(t,x))$ is Lipschitz (with constant $\|G\|_\Lambda (1 + \|v(t,\cdot)\|_\Lambda)$) and has convex compact values, Theorem 9.4.3 of SET-VALUED ANALYSIS ([5, Aubin & Frankowska]) implies that the subset G_v of Lipschitz selections ψ of the set-valued map $x \rightsquigarrow G(t,x,v(t,x))$ with Lipschitz constant less than $\nu \|G\|_\Lambda (1 + \|v(t,\cdot)\|_\Lambda)$ is not empty (where ν denotes the dimension of X.) We denote by φ_v the Lipschitz map defined by $\varphi_v(t,x) := f(t,x,v(t,x))$, with Lipschitz constant equal to $\|f\|_\Lambda(1 + \|v(t,\cdot)\|_\Lambda)$.

The fixed points to the set-valued map

$$\mathcal{R} : \mathcal{C}([0,T] \times X, Y) \rightsquigarrow \mathcal{C}([0,T] \times X, Y)$$

defined by

$$\mathcal{R}(v) \;:=\; \{\Gamma(u_0,\varphi_v,\psi)\}_{\psi \in G_v} \tag{21}$$

in Proposition 4.3 are the solutions u to inclusion (20): Indeed, if $u \in \mathcal{R}(u)$, there exists a selection $\psi \in G_u$ such that $u = \Gamma(u_0, \varphi_v, \psi)$, and thus, by Proposition 4.3, such that

$$\left\{ \begin{array}{l} 0 \ \in \ Du(t,x)(1, f(t,x,u(t,x))) - \psi(t,x) \\ \subset \ Du(t,x)(1, f(t,x,u(t,x))) - G(t,x,u(t,x)) \end{array} \right.$$

and

$$\left\{ \begin{array}{l} 0 \ \in \ Du(t,x)(-1, -f(t,x,u(t,x))) + \psi(t,x) \\ \subset \ Du(t,x)(-1, -f(t,x,u(t,x))) + G(t,x,u(t,x)) \end{array} \right.$$

Since $\|G(t,x,y)\| \ \leq \ c(1 + \|y\|)$, we deduce that any selection $\psi \in G_v$ satisfies

$$\|\psi(t,\cdot)\|_\infty \ \leq \ c(1 + \|v(t,\cdot)\|_\infty)$$

Therefore, Proposition 4.3 implies that

$$\forall \, u \in \mathcal{R}(v), \ \ \|u(t,\cdot)\|_\infty \ \leq \ \|u_0\|_\infty + c(1 + \|v(t,\cdot)\|_\infty)t$$

and

$$\left\{ \begin{array}{l} \|u(t,\cdot)\|_\Lambda \\ \leq \ (\|u_0\|_\Lambda e^{\|f\|_\Lambda(1+\|v(t,\cdot)\|_\Lambda)t} + \nu \left\|G\right\|_\Lambda (1 + \|v(t,\cdot)\|_\Lambda) e_1^{\|f\|_\Lambda(1+\|v(t,\cdot)\|_\Lambda)}(t) \end{array} \right.$$

We first observe that for any $T \leq T_1(\rho) := \frac{\rho - \|u_0\|_\Lambda}{c(1+\rho)}$

$$\left\{ \begin{array}{l} \forall \, v \in \mathcal{C}([0,T] \times X, Y) \ \ \text{such that} \ \ \sup_{t\in[0,T]} \|v(t,\cdot)\|_\infty \ \leq \ \rho, \\ \forall \, u \in \mathcal{R}(v), \ \ \sup_{t\in[0,T]} \|u(t,\cdot)\|_\infty \ \leq \rho \end{array} \right.$$

We denote by $T_2(\sigma)$ the smallest positive root of the equation

$$\|u_0\|(t,\cdot) e^{\|f\|_\Lambda(1+\sigma)t} + \nu \left\|G\right\|_\Lambda (1+\sigma) e_1^{\|f\|_\Lambda(1+\sigma)}(t) = \sigma$$

when σ is large enough for such a root to exist. Let $T := \min(T_1(\rho), T_2(\sigma))$.

We infer that

$$\left\{ \begin{array}{l} \forall \, v \in \mathcal{C}([0,T] \times X, Y) \ \ \text{such that} \ \ \sup_{t\in[0,T]} \|v(t,\cdot)\|_\Lambda \leq \sigma, \\ \forall \, u \in \mathcal{R}(v), \ \ \sup_{t\in[0,T]} \|u(t,\cdot)\|_\Lambda \ \leq \ \sigma \end{array} \right.$$

by Proposition 4.3 because u is of the form $\Gamma(u_0, \varphi_v, \psi_v)$.

Set $T := \min(T_1(\rho), T_2(\sigma))$ and let us denote by $B^1_\infty(\rho,\sigma)$ the subset defined by

$$\left\{ \begin{array}{l} B^1_\infty(\rho,\sigma) \\ := \ \left\{ v \in \mathcal{C}([0,T] \times X, Y) \mid \sup_{t\in[0,T]} \|v(t,\cdot)\|_\infty \leq \rho \ \ \& \ \ \sup_{t\in[0,T]} \|v(t,\cdot)\|_\Lambda \leq \sigma \right\} \end{array} \right.$$

which is compact (for the compact convergence topology) thanks to Ascoli's Theorem.

We have therefore proved that the set-valued map $\mathcal{R}$ sends the compact subset $B^1_\infty(\rho,\sigma)$ to itself.

It is obvious that the values of $\mathcal{R}$ are convex. Kakutani's Fixed-Point Theorem implies the existence of a fixed point $u \in \mathcal{R}(u)$ if we prove that the graph of $\mathcal{R}$ is closed.

Actually, the graph of $\mathcal{R}$ is compact. Indeed, let us consider any sequence $(v_n, u_n) \in \mathrm{Graph}(\mathcal{R})$. Since $B^1_\infty(\rho,\sigma)$ is compact, a subsequence (again denoted by) (v_n, u_n) converges to some function

$$(v,u) \in B^1_\infty(\rho,\sigma) \times B^1_\infty(\rho,\sigma)$$

But there exist bounded Lipschitz selections $\psi_n \in G_{v_n}$ with Lipschitz constant $\nu\|G\|_\Lambda(1+\sigma)$ such that

$$\forall\, n \geq 0, \quad u_n = \Gamma(u_0, \varphi_{v_n}, \psi_n)$$

Therefore a subsequence (again denoted by) ψ_n converges to some function $\psi \in G_v$. Since φ_{v_n} converges obviously to φ_v, we infer that u_n converges to $\Gamma(u_0, \varphi_v, \psi)$ where $\psi \in G_v$, i.e., that $u \in \mathcal{R}(v)$, since Γ is continuous by Proposition 4.3. $\square$

We deduce from Theorem 4.2 the following "localization property":

Theorem 5.2 *We posit the assumptions of Theorem 5.1. Let $U_0 : X \rightsquigarrow Y$, $\Phi : [0,T] \times X \rightsquigarrow X$ and $\Psi : [0,T] \times X \rightsquigarrow Y$ be Marchaud maps which are uniformly Lipschitz with respect to (x,y). We associate with them the set-valued solution $U_\propto$ to (12) defined by*

$$U_\propto(t,x) := \left\{ u_t + \int_0^t \Psi(s, x(t-s))ds \right\}_{u_t \in U_0(x(t)),\, x(\cdot) \in S^t_{-\Phi}(x)}$$

Then the bounded (contingent) single-valued solution $u(t,\cdot)$ to inclusion (20) satisfies the following estimate

$$\begin{cases} \forall\, x \in X, \ \ d(u(t,x), U_\propto(t,x)) \\ \leq \sup_{x\in X} d(u_0(x), U_0(x)) + \sup_{x \in X} \Delta(G(t,x,u(t,x)), \Psi(t,x))t \\ + \ \sup_{x\in X} d(f(t,x,u(t,x)), \Phi(t,x)) \|\Psi\|_\Lambda e_2^{\|\Phi\|_\Lambda}(t) \end{cases}$$

In particular, if we assume that

$$\forall\, (x,y) \in X \times Y, \ \ f(t,x,y) \in \Phi(t,x) \ \& \ G(t,x,y) \subset \Psi(t,x)$$

then the bounded single-valued contingent solutions $u(t,\cdot)$ to inclusion (20) is a selection of $U_\propto(t,\cdot)$.

Proof — Let u be any bounded single-valued contingent solution to inclusion (20). One can show that u can be written in the form

$$u(t,x) = u_t + \int_0^t z(s)ds \text{ where } z(s) \in G(s, x(t-s), u(x(t-s)))$$

by using the same arguments as in the first part of the proof of Theorem 4.1.

We also adapt the proof of Theorem 4.2 with $\Phi_1(t,x) := f(t,x,u(t,x))$, $z_1(s) := z(s)$, $\Phi_2 := \Phi$ and $\Psi_2 := \Psi$, to show that the estimates stated in the theorem hold true. □

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