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SETS FOR MATHEMATICS

Advanced undergraduate or beginning graduate students need a unified foundation for their study of mathematics. For the first time in a text, this book uses categorical algebra to build such a foundation, starting from intuitive descriptions of mathematically and physically common phenomena and advancing to a precise specification of the nature of categories of sets.

Set theory as the algebra of mappings is introduced and developed as a unifying basis for advanced mathematical subjects such as algebra, geometry, analysis, and combinatorics. The formal study evolves from general axioms that express universal properties of sums, products, mapping sets, and natural number recursion. The distinctive features of Cantorian abstract sets, as contrasted with the variable and cohesive sets of geometry and analysis, are made explicit and taken as special axioms. Functor categories are introduced to model the variable sets used in geometry and to illustrate the failure of the axiom of choice. An appendix provides an explicit introduction to necessary concepts from logic, and an extensive glossary provides a window to the mathematical landscape.

SETS FOR MATHEMATICS

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Portraits on the front cover are of Georg Cantor and Richard Dedekind (top) and Samuel Eilenberg and Saunders Mac Lane (bottom). The portrait of Samuel Eilenberg appears by kind permission of Columbia University.

Contents

Foreword

Why *Sets for Mathematics*?

This book is for students who are beginning the study of advanced mathematical subjects such as algebra, geometry, analysis, or combinatorics. A useful foundation for these subjects will be achieved by openly bringing out and studying what they have in common.

A significant part of what is common to all these subjects was made explicit 100 years ago by Richard Dedekind and Georg Cantor, and another significant part 50 years ago by Samuel Eilenberg and Saunders Mac Lane. The resulting idea of **categories of sets** is the main content of this book. It is worth the effort to study this idea because it provides a unified guide to approaching constructions and problems in the science of space and quantity.

More specifically, it has become standard practice to represent an object of mathematical interest (for example a surface in three-dimensional space) as a "structure." This representation is possible by means of the following two steps:

(1) First we deplete the object of nearly all content. We could think of an idealized computer memory bank that has been erased, leaving only the pure locations (that could be filled with any new data that are relevant). The bag of pure points resulting from this process was called by Cantor a *Kardinalzahl*, but we will usually refer to it as an **abstract set**.

(2) Then, just as computers can be wired up in specific ways, suitable specific **mappings** between these structureless sets will constitute a structure that reflects the complicated content of a mathematical object. For example, the midpoint operation in Euclidean geometry is represented as a mapping whose "value" at any pair of points is a special third point.

To explain the basis for these steps there is an important procedure known as the **axiomatic method**: That is, from the ongoing investigation of the ideas of sets and mappings, one can extract a few statements called *axioms*; experience has shown that these axioms are sufficient for deriving most other true statements by pure logic when that is useful. The use of this axiomatic method makes naive set theory rigorous and helps students to master the ideas without superstition. An analogous procedure was applied by Eilenberg and Steenrod to the ongoing development of algebraic topology in their 1952 book [ES52] on the foundations of that subject as well as by other practitioners of mathematics at appropriate stages in its history.

Some of the foundational questions touched on here are treated in more detail in the glossary (Appendix C) under headings such as Foundations, Set Theory, Topos, or Algebraic Topology.

Organization

In Chapters 1–5 the emphasis is on the category of abstract sets and on some very simple categorical generalities. The additional century of experience since Cantor has shown the importance of emphasizing some issues such as:

(1) Each map needs both an explicit domain and an explicit codomain (not just a domain, as in previous formulations of set theory, and not just a codomain, as in type theory).
(2) Subsets are not mere sets with a special property but are explicit inclusion maps. (This helps one to realize that many constructions involving subsets are simplified and usefully generalized when applied appropriately to maps that are not necessarily subsets.)
(3) The algebra of composition satisfies the familiar associative and identity rules; other basic concepts, such as "belonging to" (e.g., membership in, and inclusion among, subsets) and the dual "determined by" are easily expressible as "division" relative to it. It turns out that this adherence to algebra (showing that "foundation" does not need a language distinct from that of ordinary mathematics) has broad applicability; it is particularly appropriate in smoothing the transition between constant and variable sets.
(4) Because functionals play such a key role in mathematics, the algebra is explicitly strengthened to include the algebra of evaluation maps and induced maps.

All of these issues are elementary and quite relevant to the learning of basic mathematics; we hope that mathematics teachers, striving to improve mathematics education, will take them to heart and consider carefully the particular positive role that explicit formulations of mathematical concepts can play.

Beginning in Chapter 6, examples of categories of cohesive and variable sets are gradually introduced; some of these help to objectify features of the constant sets such as recursion and coequalizers.

We illustrate the use of the maximal principle of Max Zorn in Appendix B, and we include a proof of it in the form of exercises with hints. Several other results that do not hold in most categories of variable or cohesive sets, such as the Schroeder–Bernstein theorem and the total ordering of sizes, are treated in the same way. Despite our axiomatic approach, we do not use the internal language that some books on topos theory elaborate; it seemed excessively abstract and complicated for our needs here.

Appendix A presents essentially a short course in "all that a student needs to know about logic." Appendix B, as mentioned, briefly treats a few of the special topics that a more advanced course would consider in detail. Appendix C provides a glossary of definitions for reference. Some of the glossary entries go beyond bare definition, attempting to provide a window into the background.

Some exercises are an essential part of the development and are placed in the text, whereas others that are optional but recommended are for further clarification.

F. William Lawvere Robert Rosebrugh

June 2002

Contributors to Sets for Mathematics

This book began as the transcript of a 1985 course at SUNY Buffalo and still retains traces of that verbal record. Further courses in the 1990s at Buffalo and at Mount Allison followed. We are grateful to the students in all those courses for their interest, patience, and careful reading of the text, and in particular to the late Ed Barry for his detailed set of notes and his bibliographical material.

John Myhill made the original course possible and also contributed some incisive observations during the course itself. Max Zorn sent an encouraging postcard at a critical juncture. Saunders Mac Lane strongly urged that the original transcript be transformed into a book; we hope that, sixteen years later, the result of the transformation will approach fulfillment of his expectations.

Several people contributed to making this work a reality, starting with Xiao-Qing Meng, who was the original teaching assistant. By far the most creative and inspiring contributor has been Fatima Fenaroli, who worked tirelessly behind the scenes from the very beginning, and without whose constant collaboration the project would never have come so far. Indispensable have been the advice and support of Steve Schanuel; his mathematical insight, pedagogical sense, and unfailing ability to spot errors have assisted through the many stages of revision.

Ellen Wilson typed the first TEX version, which was the basis for further revisions. Some of those revisions were made in response to constructive criticism by Giuseppe Rosolini and Richard Wood, who gave courses (at Genoa and Dalhousie, respectively) using the draft manuscript, and by the anonymous referees. Federico Lastaria and Colin McLarty studied the manuscript and contributed several improvements. The drawings in xy-pic were created by Francisco Marmolejo. The transformation into the current volume was made possible by our editor Roger Astley and by Eleanor Umali, who managed the production process.

We are grateful to Rebecca Burke for her patient, understanding encouragement that was crucial to the completion of this work, and for her hospitality, which made our collaboration enjoyable.

All of these people have contributed toward the goal of concentrating the experience of the twentieth century in order to provide a foundation for twenty-first century education and research. Although our effort is only one of the first steps in that program, we sincerely hope that this work can serve as a springboard for those who carry it further.

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