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SETS OF INTEGERS CLOSED UNDER AFFINE OPERATORS-THE CLOSURE OF FINITE SETS

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We continue investigation begun in 1974 of sets of integers closed under operators of the form $(x_1, \cdots, x_r) \rightarrow m_1x_1 + \cdots + m_rx_r + c$, where $m_1, \cdots, m_r$ are integers with $gcd(m_1, \cdots, m_r) = 1$. Our main goal here is to prove the following.

THEOREM 12. **Let $r, m_1, \cdots, m_r$ be positive integers, let T be a set of integers, let c be an integer such that $(m_1 + \cdots + m_r - 1)t + c$ is positive for each $t \in T$. If $gcd(m_1, \cdots, m_r) = 1$, and if T is closed under the operator $(x_1, \cdots, x_r)$ $(x_1, \cdots, x_r)m_1x_1+\cdots+m_rx_r+c$, then the following two statements are equivalent:**

(1) T is a finite union of infinite arithmetic progressions.

(2) $T = \langle m_1x_1 + \cdots + m_rx_r + c \mid A\rangle$ for some finite set A, where $\langle m_1x_1 + \cdots + m_rx_r + c \mid A\rangle$ denotes the "smallest" set containing A, and closed under the operator $(x_1, \cdots, x_r) \rightarrow m_1x_1 + \cdots + m_rx_r + c$.

In fact, (1) and (2) are true under more general conditions; these extensions are made in [1].

NOTATION. We denote by $\boldsymbol{Z}$, $\boldsymbol{N}$, and $\boldsymbol{P}$ the set of integers, the set of nonnegative integers, and the set of positive integes, respectively. If $A, B \subseteqq \boldsymbol{Z}$, and $c \in \boldsymbol{Z}$, define $A + c = \{a + c \mid a \in A\}$, $cA = \{ca \mid a \in A\}$, and $A + B = \{a + b \mid a \in A, b \in B\}$. If $a, b \in \boldsymbol{Z}$, define $[a, b] = \{c \in Z \mid a \leqq c \leqq b\}$. If A and B are sets, we write $A \subseteqq B$ when $A \setminus B$ is finite, and $A \doteq B$ when $A \subseteqq B \subseteqq A$.

We begin by discussing sets satisfying (1).

A subset $A \subseteqq Z$ is a *periodic set* if there exists a finite set I, and for each $i \in I$, an integer a_i, and a positive integer d_i, with $A = \bigcup_{i \in I} (a_i + d_i\boldsymbol{N})$.

It is easy to see that A is periodic iff A is bounded below, and

$$(3) \qquad A + d \subseteqq A \quad \text{for some} \quad d \in \boldsymbol{P}\,.$$

For the proofs of the elementary properties of periodic sets we shall use, see [3], for though the "per-set" defined there is slightly different from the one defined here, the difference is not essential.

A $d \in \boldsymbol{P}$ satisfying (3) is called a *period* of A. However, a $d \in \boldsymbol{P}$

is an *eventual period* of a subset $A \subseteq Z$ if A is a periodic set, and $A + d \subseteq A$.

We state without proof the following elementary properties of periodic sets and their eventual periods.

LEMMA 1.
(i) *If A is a periodic set, then for some $d \in P$, dP is the set of eventual periods of A. Further, for some finite set K, both $A \cup K$ and $A \setminus K$ are periodic sets with periods dP.*

(ii) *If d_1 is an eventual period of A_1, and if d_2 is an eventual period of A_2, then $\mathrm{lcm}(d_1, d_2)$ is an eventual period of $A_1 \cup A_2$ and $A_1 \cap A_2$, $\gcd(d_1, d_2)$ is an eventual period of $A_1 + A_2$, and d_1 is an eventual period of $A_1 \setminus K$ for any finite set K.*

(iii) *(Ascending Chain Condition) Suppose for each $i \in P$, that A_i is a periodic set with an eventual period d. Suppose further that for some $b \in Z$, each A_i is bounded below by b. Then, for some $n \in P$, $\bigcup_{i \in P} A_i = \bigcup_{i=1}^{n} A_i$. In particular, $\bigcup_{i \in P} A_i$ is a periodic set with an eventual period d.*

We now consider sets defined by (2).

Let X be a set. For $r \in P$, we say f is an *r-ary operator* on X if $f: X^r \to X$. We say f is a *finitary operator* on X if f is r-ary for some $r \in P$, and we write $\rho(f) = r$. If $A \subseteq X$, and f is a finitary operator on X, let $f(A) = \{f(a) \mid a \in A^{\rho(f)}\}$. If R is a set of operators on X, let $R(A) = \bigcup_{f \in R} f(A)$. We say A is *closed under* f *(under R)* if $f(A) \subseteq A$ $(R(A) \subseteq A)$.

If $A \subseteq X$, and R is a set of finitary operators on X, let $\langle R \mid A \rangle$ be the intersection of all subsets of X containing A and closed under R. Alternatively, define a sequence $(A_n \mid n \in N)$, called the *construction sequence* of the pair (R, A), inductively as follows: let $A_n = A \cup R(A_{n-1})$ for $n \in P$. It is easy to see $\langle R \mid A \rangle = \bigcup_{n \in N} A_n$, see Theorem 2 of [3] for details, where the alternate recursion formula $A_n = A_{n-1} \cup R(A_{n-1})$ is used.

We now give two fundamental theorems. The first is a special case of Theorem 9 of [3]. For the second, we only sketch a proof, as it is essentially Theorem 4 of [3].

THEOREM 1. *Let R be a set of operators on Z of the form $(x_1, \cdots, x_r) \to m_1x_1 + \cdots + m_rx_r + c$, let $A \subseteq Z$ let $a, b \in Z$ then $a\langle R|A\rangle + b = \langle S \mid aA + b\rangle$, where $S = \{g \mid g(x) = f(x) - bf(1) + (a + b - 1)f(0) + b, f \in R\}$, and for $t \in Z$, $f(t) = f(t, t, \cdots, t)$.*

THEOREM 2. *Let $b \in Z$, let R be a set of operators on Z of the* *form* $(x_1, \cdots, x_r) \to m_1x_1 + \cdots + m_rx_r + c$, *where* $r - 1, m_1, \cdots, m_r \in \boldsymbol{P}$, $c \in \boldsymbol{Z}$, $gcd(m_1, \cdots, m_r) = 1$, *and* $(m_1 + \cdots + m_r - 1)b + c \in \boldsymbol{N}$. *Let* $A \subseteqq \boldsymbol{N} + b$, *and suppose* A *has an eventual period* $d \in \boldsymbol{P}$. *Then* $\langle R | A \rangle$ *is a periodic set with eventual period* d.

Proof. Let $(A_n | n \in \boldsymbol{N})$ be the construction sequence for (R, A). It is easy to show by induction on n, that A_n has an eventual period d, and that $A_n \subseteqq \boldsymbol{N} + b$. But $\bigcup_{n \in N} A_n = \langle R | A \rangle$, so the ascending chain condition gives the result.

Now to get down to business! Our first task, the most difficult, is to show that $\langle mx + ny | 1 \rangle$ is a periodic set whenever $m, n \in \boldsymbol{P}$, $gcd(m, n) = 1$. Curiously, we will first consider quite a different condition, namely $m = n$.

For each $l \in \boldsymbol{N}$, let $K_l = \{(c_0, \cdots, c_h) | h \in \boldsymbol{N}, c_0 \in [0, 2^l]$, and $c_i \in [0, 2c_{i-1}]$ for $i \in [1, h]\}$, and let $T_l = \{c_0 + c_1m + \cdots + c_hm^h | (c_0, \cdots, c_h) \in K_l\}$.

THEOREM 3. *Let* $m \in \boldsymbol{P}$, *let* $S = \langle mx + my + 1 | 0 \rangle$. *Then* $S = T_0$.

Proof. By the corollary to Theorem 3 of [3], we need only show that $T_0 = \{0\} \cup (mT_0 + mT_0 + 1)$. It is easy to check that $\{0\} \cup (mT_0 + mT_0 + 1) \subseteqq T_0$; for the reverse inclusion, let $t \in T_0 \backslash \{0\}$. Then $t = 1 + c_1m + \cdots + c_hm^h$, where $(1, c_1, \cdots, c_h) \in K_0$. We need only produce $(d_1, \cdots, d_h)$, $(e_1, \cdots, e_h) \in K_0$, with $d_i + e_i = c_i$ for each $i \in [1, h]$, for then $u = d_1 + d_2m + \cdots + d_hm^{h-1} \in T_0$, $v = e_1 + e_2m + \cdots + e_hm^{h-1} \in T_0$, and hence $t = mu + mv + 1 \in mT_0 + mT_0 + 1$.

We will show, by induction on s, that for all $s \in [1, h]$, there exists $(d_1, \cdots, d_s)$, $(e_1, \cdots, e_s) \in K_0$, with $d_i + e_i = c_i$ for $i \in [1, s]$. Since $c_1 \in \{0, 1, 2\}$, we can start the induction. Having found suitable $(d_1, \cdots, d_{s-1})$ and $(e_1, \cdots, e_{s-1})$ for $s \in [2, h]$, we need $d_s, e_s \in \boldsymbol{N}$ with $d_s + e_s = b_s$, $d_s \leqq 2d_{s-1}$, and $e_s \leqq 2e_{s-1}$. Since $c_s \leqq 2d_{s-1} + 2e_{s-1}$, such a selection of d_s and e_s is clearly possible, completing the induction.

THEOREM 4. *Let* $l, m \in \boldsymbol{P}$, *with* $2^{l-1} \geqq m - 1$.
Then $(2^lm^l - 1)/(2m - 1) + m^l\boldsymbol{N} \subseteqq \langle mx + my + 1 | 0 \rangle$.

Proof. If $(c_0, \cdots, c_h) \in K_l$, then $(1, 2, 4, \cdots, 2^{l-1}, c_0, \cdots, c_h) \in K_0$, thus $(2^lm^l - 1)/(2m - 1) + m^lT_l \subseteqq T_0$ for all $l \in \boldsymbol{N}$. But we claim $T_l = \boldsymbol{N}$ for $2^{l-1} \geqq m - 1$; for if not, let y be the smallest integer in $\boldsymbol{N} \backslash T_l$. By hypothesis, $[2^{l-1}, 2^l]$ contains at least m consecutive integers, thus $y = mq + r$ for some $q \in \boldsymbol{Z}$, $r \in [2^{l-1}, 2^l]$. Since $(c) \in K_l$ for $c \in [0, 2^l]$, $2^l < y$. Thus $q \in \boldsymbol{P}$. Certainly $q < y$, thus $q \in T_l$ by our choice of y. Finally, if $b \in [2^{l-1}, 2^l]$ and if $(c_0, \cdots, c_h) \in K_l$, note that $(b, c_0, \cdots, c_h) \in K_l$; thus, $mT_l + [2^{l-1}, 2^l] \subseteqq T_l$; hence, $y = mq + r \in T_l$, a contradiction. Thus, no such y exists, so $T_l = N$.

THEOREM 5. *Let $l, m, n \in P$, with $2^{l-1} \geqq mn - 1$. Then $1 + ((m + n)^2 - 1)(2mn - 1)/(2mn - 1) + ((m + n)^2 - 1)m^l n^l N \subseteqq \langle mx + ny | 1 \rangle$.*

Proof. $\langle mx + ny | 1 \rangle \supseteqq \langle m(mx_1 + ny_1) + n(mx_2 + ny_2) | 1 \rangle = \langle m^2x_1 + mny_1 + mnx_2 + n^2y_2 | 1 \rangle \supseteqq \langle mnx + mny + m^2 + n^2 | 1 \rangle = ((m + n)^2 - 1)\langle mnx + mny + 1 | 0 \rangle + 1$, by Theorem 1. The result now follows from Theorem 4.

COROLLARY 1. *Let $m, n \in P$, with $gcd(m, n) = 1$. Then, for some $a, d \in P$ with $gcd(a, d) = 1$,*

$$a + dN \subseteqq \langle mx + ny | 1 \rangle . \tag{4}$$

Proof. Let $a = 1 + ((m + n)^2 - 1)((2^l m^l n^l - 1)/(2mn - 1))$, let $d = ((m + n)^2 - 1)m^l n^l$, where $l \in P$ with $2^{l-1} \geqq mn - 1$, so that (4) holds. But $gcd(a, (m + n)^2 - 1) = 1$, and $gcd(a, mn) = gcd(1 + (m + n)^2 - 1, mn) = gcd((m + n)^2, mn) = 1$, since $gcd(m, n) = 1$.

We shall make no use of the following corollary to Theorem 5, but it is of interest in its own right. We leave the proof as an exercise.

COROLLARY 2. *Let $r \in P$, let $m_1, \cdots, m_r, c \in Z$, let $T \subseteqq Z$, with $m_1T + \cdots + m_rT + c \subseteqq T$. If at least two of the m's are nonzero and if $|T| \geqq 2$, then $a + dN \subseteqq T$ for some $a, d \in Z$, $d \neq 0$.*

THEOREM 6. *Let $m, n \in P$, with $gcd(m, n) = 1$. Then $T = \langle mx + ny | 1 \rangle$ is a periodic set.*

Proof. By Corollary 1, $a + dN \subseteqq T$ for some $a, d \in P$ with $gcd(a, d) = 1$. For each $t \in T$, let $\phi(t)$ denote the smallest element of T congruent to t modulo d. Then $k = |\phi(T)|$ is finite; and further, we may write $\phi(T) = \{a_1, \cdots, a_k\}$, where $a_1 = 1$, and for each $j \in [2, k]$, $a_j = ma_{j_1} + na_{j_2}$ for some $j_1, j_2 \in [1, j - 1]$.

We will show, by induction on j, that $aa_j + dN \subseteqq T$ for each $j \in [1, k]$. Since $a_1 = 1$, $aa_1 + dN \subseteqq T$ by hypothesis. If $j \in [2, k]$, then $aa_{j_1} + dN \subseteqq T$, and $aa_{j_2} + dN \subseteqq T$ by induction. By Lemma 5 of [3], $m(aa_{j_1} + dN) + n(aa_{j_2} + dN) \subseteqq T$; but $m(aa_{j_1} + dN) + n(aa_{j_2} + dN) = aa_j + d(mN + nN)$, completing the induction, since $mN + nN \doteq N$.

By Theorem 5 of [3], T is closed under multiplication, thus $aa_j \in T$ for each $j \in [1, k]$. Since $(a, d) = 1$, the numbers aa_j, $j \in [1, k]$, are distinct modulo d, and thus are congruent to the number a_j, $j \in [1, k]$, in some order. Hence $a_k + dN \doteq aa_k + dN \subseteqq T$ for each k, so T has an eventual period d.

COROLLARY 3. *Let* $m, n \in \boldsymbol{P}$, *with* $gcd(m, n) = 1$. *Let* $c, t \in \boldsymbol{Z}$ *with* $(m + n - 1)t + c \in \boldsymbol{P}$. *Then* $\langle mx + ny + c | t \rangle$ *is a periodic set.*

Proof. By Theorem 1,

$$\langle mx + ny + c | t \rangle = \frac{(m + n - 1)t + c}{m + n - 1} \langle mx + ny | 1 \rangle - \frac{c}{m + n - 1} .$$

With the grime still on our hands, we proceed to the next goal which is to extend Corollary 3 to operators $m_1x_1 + \cdots + m_rx_r + c$, where $gcd(m_1, \cdots, m_r) = 1$. We begin with a reduction formula.

LEMMA 2. *Let* $l, m, n \in \boldsymbol{Z}$, *with* l *odd and* $gcd(l, m, n) = 1$. *Then, for some* $\alpha \in \boldsymbol{P}$, $gcd(l, m^\alpha + n^\alpha) = 1$.

Proof. Let Q denote the finite set of primes dividing l, but not dividing mn. For each $p \in Q$, $m^{\alpha_p} \equiv n^{\beta_p} \equiv 1 \pmod p$, for some $\alpha_p, \beta_p \in \boldsymbol{P}$. Let $\alpha = lcm(\{\alpha_p | p \in Q\} \cup \{\beta_p | p \in Q\})$, thus $m^\alpha \equiv n^\alpha \equiv 1 \pmod p$ for each $p \in Q$. Now we claim $gcd(l, m^\alpha + n^\alpha) = 1$; if not, let p divide $gcd(l, m^\alpha + n^\alpha)$ for some prime p. Since $gcd(l, m, n) = 1$, $p \in Q$. But then $0 \equiv m^\alpha + n^\alpha \equiv 1 + 1 \equiv 2 \pmod p$, so $p = 2$, contradicting the assumption that l is odd.

THEOREM 7. *Let* $r \in \boldsymbol{N}+2$; *let* $m_1, \cdots, m_r \in \boldsymbol{P}$, *with* $gcd(m_1, \cdots, m_r) = 1$; *let* $c \in \boldsymbol{Z}$, *let* $T \subseteqq \boldsymbol{Z}$ *with* $m_1T + \cdots + m_rT + c \subseteqq T$. *Then, for some* $m, n \in \boldsymbol{P}$, *with* $gcd(m, n) = 1$, *and for some* $k \in \boldsymbol{Z}$, *we have* $mT + nT + k \subseteqq T$.

Proof. Let $K = \{s \in \boldsymbol{N} + 2 |$ for some $n_1, \cdots, n_s \in \boldsymbol{P}$, with $gcd(n_1, \cdots, n_s) = 1$, and for some $k \in \boldsymbol{Z}$, $n_1T + \cdots + n_sT + k \subseteqq T\}$. Thus $K \neq \emptyset$, since $r \in K$, and we must show $2 \in K$. Let $s = \min K$, and produce the appropriate $n_1, \cdots, n_s, k$. We can assume that n_1 is odd. If $s \geqq 3$, let $d = gcd(n_1, n_2, n_3)$, let $n_1 = dl$, $n_2 = dm$, and $n_3 = dn$. By Lemma 2, $gcd(l, m^\alpha + n^\alpha) = 1$ for some $\alpha \in \boldsymbol{P}$, hence $gcd(n_1, n_2^\alpha + n_3^\alpha, n_4, \cdots, n_s) = 1$.

We now prove, by induction on β, that for all $\beta \in \boldsymbol{P}$, there is a $k_\beta \in \boldsymbol{Z}$ such that $n_1T + n_2^\beta T + n_3^\beta T + n_4T + \cdots + n_sT + k_\beta \subseteqq T$. This is true for $\beta = 1$, with $k_1 = k$; suppose $n_1T + n_2^\beta T + n_3^\beta T +$

$n_4T + \cdots + n_sT + k_\beta \subseteq T$. We can assume $T \neq \emptyset$, let $t \in T$. Then $n_1T + n_2^{\beta+1}T + n_3^{\beta+1}T + n_4T + \cdots + n_sT + n_2^\beta((n_1 + n_3 + \cdots + n_s)t + k) + n_3^\beta((n_1 + n_2 + n_4 + \cdots + n_s)t + k) + k_\beta \subseteq n_1T + n_2^\beta(n_1T + \cdots + n_sT + k) + n_3^\beta(n_1T + \cdots + n_sT + k) + n_4T + \cdots + n_sT + k_\beta \subseteq n_1T + n_2^\beta T + n_3^\beta T + n_4T + \cdots + n_sT + k_\beta \subseteq T$ by induction, thus we now only take $k_{\beta+1} = n_2^\beta((n_1 + n_3 + \cdots + n_s)t + k) + n_3^\beta((n_1 + n_2 + n_4 + \cdots + n_s)t + k) + k_\beta$ to complete the induction.

In particular, $n_1T + (n_2^\alpha + n_3^\alpha)T + n_4T + \cdots + n_sT + k_\alpha \subseteq T$, and since $n_2^\alpha + n_3^\alpha \neq 0$, $s - 1 \in K$, contradicting our choice of s. Thus $s = 2$.

THEOREM 8. *Let* $r - 1$, $m_1, \cdots, m_r \in \boldsymbol{P}$, *with* $gcd(m_1, \cdots, m_r) = 1$. *Let* $c, t \in \boldsymbol{Z}$ *with* $(m_1 + \cdots + m_r - 1)t + c \in \boldsymbol{P}$. *Then* $T = \langle m_1x_1 + \cdots + m_rx_r + c \mid t\rangle$ *is a periodic set.*

Proof. It is easy to check that $\boldsymbol{N} + t$ is closed under $m_1x_1 + \cdots + m_rx_r + c$, so that $T \subseteq \boldsymbol{N} + t$ and T is bounded below. By Theorem 7, for some $m, n \in \boldsymbol{P}$, with $gcd(m, n) = 1$, and some $k \in \boldsymbol{Z}$, $mT + nT + k \subseteq T$. Since $T \subseteq \boldsymbol{N} + t$, $(m + n - 1)t + k \in \boldsymbol{N}$, but a careful examination of the proof of Theorem 7 shows in fact that m, n, and k may be chosen so that $(m + n - 1)t + k \in \boldsymbol{P}$. By Corollary 3, $S = \langle mx + ny + k \mid t\rangle$ is a periodic set; but $T = \langle m_1x_1 + \cdots + m_rx_r + c \mid S\rangle$, and so T is a periodic set by Theorem 2.

We are finally prepared to prove that statement (2) of Theorem 12 implies statement (1).

THEOREM 9. *Let* $r - 1$, $m_1, \cdots, m_r \in \boldsymbol{P}$, *with* $gcd(m_1, \cdots, m_r) = 1$. *Let* $c \in \boldsymbol{Z}$, *let* $A \subseteq \boldsymbol{Z}$, *with* A *finite, and with* $(m_1 + \cdots + m_r - 1)a + c \in \boldsymbol{P}$ *for all* $a \in A$. *Then* $T = \langle m_1x_1 + \cdots + m_rx_r + c \mid A\rangle$ *is a periodic set.*

Proof. $T = \langle m_1x_1 + \cdots + m_rx_r + c \mid A\rangle = \langle m_1x_1 + \cdots + m_rx_r + c \mid S\rangle$, where $S = \bigcup_{a \in A} \langle m_1x_1 + \cdots + m_rx_r + c \mid a\rangle$. By Theorem 8, S is a finite union of periodic sets, hence S is a periodic set. Thus T is periodic by Theorem 2.

THEOREM 10. *Let* $r - 1$, $d \in \boldsymbol{P}$, *let* $m_1, \cdots, m_r$, $c \in \boldsymbol{Z}$, *with* $gcd(d, m_1, \cdots, m_r) = 1$. *Let* $A \subseteq \boldsymbol{Z}$, *and suppose that for all* $a_1, \cdots, a_r \in A$, *there exist* $a \in A$ *with* $a \equiv m_1a_1 + \cdots + m_ra_r + c \pmod d$. *Then, for all* $a \in A$, *there exist* $a_1, \cdots, a_r \in A$ *with* $a \equiv m_1a_1 + \cdots + m_ra_r + c \pmod d$.

Proof. For each $i \in [2, r]$, choose k_i so large that $n_i = m_i +$

$k_i d \in \boldsymbol{P}$, and let $t = gcd(n_2, \cdots, n_r) \in \boldsymbol{P}$. By the Chinese remainder theorem there is a solution $k_1 \in \boldsymbol{Z}$ to all the congruences.

$$k_1 \equiv 0 \pmod{p} \text{ if } p \text{ is a prime divising } t, \text{ but not } m_1 ,$$

and

$$k_1 \equiv 1 \pmod{p} \text{ if } p \text{ is a prime dividing } gcd(t, m_1) .$$

Moreover, k_1 can be chosen so large that $n_1 = m_1 + k_1 d \in \boldsymbol{P}$. Note that $gcd(n_1, \cdots, n_r) = 1$.

Let $a \in A$, choose k so large that $(n_1 + \cdots + n_r - 1)(a + kd) + c \in \boldsymbol{P}$. Then $T = \langle n_1 x_1 + \cdots + n_r x_r + c \,|\, a + kd \rangle$ is a periodic set. Let e be a period of T. Then $a + (k + e)d \in T$, so $a + (k + e)d = n_1 t_1 + \cdots + n_r t_r + c$ for some $t_1, \cdots, t_r \in T$. But for each $i \in [1, r]$, $t_i \equiv a_i$ for some $a_i \in A$, and $a \equiv m_1 a_1 + \cdots + m_r a_r + c \pmod{d}$.

THEOREM 11. *Let $r - 1, m_1, \cdots, m_r \in \boldsymbol{P}$, with $gcd(m_1, \cdots, m_r) = 1$. Let $c \in \boldsymbol{Z}$, let $T \subseteqq \boldsymbol{Z}$, with $(m_1 + \cdots + m_r - 1)t + c \in \boldsymbol{P}$ for each $t \in T$, and assume $m_1 T + \cdots + m_r T + c \subseteqq T$. Then $T = \langle m_1 x_1 + \cdots + m_r x_r + c \,|\, A \rangle$ for some finite set $A \subseteqq \boldsymbol{Z}$.*

Proof. Let $A = T \backslash (m_1 T + \cdots + m_r T + c)$; by the corollary to Theorem 3 of [3], $T = \langle m_1 x_1 + \cdots + m_r x_r + c \,|\, A \rangle$, so we need only show A is finite.

Let d be an eventual period of T, then d is also an eventual period of $m_1 T + \cdots + m_r T + c$. Moreover, by Theorem 10, the residue classes modulo d containing elements of T are precisely the residue classes containing elements of $m_1 T + \cdots + m_r T + c$, thus $T \doteq m_1 T + \cdots + m_r T + c$. In particular, A is finite.

Theorems 9 and 11 together prove Theorem 12, our goal. We continue these investigations in [1], where we prove the following theorem.

THEOREM. *Let $r - 1 \in \boldsymbol{P}$, let $m_1, \cdots, m_r \in \boldsymbol{Z} \backslash \{0\}$, with $gcd(m_1, \cdots, m_r) = 1$, let $c \in \boldsymbol{Z}$, and let $T \subseteqq \boldsymbol{Z}$ with $m_1 T + \cdots + m_r T + c \subseteqq T$. Then $T = \langle m_1 x_1 + \cdots + m_r x_r + c \,|\, A \rangle$ for some finite set A. Further, if $|T| \geqq 2$, either T is a periodic set, or $-T$ is a periodic set, or T is a finite union of residue classes modulo some $d \in \boldsymbol{P}$. Finally, T is an affine transformation of a set $S \subseteqq \boldsymbol{Z}$, with $S + \theta \subseteqq S$, where $\theta = gcd\{m_i m_j \,|\, i, j \in [1, r], i \neq j\}$.*

An earlier version [4] of this paper was submitted years ago to Pacific Journal and was accepted subject to minor revision. Revision and strengthening of the results subsequently was carried out in connection with dissertation work [2].

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