

SETS OF NATURAL NUMBERS WITH NO MINIMAL ASYMPTOTIC BASES

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ABSTRACT. The set A of natural numbers is an asymptotic basis for S if the sets S and $2A$ eventually coincide. An asymptotic basis A for S is minimal if no proper subset of A is a basis for S . Sets S are constructed which possess infinitely many asymptotic bases, but no minimal asymptotic basis.

Let A and S be subsets of the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, and let $2A$ denote the sum set $\{a + a' | a, a' \in A\}$. Then A is an asymptotic basis (of order 2) for S if the sets $2A$ and S eventually coincide, that is, if the symmetric difference $(S \setminus 2A) \cup (2A \setminus S)$ is finite. The asymptotic basis A for S is *minimal* if no proper subset of A is an asymptotic basis for S . Minimal asymptotic bases for the natural numbers have been investigated by Erdős, Härtter, Nathanson, and Stöhr [1]–[6], [8], [9], [14]. The simplest example of a minimal asymptotic basis for \mathbb{N} is the set consisting of all finite sums of distinct powers of 3 (Nathanson [8]). Of course, a set, such as $\{2^n\}_{n=0}^\infty$, which has no asymptotic basis certainly has no minimal asymptotic basis. Let us call two sets asymptotically equivalent if their symmetric difference is finite, and asymptotically inequivalent if their symmetric difference is infinite. The object of this note is to construct a set S which has infinitely many asymptotically inequivalent asymptotic bases, but no minimal asymptotic basis.

Notation. Lower case letters denote natural numbers, and upper case letters denote sets of natural numbers. Let $A + B = \{a + b | a \in A, b \in B\}$. Let $[a, b]$ denote the interval of integers n such that $a \leq n \leq b$.

LEMMA. Let $p_1 = 2 < p_2 < p_3 < \dots$ be the sequence of primes in their natural order. Let $m_2 < m_3 < \dots$ be an increasing sequence of positive integers such that $p_k p_{k-1} | m_k$ for $k \geq 2$ and $m_{k+1} > 2m_k + p_k p_{k+3}$. Let $C_k = \{c \in [m_k, m_{k+1}] | c \equiv 0 \pmod{p_k}\}$, and let $C = \bigcup_{k=2}^\infty C_k = \{c_i\}_{i=0}^\infty$, where $c_0 < c_1 < c_2 < \dots$. Then C is an asymptotic basis for \mathbb{N} such that

- (I) $\limsup_{i \rightarrow \infty} (c_{i+1} - c_i) = \infty$, and
- (II) $C \setminus F$ is an asymptotic basis for \mathbb{N} for every finite set F .

PROOF. Clearly, $c_{i+1} - c_i \rightarrow \infty$ as $i \rightarrow \infty$.

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Let $k > 5$. Let $n \in [m_k + m_{k-2}, m_{k+1}]$. Choose $r \in [1, p_k]$ so that

$$c = m_{k-3} + rp_{k-3} \equiv n \pmod{p_k}.$$

Then $c \in C_{k-3}$, since $c \equiv 0 \pmod{p_{k-3}}$ and $m_{k-3} < c \leq m_{k-3} + p_k p_{k-3} < m_{k-2}$. Moreover, $n - c \equiv 0 \pmod{p_k}$ and $m_k < (m_k + m_{k-2}) - (m_{k-3} + p_k p_{k-3}) \leq n - c < m_{k+1}$, hence $n - c \in C_k$. Therefore, $n = (n - c) + c \in C_k + C_{k-3}$.

Let $n \in [m_k, m_k + m_{k-2}]$. Choose $r \in [1, p_{k-1}]$ so that

$$c = m_{k-2} + rp_{k-2} \equiv n \pmod{p_{k-1}}.$$

Then $c \in C_{k-2}$ and $n - c \in C_{k-1}$, hence $n = (n - c) + c \in C_{k-1} + C_{k-2}$. Therefore, for every $k \geq 5$ we have

$$[m_k, m_{k+1}] \subset 2(C_k \cup C_{k-1} \cup C_{k-2} \cup C_{k-3}).$$

This implies that $C \setminus F$ is an asymptotic basis for \mathbb{N} for every finite set F . This proves the Lemma.

THEOREM. *There exist sets S with infinitely many asymptotically inequivalent asymptotic bases, but no minimal asymptotic basis.*

PROOF. Let $C = \{c_i\}_{i=0}^\infty$ be an asymptotic basis for \mathbb{N} satisfying conditions (I) and (II) of the Lemma. It follows from (II) that for every q there is an integer $\phi(q)$ such that, if $n \geq \phi(q)$, then $n = c_i + c_j$ with $i, j \geq q$. Let

$$S = \{2c_i + 1\}_{i=0}^\infty \cup \{2n\}_{n=0}^\infty.$$

We shall construct infinitely many asymptotically inequivalent bases for S . For $t \in \mathbb{N}$, let

$$A_q(t) = \{t\} \cup \{2c_i + 1 - t\}_{i=q}^\infty$$

where q is chosen large enough that $2c_q + 1 \geq t$. Clearly,

$$2c_i + 1 = t + (2c_i + 1 - t) \in 2A_q(t)$$

for all $i \geq q$. If $n > \phi(q)$, then $n - 1 + t = c_i + c_j$ with $i, j \geq q$, hence

$$2n = (2c_i + 1 - t) + (2c_j + 1 - t) \in 2A_q(t).$$

It follows that the set $A_q(t)$ is an asymptotic basis for S . Moreover, for $s \neq t$ the bases $A_q(s)$ and $A_q(t)$ are asymptotically inequivalent, since

$$\begin{aligned} [2c_{i-1} + 2, 2c_i + 1] \cap A_q(s) &= 2c_i + 1 - s \\ &\neq 2c_i + 1 - t = [2c_{i-1} + 2, 2c_i + 1] \cap A_q(t) \end{aligned}$$

for all sufficiently large i . But $A_q(t) \not\supseteq A_{q+1}(t)$, so none of the sets $A_q(t)$ is a minimal basis for S .

Now let A be any asymptotic basis for S . Since S contains infinitely many odd numbers, then A must contain both even and odd numbers. But the difference between consecutive odd elements of S is $(2c_{i+1} + 1) - (2c_i + 1) = 2(c_{i+1} - c_i)$, which tends to infinity by condition (I) of the Lemma. This implies that A contains either exactly one odd element or exactly one even element. Let t be the unique odd (resp. even) element of A . Since $2A$ contains

all but finitely many of the odd integers $2c_i + 1$, it follows that A contains all but finitely many of the even (resp. odd) integers $2c_i + 1 - t$, and at most a finite number of other even (resp. odd) integers. Thus, A must be of the form

$$A = \{t\} \cup \{2c_i + 1 - t\}_{i=q}^{\infty} \cup F = A_q(t) \cup F,$$

where F is a finite set. Thus, every asymptotic basis for S contains a basis of the form $A_q(t)$, no subset of which is a minimal asymptotic basis for S . Therefore, there does not exist a minimal asymptotic basis for S . This proves the Theorem.

REMARK. It is not known for what infinite sets there exist minimal asymptotic bases, nor even asymptotic bases. Indeed, very little is understood about the general problem of the decomposition of sets S in the form $S = A + B$, where $|A| \geq 2$ and $|B| \geq 2$. This problem has been studied by Mann, Ostmann, Sárközy and Szemerédi [7], [10]–[13].

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