Sets of synthesis and sets of interpolation for weighted Fourier algebras

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§ 1. Introduction. Let $A_0(\mathbf{T})$ denote the Banach algebra of continuous functions with absolutely convergent Fourier series. We define

$$A_\alpha(\mathbf{T}) = \{f \in C(\mathbf{T}) : \sum_n |\hat{f}(n)| (1 + |n|)^\alpha < \infty\}, \quad \alpha > 0 .$$

We shall also be concerned with the Banach algebra of Lipschitz functions $\Lambda_\alpha(\mathbf{T}), \lambda_\alpha(\mathbf{T})$ and $(\lambda_\alpha \cap A)(\mathbf{T})$. We let $\lambda_0(\mathbf{T}) = C(\mathbf{T})$ and $\lambda_1(\mathbf{T}) = C^1(\mathbf{T})$, [6; pp. 42—43].

Let $R \subset C(\mathbf{T})$ be a regular Banach algebra such that the maximal ideal space of R is $\mathbf{T}$. For a closed subset E of $\mathbf{T}$, we define

$$I^R(E) = \{f \in R : f = 0 \text{ on } E\} ,$$

$$R(E) = R/I^R(E) \text{ is the restriction algebra of } R \text{ to } E .$$

$$\tilde{R}(E) = \left\{f \in C(E) : \sup_{\substack{\mu \in M(E) \\ \mu \neq 0}} \frac{|\int f \, d\mu|}{\|\mu\|_{R'}} < \infty\right\}$$

where R' is the dual of R. $\tilde{R}(E)$ is called the tilda algebra of $R(E)$. For $f \in \tilde{R}(E)$, $\|f\|_{\tilde{R}}$ is defined by

$$\|f\|_{\tilde{R}} = \sup_{\substack{\mu \in M(E) \\ \mu \neq 0}} \frac{|\int f \, d\mu|}{\|\mu\|_{R'}} .$$

Let I be a closed ideal in R, then hull I is defined to be the set of common zeros of all functions in I. We say that a closed subset E of $\mathbf{T}$ is of *synthesis* in R if $I^R(E)$ is the only closed ideal in R whose hull is E and that *ideal theorem holds for* E in R if every closed ideal I in R whose hull is E is the intersection of all closed primary ideals containing I. We let

$$PM_\alpha(\mathbf{T}) = (A_\alpha(\mathbf{T}))'$$

and

$$M_\alpha(\mathbf{T}) = (\lambda_\alpha(\mathbf{T}))'$$

Let us remind ourselves that when we talk of A_α, the index $\alpha \in [0, \infty[$ while in case of λ_α and $\lambda_\alpha \cap A, \alpha \in [0, 1]$.

An element of $PM_\alpha(T)$ will be called an *α-pseudomeasure* and an element of $M_\alpha(\mathbf{T})$ will be called an *α-measure*. By $PM_\alpha(E)$ $[M_\alpha(E)]$ we shall denote the set of α-pseudomeasures [α-measures] carried by E. E is called a set *without true α-pseudomeasures* if $PM_\alpha(E) = M_\alpha(E)$ and E is called an *H^α-set* if $A_\alpha(E) = \lambda_\alpha(E)$.

In Section 2, we shall discuss the Ditkin property for the algebras A_α and $\lambda_\alpha \cap A$ and prove the following:

THEOREM 1. *Let E be a closed subset of $\mathbf{T}$ such that for an infinite sequence $\{N_j\}_{j=1}^\infty$ of integers, the points $2\pi j/N$ $(N \in \{N_j\})$ either belong to E or are at least at a distance $2\pi/N$ from E. Then E is a set of synthesis for $\lambda_\alpha \cap A$, $0 < \alpha < 1$.*

In Section 3, we describe a totally disconnected perfect set of synthesis for $A_\alpha(\mathbf{T})$, $0 \leq \alpha < 1/2$.

Finally in Section 4, we discuss H^α-sets and sets without true α-pseudomeasures. We prove that only finite sets are H^α-sets. In this process we shall also prove that every function $f \in A_\alpha(E)$ can be extended to $A_\alpha(\mathbf{T})$ without increasing its norm and that the tilda algebra of $A_\alpha(E)$ is $A_\alpha(E)$.

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§ 2. Carl Herz [1] has proved that if $f \in A_\alpha$ is such that $f(t_0) = f'(t_0) = \ldots = f^{[\alpha]}(t_0) = 0$ then there exists a sequence of functions $f_n \in A_\alpha$ such that $f_n = 0$ in a neighbourhood of t_0 and

$$\|f - ff_n\|_{A_\alpha} \to 0 \quad \text{as} \quad n \to \infty .$$

We state the following consequence of his result.

THEOREM. *Let E and F be closed subsets of $\mathbf{T}$ such that E is of synthesis in A_α and F has countable boundary. Then ideal theorem holds for $E \cup F$ in A_α. In particular, ideal theorem holds for a closed subset with countable boundary for the algebra A_α.*

If $0 < \alpha < 1$, we observe that usual Ditkin property holds for the algebra $\lambda_\alpha \cap A$. We shall now give the proof of Theorem 1.

For a positive integer n and $f \in \lambda_\alpha \cap A$, let us define $f_n \in \lambda_\alpha \cap A$ as follows: $f_n(2\pi j/n) = f(2\pi j/n)$, $0 \leq j \leq n$ and linear in each interval $[2\pi j/n, 2\pi(j+1)/n]$.

Let $T_n : \lambda_\alpha \cap A \to \lambda_\alpha \cap A$ be the linear operator defined by $T_n(f) = f_n$. The routine computation shows that $\|T_n\| \leq 3$ for every n. Since piecewise linear functions are dense in $\lambda_\alpha \cap A$ it follows that if $N \in$ an infinite sequence $\{N_j\}$ of integers, $T_N f \to f$ in $\lambda_\alpha \cap A$ for all $f \in \lambda_\alpha \cap A$ as $N \to \infty$.

Let E and $\{N_j\}$ be as in the statement of Theorem 1. The condition on E implies that if $N \in \{N_j\}$ then $T_N f = 0$ on E for all $f \in I^{\lambda_\alpha \cap A}(E)$. Moreover, $T_N f$ vanishes on an open set which contains all but finitely many points of E.

Given $\varepsilon > 0$, choose $N \in \{N_j\}$ so large that

$$\|f - T_N f\|_{\lambda_\alpha \cap A} < \varepsilon/2 .$$

Now the Ditkin property for $\lambda_\alpha \cap A$ implies the existence of a $g \in \lambda_\alpha \cap A$ such that $g \cdot T_N f$ vanishes in a neighbourhood of E and

$$\|T_N f - g \cdot T_N f\|_{\lambda_\alpha \cap A} < \varepsilon/2 .$$

Therefore $\|f - g \cdot T_N f\| < \varepsilon$ where $gT_N f = 0$ in a neighbourhood of E. This proves that E is of synthesis for the algebra $\lambda_\alpha \cap A$.

§ 3. We shall follow McGehee [4] in constructing a perfect totally disconnected set of synthesis for the algebra A_α, $0 \leq \alpha < 1/2$.

We recall the following lemma due to McGehee [4] about finitely supported measures.

3.1. LEMMA. *Let $F = \{x_j : 1 \leq j \leq k\}$ be a finite set of distinct points of* $\mathbf{T}$. *Then given $\varepsilon > 0$ there exists a number $N = N(x_1, \ldots, x_k; \varepsilon)$ such that for every $\mu \in M(F)$*

$$\max_{|n-m| \leq N} |\hat{\mu}(n)| \geq (1 - \varepsilon) \|\mu\|_{PM} \quad \text{for each } m;$$

where

$$\|\mu\|_{PM} = \sup_n |\hat{\mu}(n)|$$

We construct E as the intersection of a decreasing sequence of closed sets E^k, each E^k being the union of s_k disjoint closed intervals each of length l_k. Let $E_k = \{x_k^{(1)}, \ldots, x_k^{(s_k)}\}$ be the set of the left end points of the intervals constituting E^k. Given E_k, choose N_k (by Lemma 3.1) such that for every measure $\mu \in M(E_k)$,

$$\sup_{|n| \leq N_k} |\hat{\mu}(n)| \geq \tfrac{1}{2}\|\mu\|_{PM}$$

Now choose the length l_k of the intervals in E^k such that the points of E_k are at least $2l_k$ apart and such that

$$N_k(s_k l_k)^{1/2} = o(1) \tag{1}$$

We shall now show that E is a set of synthesis for the algebra A_α, $0 \leq \alpha < 1/2$.

3.2. LEMMA. *To each* $S \in PM_\alpha(E)$ *we can associate a sequence of measures* $\mu_k \in M(E_k)$ *such that*

$$|\hat{S}(n) - \hat{\mu}_k(n)| \leq C(\alpha)|n|(s_k l_k)^{1/2}\|S\|_{PM_\alpha} \text{ for all } k \text{ and } n \tag{2}$$

where $C(\alpha)$ *is a constant depending only on* α.

In particular,

$$\lim_{k\to\infty} \hat{\mu}_k(n) = \hat{S}(n) \text{ for all } n . \tag{3}$$

Proof. We observe that the formal integral of S is the L^2-function $\sigma(x) \sim \sum_{n\neq 0} \frac{\hat{S}(n)}{in} e^{inx}$ (where we have assumed that $\hat{S}(0) = 0$) with the norm $\|\sigma\|_2 \leq C(\alpha)\,\|S\|_{PM_\alpha}$ where $C(\alpha)$ is a positive constant depending only on α. The proof of the lemma is now exactly the same as that of the lemma 4, p. 141—143 in [4].

3.3. THEOREM. E *is a set of synthesis for the algebra* A_α, $0 \leq \alpha < 1/2$.

Proof. Let $S \in PM_\alpha(E)$. By (2) in Lemma 3.2, we have

$$|\hat{S}(n) - \hat{\mu}_k(n)| \leq C(\alpha)|n|(s_k l_k)^{1/2}\|S\|_{PM_\alpha}$$

for all n and k. Now by our choice

$$\sup_{|n|\leq N_k} |\hat{\mu}_k(n)| \geq \tfrac{1}{2}\,\|\mu_k\|_{PM}$$

Therefore

$$\|\mu_k\|_{PM_\alpha} \leq 2 \sup_{|n|\leq N_k} \frac{|\hat{\mu}_k(n)|}{(1+|n|)^\alpha} \leq 2[1 + C(\alpha)N_k(s_k l_k)^{1/2}]\,\|S\|_{PM_\alpha}$$

Now since $N_k(s_k l_k)^{1/2} = o(1)$, it follows that $\sup_k \|\mu_k\|_{PM_\alpha} < \infty$. This together with (3) of Lemma 3.2 implies that μ_k converges to S in the weak*-topology of PM_α. This proves that E is of synthesis for the algebra A_α.

Remark. By changing the thinness condition (1), we can construct sets E which are of synthesis for the algebras A_α, where $\alpha \in [n, n + 1/2[$, n is a positive integer.

§ 4. We start by stating the following proposition which relates the sets without true α-pseudomeasures and the H^α-sets.

4.1. PROPOSITION. *Let* E *be a closed subset of* $\mathbf{T}$ *without true* α*-pseudomeasures. Then* E *is an* H^α*-set and the ideal theorem holds for* E *in* A_α.

We now proceed to determine the H^{α}-sets. We need the following easy lemma.

4.2. LEMMA. *Let* $\alpha \geq 0$ *and*

$$l_{\alpha}^{\infty} = \left\{ \textit{sequences} \quad c = \{c_n\} : \sup_n \frac{|c_n|}{(1+|n|)^{\alpha}} < \infty \right\},$$

$$l_{\alpha}^{1} = \{\textit{sequences} \quad a = \{a_n\} : \sum_n |a_n|(1+|n|)^{\alpha} < \infty\},$$

$$c_{0,\alpha} = \{c \in l_{\alpha}^{\infty} : c_n = o(|n|^{\alpha})\}.$$

Then $(c_{0,\alpha})' = l_{\alpha}^{1}$.

The following proposition is of independent interest. However, we shall use the method of its proof.

4.3. PROPOSITION. *Let* E *be a closed subset of* $\mathbf{T}$. *Then for each* $f \in A_{\alpha}(E)$ *there exists* $F \in A_{\alpha}(\mathbf{T})$ *such that* $F|_E = f$ *and* $\|F\|_{A_{\alpha}(\mathbf{T})} = \|f\|_{A_{\alpha}(E)}$.

Proof. Let $N_{\alpha}(E)$ be the annihilator of $I^{A_{\alpha}(T)}(E)$ in $PM_{\alpha}(\mathbf{T})$ and $L(E) = c_{0,\alpha} \cap N_{\alpha}(E)$. Then f induces a bounded linear functional f' on $L(E)$ with the norm $\leq \|f\|_{A_{\alpha}(E)}$. By the Hahn-Banach theorem there is an extension F' of f' to $c_{0,\alpha}$ such that $\|F'\| \leq \|f\|_{A_{\alpha}(E)}$. By Lemma 4.2 this corresponds to a function $F \in A_{\alpha}(\mathbf{T})$ such that $\|F\|_{A_{\alpha}(\mathbf{T})} \leq \|f\|_{A_{\alpha}(E)}$. We shall now show that $F|_E = f$. Let t_0 be an arbitrary point of E. Then $\delta_{t_0} \in L(E)$. Therefore

$$f(t_0) = f'(\delta_{t_0}) = F'(\delta_{t_0}) = F(t_0).$$

This completes the proof of the proposition.

Remark. The idea of the above proof goes back to Y. Katznelson and K. DeLeeuw [3].

We now prove the main proposition needed for our study of H^{α}-sets. The proposition is interesting for its own sake.

4.4. PROPOSITION. *Let* E *be a closed subset of* $\mathbf{T}$. *Then* $\widetilde{A_{\alpha}(E)} = A_{\alpha}(E)$.

Proof. Let $f \in \widetilde{A_{\alpha}(E)}$. Then f induces a bounded linear functional f' on $M(E)$ as a subspace of $N_{\alpha}(E)$. As in Proposition 4.3 we can extend f' to $c_{0,\alpha}$ and thus get $F \in A_{\alpha}(\mathbf{T})$ such that $F|_E = f$, i.e. $f \in A_{\alpha}(E)$. This completes the proof.

We need the following easy facts: (i) Let E be a closed subset of $\mathbf{T}$. Then for $0 < \alpha \leq 1$, $\widetilde{\lambda_{\alpha}(E)} = \Lambda_{\alpha}(E)$.

(ii) If E is an infinite compact subset of $\mathbf{T}$, then

$$\lambda_{\alpha}(E) \neq \Lambda_{\alpha}(E) \quad \text{for} \quad 0 < \alpha \leq 1.$$

The above facts together with Proposition 4.4 imply

4.5. PROPOSITION. *If* E *is an infinite compact subset of* $\mathbf{T}$ *then* $A_\alpha(E) \neq \lambda_\alpha(E)$.

Our findings about H^α-sets can be summed up in the following:

4.6. THEOREM. *Let* E *be a compact subset of* $\mathbf{T}$ *and* $0 < \alpha \leq 1$. *Then the following are equivalent:*

(i) E *is a set without true* α*-pseudomeasures.*
(ii) E *is an* H^α*-set.*
(iii) E *is finite.*

References

1. HERZ, C. S., *Two problems in the spectral synthesis of unbounded functions.* Mimeographed notes (1964).
2. KAHANE, J. P., & SALEM, R., *Ensembles parfaits et séries trigonometriques,* Hermann, Paris, 1963.
3. DELEEUW, K., & KATZNELSON, Y., On certain homomorphisms of quotients of group algebras. *Israel J. Math.* 2 (1964), 120–126.
4. MCGEHEE, O. C., Certain isomorphisms between quotients of a group algebra. *Pacific J. Math.* 21 (1967), 133–151.
5. VAROPOULOS, N. TH., On a problem of A. Beurling. *J. Functional Analysis* 2 (1968), 24–30.
6. ZYGMUND, A., *Trigonometric series,* 2nd ed., Cambridge University Press, 1959.

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