SETTING OF TOLERANCE LIMITS WHEN THE SAMPLE IS LARGE

By Abraham Wald

Columbia University

1. Introduction. Let $f(x_1, \cdots, x_p, \theta_1, \cdots, \theta_k)$ be the joint probability density function of the variates $x_1, \cdots, x_p$ involving k unknown parameters $\theta_1, \cdots, \theta_k$. A sample of size n is drawn from this population. Denote by $x_{i\alpha}(i = 1, \cdots, p; \alpha = 1, \cdots, n)$ the α-th observation on x_i. We will deal here with the following two problems of setting tolerance limits, which are of importance in the mass production of a product:

Problem 1. *For any two positive numbers $\beta < 1$ and $\gamma < 1$ we have to construct p pairs of functions of the observations $L_i(x_{11}, \cdots, x_{pn})$ and $U_i(x_{11}, \cdots, x_{pn})$ $(i = 1, \cdots, p)$ such that*

$$(1) \quad P\left\{\int_{L_p}^{U_p} \cdots \int_{L_1}^{U_1} f(x_1, \cdots, x_p, \theta_1, \cdots, \theta_k)\, dx_1 \cdots dx_p \geq \gamma \mid \theta_1, \cdots, \theta_k\right\} = \beta,$$

where for any relation R, $P(R \mid \theta_1, \cdots, \theta_k)$ denotes the probability that R holds, calculated under the assumption that $\theta_1, \cdots, \theta_k$ are the true values of the parameters.

Problem 2. *For any positive numbers $\beta < 1, \lambda < 1$ and for any positive integer N we have to construct p pairs of functions of the observations $L_i(x_{11}, \cdots, x_{pn})$ and $U_i(x_{11}, \cdots, x_{pn})$ with the following property: Let $y_{i\alpha}(i = 1, \cdots, p; \alpha = 1, \cdots, N)$ be the α-th observation on the variate x_i in a second sample of size N drawn from the same population as the first sample has been drawn. Denote by M the number of different values of α for which the p inequalities*

$$L_i(x_{11}, \cdots, x_{pn}) \leq y_{i\alpha} \leq U_i(x_{11}, \cdots, x_{pn}) \quad (i = 1, \cdots, p),$$

are fulfilled. Then

$$(2) \qquad P(M \geq \lambda N \mid \theta_1, \cdots, \theta_k) = \beta,$$

where $\theta_1, \cdots, \theta_k$ denote the unknown parameter values of the population from which the observations $x_{i\alpha}$ and $y_{i\alpha}$ have been drawn.

The functions L_i and U_i are called the tolerance limits for the variate x_i. We will say that L_i is the lower, and U_i the upper tolerance limit of x_i. In general, there exist infinitely many tolerance limits L_i and U_i which are solutions of Problem 1 or Problem 2. It is clear that the tolerance limits L_i and U_i are the more favorable the smaller the difference $U_i - L_i$. Hence if there exist several solutions for the tolerance limits L_i and U_i we should select that one for which the difference $U_i - L_i$ becomes a minimum in some sense.

S. S. Wilks[1] gave a solution of Problems 1 and 2 in the univariate case, i.e.

[1] S. S. Wilks, "Determination of sample sizes for setting tolerance limits," *Annals of Math. Stat.*, Vol. 12 (1941). See also his paper on the same subject presented at the meeting of the Institute of Mathematical Statistics in Poughkeepsie, September, 1942.

if $p = 1$. It seems that Wilks' solution is the best possible one if nothing is known about the probability density function except that it is continuous. However, if it is known a priori that the unknown density function is an element of a k-parameter family of functions, it will in general be possible to derive tolerance limits which are considerably better than those proposed by Wilks.

Wilks' results can easily be extended to the multivariate case provided the variates $x_1, \cdots, x_p$ are known to be independently distributed.[2] This is a serious restriction, since in many practical cases the independence of the variates $x_1, \cdots, x_p$ cannot be assumed. The case of dependent variates has not been treated by Wilks.

In this paper we give a solution of problems 1 and 2 when the size n of the sample is large. In the next section a lemma is proved which will be used in the derivation of tolerance limits. In section 3 the univariate case is treated and in section 4 the results are extended to the multivariate case.

2. A lemma. We will prove the following

LEMMA: *Let* $\{x_{1n}\}, \cdots, \{x_{rn}\}$ $(n = 1, 2, \cdots, \text{ad inf.})$ *be r sequences of random variables and let* $a_1, \cdots, a_r$ *be r constants such that the joint distribution of* $\sqrt{n}(x_{1n} - a_1), \cdots, \sqrt{n}(x_{rn} - a_r)$ *converges with* $n \to \infty$ *towards the r-variate normal distribution with zero means and finite non-singular covariance matrix* $\| \sigma_{ij} \|$ $(i, j = 1, \cdots, r)$. *Furthermore, let* $g(u_1, \cdots, u_r)$ *be a function of r variables* $u_1, \cdots, u_r$ *which admits continuous first derivatives in the neighborhood of the point* $u_1 = a_1, \cdots, u_r = a_r$. *Assume that at least one of the first partial derivatives of* $g(u_1, \cdots, u_r)$ *is not zero at the point* $u_1 = a_1, \cdots, u_r = a_r$. *Then the distribution of* $\sqrt{n}[g(x_{1n}, \cdots, x_{rn}) - g(a_1, \cdots, a_r)]$ *converges with* $n \to \infty$ *towards the normal distribution with zero mean and variance* $\sigma_g^2 = \sum_j \sum_i \sigma_{ij} g_i g_j$ *where* g_i *denotes the partial derivative of* $g(u_1, \cdots, u_r)$ *with respect to* u_i *taken at* $u_1 = a_1, \cdots, u_r = a_r$.

Proof: Since the joint distribution of $\sqrt{n}(x_{1n} - a_1), \cdots, \sqrt{n}(x_{rn} - a_r)$ approaches an r-variate normal distribution with zero means and finite non-singular covariance matrix, the probability that

$$a_i - \frac{1}{\sqrt[3]{n}} \leq x_{in} \leq a_i + \frac{1}{\sqrt[3]{n}} \qquad (i = 1, \cdots, r) \tag{3}$$

holds, converges to 1 with $n \to \infty$. From (3) and the continuity of the first derivatives of $g(u_1, \cdots, u_r)$ it follows easily that for any positive ϵ the probability that

$$\begin{aligned} &\sum_{i=1}^{r} \sqrt{n}\,(x_{in} - a_i)g_i - \epsilon \\ &\quad \leq \sqrt{n}\,[g(x_{1n}, \cdots, x_{rn}) - g(a_1, \cdots, a_r)] \leq \sum_i \sqrt{n}\,(x_{in} - a_i)g_i + \epsilon \end{aligned} \tag{4}$$

[2] This was mentioned by Wilks in his paper presented at the meeting of the Institute of Mathematical Statistics in Poughkeepsie, N. Y., September, 1942.

holds, converges to 1 with $n \to \infty$. Since the limit distribution of $\sum_i \sqrt{n}(x_{in} - a_i)g_i$ is normal with zero mean and variance equal to $\Sigma\Sigma\sigma_{ij}g_i g_j$, our Lemma follows easily from the fact that the quantity ϵ in (4) can be chosen arbitrarily small.

3. The univariate case. In this section we assume that $p = 1$. Hence the probability density function $f(x_1, \cdots, x_p, \theta_1, \cdots, \theta_k)$ is replaced by the univariate density function $f(x, \theta_1, \cdots, \theta_k)$. In order to simplify the notations, the letter θ without any subscript will be used to denote the set of parameter values $\theta_1, \cdots, \theta_k$.

For any positive $\xi < 1$ let $\varphi(\theta, \xi)$ and $\psi(\theta, \xi)$ be two functions of θ such that

$$\int_{\varphi(\theta, \xi)}^{\psi(\theta, \epsilon)} f(x, \theta)\, dx = \xi. \tag{5}$$

If $f(x, \theta)$ is a continuous function of x, functions $\varphi(\theta, \xi)$ and $\psi(\theta, \xi)$ satisfying (5) exist. It is clear that for any function $\varphi(\theta, \xi)$ subject to the condition

$$\int_{-\infty}^{\varphi(\theta, \epsilon)} f(x, \theta)\, dx < 1 - \xi$$

there exists a function $\psi(\theta, \xi)$ such that (5) holds. We will choose $\varphi(\theta, \xi)$ and $\psi(\theta, \xi)$ so that (5) is satisfied and

$$\psi(\theta, \xi) - \varphi(\theta, \xi) \leq \bar{\psi}(\theta, \xi) - \bar{\varphi}(\theta, \xi) \tag{6}$$

for any value of θ and for any functions $\bar{\varphi}(\theta, \xi)$ and $\bar{\psi}(\theta, \xi)$ which satisfy (5).

Let $\hat{\theta}_i$ $(i = 1, \cdots, k)$ be the maximum likelihood estimate of θ_i calculated from the observations $x_{11}, \cdots, x_{pn}$. We propose the use of the tolerance limits

$$L = \varphi(\hat{\theta}, \xi) \quad \text{and} \quad U = \psi(\hat{\theta}, \xi) \tag{7}$$

where the value of the constant ξ has to be properly determined. Problem 1 is solved if we can determine ξ as a function of β and γ such that

$$P\left\{\int_{\varphi(\hat{\theta}, \xi)}^{\psi(\hat{\theta}, \xi)} f(x, \theta)\, dx \geq \gamma \mid \theta\right\} = \beta. \tag{8}$$

Problem 2 is solved if we determine ξ as a function of β, λ and N such that

$$P(M \geq \lambda N \mid \theta) = \beta \tag{9}$$

where M denotes the number of observation in the second sample which lie between the tolerance limits $\varphi(\hat{\theta}, \xi)$ and $\psi(\hat{\theta}, \xi)$. The use of tolerance limits of the form (7) seems to be well justified by the fact that the functions $\varphi(\theta, \xi)$ and $\psi(\theta, \xi)$ satisfy (5) and (6) and that $\hat{\theta}_i$ is an optimum estimate of θ_i $(i = 1, \cdots, k)$.

Now we will derive the large sample distribution of

$$I(\hat{\theta}, \theta, \xi) = \int_{\varphi(\hat{\theta}, \xi)}^{\psi(\hat{\theta}, \xi)} f(x, \theta)\, dx. \tag{10}$$

We obviously have

$$I(\theta, \theta, \xi) = \xi. \tag{11}$$

We will assume that the limit joint distribution of $\sqrt{n}(\hat{\theta}_1 - \theta_1), \cdots, \sqrt{n}(\hat{\theta}_k - \theta_k)$ is normal with mean values 0 and non-singular covariance matrix $\| \sigma_{ij}(\theta) \| = \| c_{ij}(\theta) \|^{-1}$ where $c_{ij}(\theta)$ denotes the expected value of $-\frac{\partial^2 \log f(x, \theta)}{\partial \theta_i \, \partial \theta_j}$ $(i, j = 1, \cdots, k)$. This is known to be true if $f(x, \theta)$ satisfies some regularity conditions.[3] Furthermore we assume that $\varphi(\theta, \xi)$ and $\psi(\theta, \xi)$ admit continuous first partial derivatives with respect to $\theta_1, \cdots, \theta_k$ and that $f(x, \theta)$ is a continuous function of x in the neighborhood of $x = \varphi(\theta, \xi)$ and $x = \psi(\theta, \xi)$. We have

$$\left. \frac{\partial I(\hat{\theta}, \theta, \xi)}{\partial \hat{\theta}_i} \right|_{\hat{\theta} = \theta} = \frac{\partial \psi(\theta, \xi)}{\partial \theta_i} f[\psi(\theta, \xi), \theta] - \frac{\partial \varphi(\theta, \xi)}{\partial \theta_i} f[\varphi(\theta, \xi), \theta] \tag{12}$$

Assuming that at least one of the derivatives $\left. \frac{\partial I(\hat{\theta}, \theta, \xi)}{\partial \hat{\theta}_i} \right|_{\hat{\theta} = \theta}$ is not zero, it follows from our Lemma that $\sqrt{n}[I(\hat{\theta}, \theta, \xi) - I(\theta, \theta, \xi)] = \sqrt{n}[I(\hat{\theta}, \theta, \xi) - \xi]$ is in the limit normally distributed with zero mean and variance

$$\begin{aligned} \sigma^2(\theta, \xi) = {} & \{f[\psi(\theta, \xi), \theta]\}^2 \sum_j \sum_i \frac{\partial \psi(\theta, \xi)}{\partial \theta_i} \frac{\partial \psi(\theta, \xi)}{\partial \theta_j} \sigma_{ij}(\theta) \\ & - 2 f[\psi(\theta, \xi), \theta] f[\varphi(\theta, \xi), \theta] \sum_j \sum_i \frac{\partial \psi(\theta, \xi)}{\partial \theta_i} \frac{\partial \varphi(\theta, \xi)}{\partial \theta_j} \sigma_{ij}(\theta) \\ & + \{f[\varphi(\theta, \xi), \theta]\}^2 \sum_j \sum_i \frac{\partial \varphi(\theta, \xi)}{\partial \theta_i} \frac{\partial \varphi(\theta, \xi)}{\partial \theta_j} \sigma_{ij}(\theta). \end{aligned} \tag{13}$$

For any positive $\beta < 1$ denote by λ_β the value for which

$$\frac{1}{\sqrt{2\pi}} \int_{\lambda_\beta}^{\infty} \theta^{-\frac{1}{2}t^2}\, dt = \beta. \tag{14}$$

Then the probability that

$$I(\hat{\theta}, \theta, \xi) \geq \xi + \lambda_\beta \frac{\sigma(\theta, \xi)}{\sqrt{n}}, \tag{15}$$

converges with $n \to \infty$ towards β.

Let

$$\bar{\xi}(\beta, \gamma, \hat{\theta}) = \gamma - \lambda_\beta \frac{\sigma(\hat{\theta}, \gamma)}{\sqrt{n}} \tag{16}$$

---

[3] See for instance J. L. Doob, "Probability and statistics," *Trans. Amer. Math. Soc.*, October, 1934.

If $\sigma(\theta, \xi)$ is continuous in θ and ξ, it follows easily from (15) that the probability that

$$(17) \qquad I[\hat{\theta}, \theta, \bar{\xi}(\beta, \gamma, \hat{\theta})] \geq \gamma$$

holds, converges to β with $n \to \infty$. Hence we can summarize our results in the following

THEOREM 1: *Let $\varphi(\theta, \xi)$ and $\psi(\theta, \xi)$ be two functions satisfying* (5) *and* (6). *Furthermore, let the functions $I(\hat{\theta}, \theta, \xi)$, $\sigma^2(\theta, \xi)$ and $\bar{\xi}(\beta, \gamma, \hat{\theta})$ be defined by* (10), (13) *and* (16) *respectively. Denote by $\theta_1^0, \cdots, \theta_k^0$ the true values of the parameters. It is assumed that there exist two positive numbers ϵ and δ such that the following three conditions are fulfilled*:

(a) *For any point θ for which $\sum_{i=1}^{k} (\theta_i - \theta_i^0)^2 \leq \epsilon$ the limit joint distribution of $\sqrt{n}(\hat{\theta}_1 - \theta_1), \cdots, \sqrt{n}(\hat{\theta}_k - \theta_k)$, calculated under the assumption that θ is the true parameter point, is normal with zero means and a finite non-singular covariance matrix $\| \sigma_{ij}(\theta) \|$ where $\sigma_{ij}(\theta)$ is a continuous function of θ in the domain $\sum_i (\theta_i - \theta_i^0)^2 \leq \epsilon$.*

(b) *The partial derivatives $\left. \frac{\partial I(\hat{\theta}, \theta, \xi)}{\partial \hat{\theta}_i} \right|_{\hat{\theta}=\theta}$ $(i = 1, \cdots, k)$ are continuous functions of θ and ξ in the domain*

$$\sum_{i=1}^{k} (\theta_i - \theta_i^0)^2 \leq \epsilon \quad \text{and} \quad |\xi - \gamma| \leq \delta.$$

(c) *At least one of the partial derivatives $\left. \frac{\partial I(\hat{\theta}, \theta^0, \gamma)}{\partial \hat{\theta}_i} \right|_{\hat{\theta}=\theta^0}$ $(i = 1, \cdots, k)$ is not equal to zero.*

Then the probability that

$$I[\hat{\theta}, \theta^0, \bar{\xi}(\beta, \gamma, \hat{\theta})] \geq \gamma,$$

holds, converges to β with $n \to \infty$.

From Theorem 1 we obtain the following

LARGE SAMPLE SOLUTION OF PROBLEM 1. *For large n we can approximate the lower and upper tolerance limits by $\varphi[\hat{\theta}, \bar{\xi}(\beta, \gamma, \hat{\theta})]$ and $\psi[\hat{\theta}, \bar{\xi}(\beta, \gamma, \hat{\theta})]$ respectively, where $\bar{\xi}(\beta, \gamma, \hat{\theta})$ is given by* (16).

Now we will deal with Problem 2. We distinguish two cases

$$\text{(a)} \qquad \lim_{n \to \infty} \frac{N}{n} = \infty.$$

It is easy to see that in this case the solution of Problem 2 is obtained from that of Problem 1 by substituting λ for γ. Hence for large n the tolerance limits can be approximated by $\varphi[\hat{\theta}, \bar{\xi}(\beta, \lambda, \hat{\theta})]$ and $\psi[\hat{\theta}, \bar{\xi}(\beta, \lambda, \hat{\theta})]$ respectively.

For these tolerance limits condition 2 is fulfilled in the limit, i.e. $\lim_{n=\infty} P(M \geq \lambda N \mid \theta_1, \cdots, \theta_k) = \beta$

(b) The integers n and N approach infinity while $\frac{N}{n}$ remains bounded.

Denote $\sqrt{n}\,[I(\hat{\theta}, \theta, \xi) - \xi]$ by u and $\sqrt{N}\left(\frac{M(\xi)}{N} - \xi\right)$ by v, where $M(\xi)$ denotes the number of observations in the second sample which fall between the limits $\varphi(\hat{\theta}, \xi)$ and $\psi(\hat{\theta}, \xi)$. For any fixed value of u the conditional expected value of $\frac{M(\xi)}{N}$ is given by $\xi + \frac{u}{\sqrt{n}}$ and the conditional variance of $\frac{M(\xi)}{N}$ is given by $\frac{1}{N}\left(\xi + \frac{u}{\sqrt{n}}\right)\left(1 - \xi - \frac{u}{\sqrt{n}}\right)$. Hence the conditional expected value of v is equal to $u\sqrt{\frac{N}{n}}$ and the conditional variance of v is equal to $\left(\xi + \frac{u}{\sqrt{n}}\right)\left(1 - \xi - \frac{u}{\sqrt{n}}\right)$. Since the limit distribution of u is normal with zero mean and standard deviation $\sigma(\theta, \xi)$ given in (13), we find that the limit bivariate distribution of u and v is given by

$$(18) \qquad \frac{1}{2\pi\sigma(\theta, \xi)\sqrt{\xi(1 - \xi)}} \exp\left[-\frac{u^2}{2\sigma^2(\theta, \xi)} - \frac{\left(v - \sqrt{\frac{N}{n}}u\right)^2}{2\xi(1 - \xi)}\right] du\, dv.$$

From (18) it follows that the limit distribution of v is normal with zero mean and variance

$$(19) \qquad \begin{aligned} \sigma_v^2 &= \sigma^2(\theta, \xi)\left(\frac{1}{\sigma^2(\theta, \xi)} + \frac{N}{n\xi(1 - \xi)}\right)\xi(1 - \xi) \\ &= \frac{n\xi(1 - \xi) + N\sigma^2(\theta, \xi)}{n}. \end{aligned}$$

From (19) it follows easily that the probability that

$$(20) \qquad \frac{M(\xi)}{N} \geq \xi + \frac{\lambda_\beta \sigma_v}{\sqrt{N}}$$

converges to β with $n \rightarrow \infty$. Let

$$(21) \qquad \xi^*(\beta, \lambda, \hat{\theta}) = \lambda - \frac{\lambda_\beta}{\sqrt{N}} \sqrt{\frac{n\lambda(1 - \lambda) + N\sigma^2(\hat{\theta}, \lambda)}{n}}.$$

From (20) it follows that the probability that

$$\frac{M}{N} \geq \lambda,$$

converges to β with $n \to \infty$. The letter M denotes the number of observations in the second sample which lie between the limits $\varphi[\theta, \xi^*(\beta, \lambda, \hat{\theta})]$ and $\psi[\hat{\theta}, \xi^*(\beta, \lambda, \hat{\theta})]$.

We can summarize our results in the following

THEOREM 2. *Let $\varphi(\theta, \xi)$ and $\psi(\theta, \xi)$ be two functions satisfying* (5) *and* (6). *Two samples of size n and N respectively are drawn and the maximum likelihood estimate $\hat{\theta}$ is calculated from the first sample only. Assume that conditions* (a), (b) *and* (c) *of Theorem* 1 *are satisfied. Let $\bar{\xi}(\beta, \gamma, \hat{\theta})$ and $\xi^*(\beta, \lambda, \hat{\theta})$ be defined by* (16) *and* (21) *respectively.*

If n and $\frac{N}{n}$ both approach infinity, the probability that $\frac{M}{N} \geq \lambda$ holds, converges to β, where M denotes the number of observations in the second sample which lie between the limits $\varphi[\hat{\theta}, \bar{\xi}(\beta, \lambda, \hat{\theta})]$ and $\psi[\hat{\theta}, \bar{\xi}(\beta, \lambda, \hat{\theta})]$.

If n and N approach infinity while $\frac{N}{n}$ remains bounded, the probability that $\frac{M}{N} \geq \lambda$ holds, converges to β, where M denotes the number of observations in the second sample which lie between the limits $\varphi[\hat{\theta}, \xi^*(\beta, \lambda, \hat{\theta})]$ and $\psi[\hat{\theta}, \xi^*(\beta, \lambda, \hat{\theta})]$.

From Theorem 2 we obtain the following

LARGE SAMPLE SOLUTION OF PROBLEM 2. *If n and $\frac{N}{n}$ both approach infinity the lower and upper tolerance limits can be approximated by $\varphi[\hat{\theta}, \bar{\xi}(\beta, \lambda, \hat{\theta})]$ and $\psi[\hat{\theta}, \bar{\xi}(\beta, \lambda, \hat{\theta})]$ respectively. If n and N both approach infinity while $\frac{N}{n}$ remains bounded, the tolerance limits can be approximated by $\varphi[\hat{\theta}, \xi^*(\beta, \lambda, \hat{\theta})]$ and $\psi[\hat{\theta}, \xi^*(\beta, \lambda, \hat{\theta})]$ respectively. The expressions $\bar{\xi}(\beta, \lambda, \hat{\theta})$ and $\xi^*(\beta, \lambda, \hat{\theta})$ are given by* (16) *and* (21) *respectively.*

4. The multivariate case. For any positive $\xi < 1$ let $\varphi_i(\theta, \xi)$ and $\psi_i(\theta, \xi)$ $(i = 1, \cdots, p)$ be p pairs of functions of θ such that

$$\int_{\varphi_p(\theta,\xi)}^{\psi_p(\theta,\xi)} \cdots \int_{\varphi_1(\theta,\xi)}^{\psi_1(\theta,\xi)} f(x_1, \cdots, x_p, \theta)\, dx_1 \cdots dx_p = \xi. \tag{22}$$

If $f(x_1, \cdots, x_p, \theta)$ is a continuous function of $x_1, \cdots, x_p$, functions $\varphi_i(\theta, \xi)$ and $\psi_i(\theta, \xi)$ $(i = 1, \cdots, p)$ satisfying (22) certainly exist. As in the univariate case, there will be infinitely many sets of p pairs of functions $\varphi_i(\theta, \xi)$ and $\psi_i(\theta, \xi)$ which satisfy (22). Since we wish to have tolerance limits as narrow as possible, we will try to choose the functions $\varphi_i(\theta, \xi)$ and $\psi_i(\theta, \xi)$ so that $\psi_i(\theta, \xi) - \varphi_i(\theta, \xi)$ should be as small as possible. Since it is impossible to minimize all p differences $\psi_1(\theta, \xi) - \varphi_1(\theta, \xi), \cdots, \psi_p(\theta, \xi) - \varphi_p(\theta, \xi)$ simultaneously, we will have to be satisfied with some compromise solution. For example, we could minimize the product $\prod_i [\psi_i(\theta, \xi) - \varphi_i(\theta, \xi)]$ or some other function of the p differences $\psi_i(\theta, \xi) - \varphi_i(\theta, \xi)$. Another reasonable procedure would be to minimize

$\prod_i [\psi_i(\theta, \xi) - \varphi_i(\theta, \xi)]$ subject to (22) and the condition that for any i and j, $\dfrac{\psi_i(\theta, \xi) - \varphi_i(\theta, \xi)}{\psi_j(\theta, \xi) - \varphi_j(\theta, \xi)}$ is equal to the ratio of the standard deviation of x_i to that of x_j.

Here we will deal with the problem of deriving tolerance limits for the variates $x_1, \cdots, x_p$ after the functions $\varphi_i(\theta, \xi)$ and $\psi_i(\theta, \xi)$ have been chosen. Since the theory of the multivariate case is very similar to that of the univariate case, we will merely outline it briefly.

As tolerance limits for x_i we will use the functions $\varphi_i(\hat{\theta}, \xi)$ and $\psi_i(\hat{\theta}, \xi)$ where the value of ξ has to be properly determined. Problem 1 is solved if we can determine ξ as a function of β and γ so that

$$P\left\{\int_{\varphi_p(\hat{\theta},\xi)}^{\psi_p(\hat{\theta},\xi)} \cdots \int_{\varphi_1(\hat{\theta},\xi)}^{\psi_1(\hat{\theta},\xi)} f(x_1, \cdots, x_p, \theta)\, dx_1 \cdots dx_p \geq \gamma \mid \theta\right\} = \beta. \tag{23}$$

Problem 2 is solved if we determine ξ as a function of β, λ and N such that condition 2 is fulfilled. Let

$$I(\hat{\theta}, \theta, \xi) = \int_{\varphi_p(\hat{\theta},\xi)}^{\psi_p(\hat{\theta},\xi)} \cdots \int_{\varphi_1(\hat{\theta},\xi)}^{\psi_1(\hat{\theta},\xi)} f(x_1, \cdots, x_p, \theta)\, dx_1 \cdots dx_p \tag{24}$$

and let

$$\begin{aligned} I_i(\hat{\theta}, \theta, \xi, x_i) = & \int_{\varphi_p(\hat{\theta},\xi)}^{\psi_p(\hat{\theta},\xi)} \cdots \int_{\varphi_{i+1}(\hat{\theta},\xi)}^{\psi_{i+1}(\hat{\theta},\xi)} \int_{\varphi_{i-1}(\hat{\theta},\xi)}^{\psi_{i-1}(\hat{\theta},\xi)} \\ & \cdots \int_{\varphi_1(\hat{\theta},\xi)}^{\psi_1(\hat{\theta},\xi)} f(x_1, \cdots, x_p, \theta)\, dx_1 \cdots dx_{i-1}\, dx_{i+1} \cdots dx_p . \end{aligned} \tag{25}$$

We have

$$\begin{aligned} \left.\frac{\partial I(\hat{\theta}, \theta, \xi)}{\partial \hat{\theta}_i}\right|_{\hat{\theta}=\theta} = & \sum_{s=1}^{p} \frac{\partial \psi_s(\theta, \xi)}{\partial \theta_i} I_s[\theta, \theta, \xi, \psi_s(\theta, \xi)] \\ & - \sum_{s=1}^{p} \frac{\partial \varphi_s(\theta, \xi)}{\partial \theta_i} I_s[\theta, \theta, \xi, \varphi_s(\theta, \xi)]. \end{aligned} \tag{26}$$

Assuming that the partial derivatives $\left.\dfrac{\partial I(\hat{\theta}, \theta, \xi)}{\partial \hat{\theta}_i}\right|_{\hat{\theta}=\theta}$ $(i = 1, \cdots, k)$ are continuous functions and that $\left.\dfrac{\partial I(\hat{\theta}, \theta, \xi)}{\partial \hat{\theta}_i}\right|_{\hat{\theta}=\theta}$ is not zero for at least one value of i, it follows from our Lemma that $\sqrt{n}\,[I(\hat{\theta}, \theta, \xi) - I(\theta, \theta, \xi)] = \sqrt{n}\,[I(\hat{\theta}, \theta, \xi) - \xi]$ is in the limit normally distributed with mean value zero and variance

$$\begin{aligned} \bar{\sigma}^2(\theta, \xi) = & \sum_{q=1}^{p} \sum_{s=1}^{p} \sum_{j=1}^{k} \sum_{i=1}^{k} \frac{\partial \psi_s(\theta, \xi)}{\partial \theta_i} \frac{\partial \psi_q(\theta, \xi)}{\partial \theta_j} I_s[\theta, \theta, \xi, \psi_s(\theta, \xi)] I_q[\theta, \theta, \xi, \psi_s(\theta, \xi)] \sigma_{ij}(\theta) \\ & - 2 \sum_{s} \sum_{q} \sum_{i} \sum_{j} \frac{\partial \psi_s(\theta, \xi)}{\partial \theta_i} \frac{\partial \varphi_q(\theta, \xi)}{\partial \theta_j} \\ & \cdot I_s[\theta, \theta, \xi, \psi_s(\theta, \xi)] I_q[\theta, \theta, \xi, \varphi_q(\theta, \xi)] \sigma_{ij}(\theta) \\ & + \sum_{s} \sum_{q} \sum_{j} \sum_{i} \frac{\partial \varphi_s(\theta, \xi)}{\partial \theta_i} \frac{\partial \varphi_q(\theta, \xi)}{\partial \theta_j} \\ & \cdot I_s[\theta, \theta, \xi, \varphi_s(\theta, \xi)] I_q[\theta, \theta, \xi, \varphi_q(\theta, \xi)] \sigma_{ij}(\theta) \end{aligned} \tag{27}$$

where $\|\sigma_{ij}(\theta)\|$ is the limit covariance matrix of $\sqrt{n}(\hat{\theta}_1 - \theta_1), \cdots, \sqrt{n}(\hat{\theta}_k - \theta_k)$.

For any positive $\beta > 1$, let λ_β be the real value defined by the equation

$$\frac{1}{\sqrt{2\pi}} \int_{\lambda_\beta}^{\infty} e^{-\frac{1}{2}t^2} \, dt = \beta. \tag{28}$$

Let

$$\bar{\zeta}(\beta, \gamma, \hat{\theta}) = \gamma - \lambda_\beta \frac{\bar{\sigma}(\hat{\theta}, \gamma)}{\sqrt{n}} \tag{29}$$

and

$$\zeta^*(\beta, \lambda, \hat{\theta}) = \lambda - \frac{\lambda_\beta}{\sqrt{N}} \sqrt{\frac{n\lambda(1-\lambda) + N\bar{\sigma}^2(\hat{\theta}, \lambda)}{n}}. \tag{30}$$

We can easily prove the following two theorems:

THEOREM 3. *Let $\varphi_i(\theta, \xi)$ and $\psi_i(\theta, \xi)$ $(i = 1, \cdots, p)$ be p pairs of functions which satisfy* (22). *Let the functions $I(\hat{\theta}, \theta, \xi)$, $\bar{\sigma}^2(\theta, \xi)$ and $\bar{\zeta}(\beta, \gamma, \hat{\theta})$ be defined by* (24), (27) *and* (29) *respectively. Denote by $\theta_1^0, \cdots, \theta_k^0$ the true values of the parameters $\theta_1, \cdots, \theta_k$. It is assumed that there exist two positive numbers ϵ and δ such that the following three conditions are fulfilled:*

(a) *For any point θ for which $\sum_{i=1}^{k} (\theta_i - \theta_i^0)^2 \leq \epsilon$ the limit joint distribution of $\sqrt{n}(\hat{\theta}_1 - \theta_1), \cdots, \sqrt{n}(\hat{\theta}_k - \theta_k)$, calculated under the assumption that θ is the true parameter point, is normal with zero means and a finite non-singular covariance matrix $\|\sigma_{ij}(\theta)\|$ where $\sigma_{ij}(\theta)$ is a continuous function of θ in the domain $\sum_i (\theta_i - \theta_i^0)^2 \leq \epsilon$.*

(b) *The partial derivatives $\left.\frac{\partial I(\hat{\theta}, \theta, \xi)}{\partial \hat{\theta}_i}\right|_{\hat{\theta}=\theta}$ $(i = 1, \cdots, k)$ are continuous functions of θ and ξ in the domain $\sum_{i=1}^{k} (\theta_i - \theta_i^0)^2 \leq \epsilon$ and $|\xi - \gamma| \leq \delta$.*

(c) *At least one of the partial derivatives $\left.\frac{\partial I(\hat{\theta}, \theta^0, \gamma)}{\partial \hat{\theta}_i}\right|_{\hat{\theta}=\theta^0}$ $(i = 1, \cdots, k)$ is not equal to zero.*

Then the probability that

$$I[\hat{\theta}, \theta^0, \bar{\zeta}(\beta, \gamma, \hat{\theta})] \geq \gamma$$

holds, converges to β with $n \to \infty$.

THEOREM 4. *Let $\varphi_i(\theta, \xi)$ and $\psi_i(\theta, \xi)$ $(i = 1, \cdots, p)$ be p pairs of functions which satisfy* (22). *Two samples of size n and N respectively are drawn and the maximum likelihood estimate $\hat{\theta}$ is calculated from the first sample only. Assume that conditions* (a), (b) *and* (c) *of Theorem* 3 *are fulfilled and let $\bar{\zeta}(\beta, \gamma, \hat{\theta})$ and $\zeta^*(\beta, \lambda, \hat{\theta})$ be defined by* (29) *and* (30) *respectively. Denote by $y_{i\alpha}$ the outcome of the α-th observation on the i-th variate in the second sample.*

If n and $\frac{N}{n}$ both approach infinity, the probability that $M \geq \lambda N$ holds converges to β, where M denotes the number of different values of α for which

$$\varphi_i[\hat{\theta}, \bar{\zeta}(\beta, \lambda, \hat{\theta})] \leq y_{i\alpha} \leq \psi_i[\hat{\theta}, \bar{\zeta}(\beta, \lambda, \hat{\theta})] \qquad (i = 1, \cdots, p).$$

If n and N approach infinity while $\frac{N}{n}$ remains bounded, the probability that $M \geq \lambda N$ holds converges to β where M denotes the number of different values of α for which

$$\varphi_i[\hat{\theta}, \zeta^*(\beta, \lambda, \hat{\theta})] \leq y_{i\alpha} \leq \psi_i[\hat{\theta}, \zeta^*(\beta, \lambda, \hat{\theta})] \qquad (i = 1, \cdots, p).$$

The proofs of Theorems 3 and 4 are omitted since they are similar to the proofs of Theorems 1 and 2.

From Theorem 3 we obtain the following

LARGE SAMPLE SOLUTION OF PROBLEM 1. *For large n we can approximate the lower and upper tolerance limits for x_i by $\varphi_i[\hat{\theta}, \bar{\zeta}(\beta, \gamma, \hat{\theta})]$ and $\psi_i[\hat{\theta}, \bar{\zeta}(\beta, \gamma, \hat{\theta})]$ respectively where $\bar{\zeta}(\beta, \gamma, \theta)$ is given by* (29).

From Theorem 4 we obtain the following

LARGE SAMPLE SOLUTION OF PROBLEM 2. *If n and $\frac{N}{n}$ approach infinity, the lower and upper tolerance limits for x_i can be approximated by $\varphi_i[\hat{\theta}, \bar{\zeta}(\beta, \lambda, \hat{\theta})]$ and $\psi_i[\hat{\theta}, \bar{\zeta}(\beta, \lambda, \hat{\theta})]$ respectively. If n and N both approach infinity while $\frac{N}{n}$ remains bounded, the tolerance limits for x_i can be approximated by $\varphi_i[\hat{\theta}, \zeta^*(\beta, \lambda, \hat{\theta})]$ and $\psi_i[\hat{\theta}, \zeta^*(\beta, \lambda, \hat{\theta})]$ respectively. The expressions $\bar{\zeta}(\beta, \lambda, \hat{\theta})$ and $\zeta^*(\beta, \lambda, \hat{\theta})$ are defined in* (29) *and* (30) *respectively.*

5. An example. Let x be a normally distributed variate with mean value θ_1 and standard deviation θ_2, i.e. the probability density function of x is given by

$$f(x, \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi}\,\theta_2} e^{-\frac{1}{2}(x-\theta_1)^2/\theta_2^2}.$$

For any positive $\xi < 1$ let $\rho(\xi)$ be the value for which

$$\frac{1}{\sqrt{2\pi}} \int_{-\rho(\xi)}^{\rho(\xi)} e^{-\frac{1}{2}t^2}\, dt = \xi.$$

Then the functions

$$\varphi(\theta, \xi) = \theta_1 - \rho(\xi)\theta_2$$

and

$$\psi(\theta, \xi) = \theta_1 + \rho(\xi)\theta_2$$

satisfy conditions (5) and (6).

We have

$$\hat{\theta}_1 = \frac{x_1 + \cdots + x_n}{n} = \bar{x} \quad \text{and} \quad \hat{\theta}_2 = \sqrt{\frac{\sum_{\alpha=1}^{n} (x_\alpha - \bar{x})^2}{n}}.$$

The variance of $\sqrt{n}(\hat{\theta}_1 - \theta_1)$ is equal to θ_2^2 and the limit variance of $\sqrt{n}(\hat{\theta}_2 - \theta_2)$ is equal to $\frac{1}{2}\theta_2^2$. Since the covariance of $\hat{\theta}_1$ and $\hat{\theta}_2$ is equal to zero, we obtain from (13)

$$\sigma^2(\theta, \xi) = 2\left\{\frac{1}{\sqrt{2\pi}\,\theta_2} e^{-\frac{1}{2}[\rho(\xi)]^2}\right\}\{\theta_2^2 + \tfrac{1}{2}\theta_2^2[\rho(\xi)]^2\} - 2\left\{\frac{1}{\sqrt{2\pi}\,\theta_2} e^{-\frac{1}{2}[\rho(\xi)]^2}\right\}\{\theta_2^2 - \tfrac{1}{2}\theta_2^2[\rho(\xi)]^2\}$$

$$= \frac{1}{\pi}[\rho(\xi)]^2 e^{-[\rho(\xi)]^2}.$$

Hence for large n the tolerance limits satisfying (1) can be approximated by $\hat{\theta}_1 - \rho(\bar{\xi})\hat{\theta}_2$ and $\hat{\theta}_1 + \rho(\bar{\xi})\hat{\theta}_2$ respectively where

$$\bar{\xi} = \gamma - \lambda_\beta \frac{\rho(\gamma)}{\sqrt{n\pi}} e^{-\frac{1}{2}\rho[(\gamma)]^2}$$

and λ_β is the value determined by the equation

$$\frac{1}{\sqrt{2\pi}} \int_{\lambda_\beta}^{\infty} e^{-\frac{1}{2}t^2}\, dt = \beta.$$

If n and N are large, the tolerance limits satisfying (2) can be approximated by $\hat{\theta}_1 - \rho(\xi^*)\hat{\theta}_2$ and $\hat{\theta}_1 + \rho(\xi^*)\hat{\theta}_2$ respectively where

$$\xi^* = \lambda - \lambda_\beta \sqrt{\frac{\lambda(1-\lambda)}{N} + \frac{[\rho(\lambda)]^2}{n\pi} e^{-[\rho(\lambda)]^2}}.$$