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SEVENTH POWER MOMENTS OF KLOOSTERMAN SUMS

BY

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ABSTRACT

Evaluations of the n-th power moments S_n of Kloosterman sums are known only for $n \leqslant 6$. We present here substantial evidence for an evaluation of S_7 in terms of Hecke eigenvalues for a weight 3 newform on $\Gamma_0(525)$ with quartic nebentypus of conductor 105. We also prove some congruences modulo 3, 5 and 7 for the closely related quantity T_7, where T_n is a sum of traces of n-th symmetric powers of the Kloosterman sheaf.

1. Introduction

For an odd prime p, let $\mathbb{F}_p$ denote a field of p elements, and write $\zeta_p = \exp(2\pi i/p)$. Consider the Kloosterman sums

$$K(a) = \sum_{x=1}^{p-1} \zeta_p^{x+a/x}, \quad a \in \mathbb{F}_p, \tag{1.1}$$

and their n-th power moments

$$S_n = \sum_{a=0}^{p-1} K(a)^n, \quad n \in \mathbb{N}. \tag{1.2}$$

It is well-known [5, §4.4] that

$$S_1 = 0, \quad S_2 = p^2 - p, \quad S_3 = \left(\frac{p}{3}\right)p^2 + 2p, \quad S_4 = 2p^3 - 3p^2 - 3p. \tag{1.3}$$

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The work in [8], [9] shows that S_5 can be expressed in terms of the p-th eigenvalue for a weight 3 newform on $\Gamma_0(15)$. The work in [4] shows that S_6 can be expressed in terms of the p-th eigenvalue for a weight 4 newform on $\Gamma_0(6)$. See also [1].

In Conjecture 1.1 below, we propose an evaluation of S_7 in terms of the p-th eigenvalue for a weight 3 newform on $\Gamma_0(525)$. This conjecture is based on substantial numerical evidence.

Write

$$K(a) = -g(a) - h(a), \quad a \neq 0, \tag{1.4}$$

where $g(a)$, $h(a)$ are the two Frobenius eigenvalues for the Kloosterman sheaf at a, given by

$$g(a) = p^{1/2}\exp(i\theta_p(a)), \quad h(a) = p^{1/2}\exp(-i\theta_p(a)), \tag{1.5}$$

with $\theta_p(a) \in [0, \pi]$. (In fact, $\theta_p(a) \in (0, \pi)$; see [2, Theorem 6.1].) By (1.2) and (1.4),

$$S_n = (-1)^n + (-1)^n \sum_{a=1}^{p-1} (g(a) + h(a))^n. \tag{1.6}$$

As noted in [5, p. 63], one should study the "more natural" related expressions

$$T_n = \sum_{a=1}^{p-1} (g(a)^n + g(a)^{n-1}h(a) + \cdots + h(a)^n). \tag{1.7}$$

The summand in (1.7) is the trace of the n-th symmetric power of the Kloosterman sheaf at a, and equals

$$p^{n/2} U_n(2\cos\theta_p(a)), \tag{1.8}$$

where U_n is the n-th monic Chebyshev polynomial of the second kind. We have the bound [3, Theorem 0.2], [6]

$$|1 + T_n| \leqslant \left[\frac{n-1}{2}\right] p^{(n+1)/2}, \quad \text{if } p > n > 0, \tag{1.9}$$

whose proof is based on Deligne's theory of exponential sums for varieties over $\mathbb{F}_p$. (A slightly weaker bound which holds for all $p > 2$ is given in [5, Theorem 4.6].)

The expressions S_n and T_n are related by the formula

$$(-1)^n S_n - 1 = \sum_{k=0}^{[n/2]} \left\{ \binom{n}{k} - \binom{n}{k-1} \right\} p^k T_{n-2k}. \tag{1.10}$$

In [1, (1.11)], it is proved that S_n is an integer multiple of p satisfying

$$S_n \equiv p(n-1)(-1)^{n-1} \pmod{p^2}. \tag{1.11}$$

From (1.10)–(1.11), it follows by induction that

$$T_n \equiv -1 (\mathrm{mod}\ p^2), \quad n > 0. \tag{1.12}$$

By (1.3) and (1.10), we have

$$T_0 + 1 = p, \quad T_1 + 1 = T_2 + 1 = 0, \quad T_3 + 1 = -\left(\frac{p}{3}\right)p^2, \quad T_4 + 1 = -p^2. \tag{1.13}$$

By [1, (1.8)], we have for $p > 5$,

$$a_p := \frac{-1 - T_5}{p^2} = \begin{cases} 2p - 12u^2, & \text{if } p = 3u^2 + 5v^2 \\ 4x^2 - 2p, & \text{if } p = x^2 + 15y^2 \\ 0, & \text{if } (\frac{p}{15}) = -1. \end{cases} \tag{1.14}$$

Define

$$c_p := (-1 - T_7)/p^2. \tag{1.15}$$

By (1.12), a_p and c_p are integers, and by (1.9), we have

$$|a_p| \leqslant 2p, \quad |c_p| \leqslant 3p^2. \tag{1.16}$$

Putting $n = 7$ in (1.10) yields

$$S_7 = p^2 c_p + 6p^3 a_p + 14\left(\frac{p}{3}\right)p^4 + 14p^3 + 14p^2 + 6p. \tag{1.17}$$

Hence by (1.16),

$$|S_7| \leqslant 29p^4 + 14p^3 + 14p^2 + 6p. \tag{1.18}$$

In view of (1.14) and (1.17), an evaluation of c_p would yield an evaluation of S_7. Hence we focus on c_p in Conjecture 1.1 below, and in the sequel.

Let χ_5 denote the quartic Dirichlet character (mod 5) defined by $\chi_5(2) = -i$, and let ψ denote the quartic character of conductor 105 defined by

$$\psi(d) = \left(\frac{d}{21}\right)\chi_5(d), \quad d \in \mathbb{Z}. \tag{1.19}$$

Conjecture 1.1: *For $p > 7$,*

$$c_p = \left(\frac{p}{105}\right)(-p^2 + b(p)^2\ \overline{\psi}(p)) = \left(\frac{p}{105}\right)(-p^2 + |b(p)|^2), \tag{1.20}$$

where $b(p)$ is the p-th Hecke eigenvalue for a weight 3 newform f on $\Gamma_0(525)$ with nebentypus ψ and eigenfield $\mathbb{Q}(i, \sqrt{6}, \sqrt{14})$.

In Section 2, we motivate Conjecture 1.1 and discuss the evidence for it. In Section 3, we examine the integers c_p modulo 3, 5, and 7, proving in the process some observations of Katz [7]. Section 4, the Appendix, records a Sage [10] session which exhibits numerical evidence for Conjecture 1.1.

2. Motivation and evidence for Conjecture 1.1

The following conjecture has been verified for each of the 396 primes p in the interval $7 < p \leq 2741$.

Conjecture 2.1: *Let $p > 7$, and define the signature $\alpha_p := ((\frac{p}{3}), (\frac{p}{5}), (\frac{p}{7}))$. Then*

$$\left(\frac{p}{105}\right)c_p + p^2 = x(p)^2 \tag{2.1}$$

for a nonnegative number $x(p)$ of the form:

$2m\sqrt{7}$ *with* $m \equiv \pm 1 \pmod{10}$, $3 \nmid m$, *if* $\alpha_p = (1, -1, -1)$;

$4m\sqrt{3}$ *with* $m \equiv \pm 1 \pmod{10}$, *if* $\alpha_p = (-1, -1, 1)$;

$2m\sqrt{42}$ *with* $m \equiv \pm 1 \pmod{5}$, *if* $\alpha_p = (1, -1, 1)$;

$6m\sqrt{2}$ *with* $m \equiv \pm 2 \pmod{5}$, *if* $\alpha_p = (-1, -1, -1)$;

$2m$ *with* $m \equiv \pm(3 - 2\chi_5(p)) \pmod{10}$, $3 \nmid m$, *if* $\alpha_p = (1, 1, 1)$;

$4m\sqrt{21}$ *with* $m \equiv \pm(1 + \chi_5(p)) \pmod{5}$, *if* $\alpha_p = (-1, 1, -1)$;

$2m\sqrt{6}$ *with* $m \equiv \pm(2 - 2\chi_5(p)) \pmod{5}$, *if* $\alpha_p = (1, 1, -1)$;

$6m\sqrt{14}$ *with* $m \equiv \pm(2 - 2\chi_5(p)) \pmod{5}$, *if* $\alpha_p = (-1, 1, 1)$

where m is a positive integer.

The values of $x(p)$ for $7 < p < 100$ are given in Table 2.1 below.

Motivated by our Conjecture 2.1, Katz [7] proposed the following scenario. For $p > 7$, the number c_p/p^2 (which lies in $[-3, 3]$ by (1.16)) is the trace of Frob_p in a representation towards O(3) (the orthogonal group with respect to

p	11	13	17	19	23	29	31
$x(p)$	0	$2\sqrt{7}$	$18\sqrt{2}$	$8\sqrt{6}$	$4\sqrt{3}$	$6\sqrt{14}$	$10\sqrt{6}$

p	37	41	43	47	53	59	61
$x(p)$	$2\sqrt{42}$	$12\sqrt{21}$	$8\sqrt{42}$	$12\sqrt{2}$	$36\sqrt{3}$	$20\sqrt{21}$	$30\sqrt{6}$

p	67	71	73	79	83	89	97
$x(p)$	$12\sqrt{42}$	$30\sqrt{14}$	$38\sqrt{7}$	50	$78\sqrt{2}$	$20\sqrt{21}$	$38\sqrt{7}$

Table 1

a trace form). This Frob_p has determinant $(\frac{p}{105})$, so $(\frac{p}{105})c_p/p^2$ is the trace of Frob_p in a representation towards SO(3). For some Dirichlet character χ, this representation is $\overline{\chi}(p)\otimes\mathrm{Sym}^2(V)$ for a 2-dimensional representation V, where Frob_p in V has eigenvalues α, β with $|\alpha| = |\beta| = 1$ and $\alpha\beta = \chi(p)$. After equating traces, we obtain

$$\chi(p)\Big(\frac{p}{105}\Big)c_p/p^2 = \chi(p) + \alpha^2 + \beta^2,$$

so

$$\chi(p)\Big\{\Big(\frac{p}{105}\Big)c_p + p^2\Big\} = p^2(\alpha+\beta)^2.$$

Define $b(p) := p(\alpha+\beta)$, so that $|b(p)| \leqslant 2p$ and $b(p)/p$ is the trace of Frob_p in V. In the notation of (2.1), it follows that

$$\chi(p)x(p)^2 = b(p)^2, \quad p > 7. \tag{2.2}$$

Assuming the validity of Katz's scenario, we hoped to find a Dirichlet character χ, a level N, and a weight 3 newform

$$f(z) = \sum_{m=1}^{\infty} \widehat{f}(m)e^{2\pi imz}, \quad \widehat{f}(p) = b(p) \tag{2.3}$$

on $\Gamma_0(N)$ with nebentypus χ such that $x(p) = |b(p)|$ for $p > 7$. The equality $x(p) = |b(p)|$ is equivalent to (2.2), by [5, (6.57)]. Our search for N, χ, f culminated with the discovery of a weight 3 newform (2.3) on $\Gamma_0(525)$ with nebentypus ψ and eigenfield $\mathbb{Q}(i, \sqrt{6}, \sqrt{14})$ such that $x(p) = |b(p)|$ for $7 < p < 100$. Equivalently,

$$\psi(p)x(p)^2 = b(p)^2, \quad 7 < p < 100,$$

which is powerful evidence that (1.20) in fact holds for *all* $p > 7$.

We proceed to describe how this newform f of level 525 was discovered. While browsing William Stein's *Modular Forms Explorer* found in his *Modular Forms Database* [11], we had encountered a weight 3 newform $g(z)$ on $\Gamma_0(168)$ with quadratic nebentypus of conductor 168 and eigenfield $\mathbb{Q}(i\sqrt{2}, \sqrt{3},\ i\sqrt{7})$. For each p with $7 < p < 100$, $|\widehat{g}(p)|$ appeared to be an integer multiple of one of $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{6}, \sqrt{7}, \sqrt{14}, \sqrt{21}, \sqrt{42}$, just as was the case for $x(p)$ (cf. Table 2.1). Moreover, analogous to the situation in Conjecture 2.1, the particular choice of square root occurring in $|\widehat{g}(p)|$ seemed to be completely determined by the signature $((\frac{p}{3}), (\frac{p}{7}), (\frac{-8}{p}))$. The product of the conductors of the three quadratic characters in this signature is $3 \cdot 7 \cdot 8 = 168$, which equals the conductor of the nebentypus of g. It seemed reasonable to guess by analogy that the product of the conductors of the three quadratic characters in α_p, namely $3 \cdot 5 \cdot 7 = 105$, should be the conductor of the nebentypus χ of the newform f that we were seeking. Since f has odd weight, χ is odd. The simplest odd character of conductor 105 is the quartic character ψ defined in (1.19). Thus we took $\chi = \psi$ as a first guess, and the evidence strongly suggests that this was the right choice.

As a first guess for the level N, we took $N = 105$, hoping that the level would equal the conductor of the nebentypus as was the case for the newform g on $\Gamma_0(168)$. However, for newforms f on $\Gamma_0(105)$, there were already small primes $p > 7$ for which $|\widehat{f}(p)|$ failed to equal $x(p)$. Our next guess was that the level equals 105 times a small prime factor. The levels $2 \cdot 105$ and $3 \cdot 105$ each failed, but the level $5 \cdot 105 = 525$ provided a happy ending. Indeed the Sage session in the Appendix shows the existence of a weight 3 newform f on $\Gamma_0(525)$ with nebentypus ψ and eigenfield $\mathbb{Q}(i, \sqrt{6}, \sqrt{14})$ such that $|\widehat{f}(p)| = x(p)$ for all p with $7 < p < 100$. As was noted above, this is powerful evidence for Conjecture 1.1.

3. Congruences for c_p

Let $p > 7$. It follows from [1, Theorem 2.1] that $S_7 \equiv -(\frac{p}{105}) \pmod 4$. Thus, by (1.17),

$$2 \nmid c_p. \tag{3.1}$$

Katz [7] observed that numerical evidence moreover suggests

$$5 \nmid c_p \tag{3.2}$$

and

$$7 \nmid c_p. \tag{3.3}$$

On the other hand, we conjecture that for every prime $q \notin \{2, 5, 7\}$, one has $q \mid c_p$ for infinitely many primes p.

In Theorem 3.1, we prove (3.3). In Theorem 3.2, we give an evaluation of c_p (mod 5) which in particular proves (3.2). In Theorem 3.3, we evaluate c_p (mod 3).

Our proofs will make use of the simple fact that

$$n \mid S_n, \quad \text{for prime } n. \tag{3.4}$$

To justify (3.4), note that

$$S_n = \sum_{a=0}^{p-1} \left(\sum_{x=1}^{p-1} \zeta_p^{x+a/x} \right)^n \equiv \sum_{x=1}^{p-1} \sum_{a=0}^{p-1} \zeta_p^{n(x+a/x)} \equiv 0 \pmod{n}.$$

THEOREM 3.1: *For each $p > 7$, we have $7 \nmid c_p$.*

Proof. By (1.17),

$$p^2 c_p \equiv S_7 + p^3 a_p + p \pmod{7}.$$

Since $7 \mid S_7$ by (3.4),

$$p^2 c_p \equiv \left(\frac{p}{7}\right) a_p + p \pmod{7}. \tag{3.5}$$

It remains to prove that

$$\left(\frac{p}{7}\right) a_p + p \not\equiv 0 \pmod{7}. \tag{3.6}$$

We may assume that $a_p \neq 0$, since otherwise (3.6) is clear. By (1.14), either

$$a_p = 10v^2 - 6u^2 \quad \text{with} \quad p = 3u^2 + 5v^2. \tag{3.7}$$

or

$$a_p = 2x^2 - 30y^2 \quad \text{with} \quad p = x^2 + 15y^2. \tag{3.8}$$

In the case (3.7),

$$\left(\frac{p}{7}\right) a_p + p = u^2 \left(3 - 6\left(\frac{p}{7}\right)\right) + v^2 \left(5 + 10\left(\frac{p}{7}\right)\right) \not\equiv 0 \pmod{7},$$

since $(-3 + 6(\frac{p}{7}))(5 + 10(\frac{p}{7}))$ is a nonsquare (mod 7). In the case (3.8),

$$\left(\frac{p}{7}\right) a_p + p = x^2 \left(1 + 2\left(\frac{p}{7}\right)\right) + y^2 \left(15 - 30\left(\frac{p}{7}\right)\right) \not\equiv 0 \pmod{7},$$

since $(1+2(\frac{p}{7}))(-15+30(\frac{p}{7}))$ is a nonsquare (mod 7). ∎

THEOREM 3.2: *For $p>7$,*

$$c_p \equiv p + p\left(\frac{p}{5}\right) + \left(\frac{p}{21}\right) \pmod 5.$$

In particular, $5 \nmid c_p$.

Proof. All congruences in this proof are modulo 5. By (1.17),

$$p^2 c_p \equiv S_7 - p^3 a_p + \left(\frac{p}{3}\right)p^4 + p^3 + p^2 - p.$$

Since $p^2 \equiv \left(\frac{p}{5}\right)$, we have

$$c_p \equiv \left(\frac{p}{5}\right)S_7 - pa_p + \left(\frac{p}{15}\right) + p + 1 - \left(\frac{p}{5}\right)p. \tag{3.9}$$

It remains to prove

$$a_p \equiv \left(\frac{p}{3}\right)p + \left(\frac{p}{5}\right)p \tag{3.10}$$

and

$$S_7 \equiv 2p + \left(\frac{p}{105}\right), \tag{3.11}$$

since the theorem follows from (3.9)–(3.11).

By (1.10) and (1.13),

$$p^2 a_p = S_5 - 4p^3\left(\frac{p}{3}\right) - 5p^2 - 4p. \tag{3.12}$$

Thus

$$a_p \equiv \left(\frac{p}{5}\right)S_5 + \left(\frac{p}{3}\right)p + \left(\frac{p}{5}\right)p. \tag{3.13}$$

This proves (3.10), since $5 \mid S_5$ by (3.4).

To prove (3.11), observe that

$$\begin{aligned} S_7 &= \sum_{a=0}^{p-1} K(a)^7 \equiv \sum_{a=0}^{p-1} K(a)^2 \sum_{x=1}^{p-1} \zeta_p^{5(x+a/x)} = \sum_{a=0}^{p-1} K(a)^2 K(25a) \\ &= \sum_{a=0}^{p-1} \sum_{x,y,z\neq 0} \zeta_p^{x+y+z+a(\frac{1}{x}+\frac{1}{y}+\frac{25}{z})} \\ &= p \sum_{\substack{x,y,z\neq 0 \\ \frac{1}{x}+\frac{1}{y}+\frac{25}{z}=0}} \zeta_p^{x+y+z} \\ &= p \sum_{\substack{x,y\neq 0 \\ x+y\neq 0}} \zeta_p^{x+y-25xy/(x+y)}. \end{aligned}$$

With the change of variables

$$r = x+y, \quad s = xy,$$

this becomes

$$S_7 \equiv p \sum_{r,s\neq 0} \zeta_p^{r-25s/r} \left\{1 + \left(\frac{r^2-4s}{p}\right)\right\} = p \sum_{r,s\neq 0} \zeta_p^{r(1-25s)} \left\{1 + \left(\frac{1-4s}{p}\right)\right\},$$

where in the last step we replaced s by sr^2. Replacing s by $(1-s)/4$, we obtain

$$\begin{aligned} S_7 &\equiv p \sum_{r\neq 0, s\neq 1} \zeta_p^{r(\frac{-21}{4}+\frac{25s}{4})} \left\{1 + \left(\frac{s}{p}\right)\right\} \\ &= 2p - p^2 + p \sum_{r,s} \zeta_p^{r(\frac{-21}{4}+\frac{25s}{4})} \left\{1 + \left(\frac{s}{p}\right)\right\} \\ &= 2p - p^2 + p^2 \left\{1 + \left(\frac{21}{p}\right)\right\} = 2p + p^2 \left(\frac{p}{21}\right) \equiv 2p + \left(\frac{p}{105}\right). \end{aligned}$$

This completes the proof of (3.11). ∎

THEOREM 3.3: *For* $p > 7$,

$$c_p \equiv 1 + \left(\frac{p}{3}\right) + \left(\frac{p}{35}\right) \pmod 3.$$

In particular, $3 \mid c_p$ *if and only if* $\left(\frac{p}{3}\right) = \left(\frac{p}{35}\right) = 1$.

Proof. By (1.17),

$$c_p \equiv S_7 + \left(\frac{p}{3}\right) + p + 1 \pmod 3.$$

Thus it remains to show that

$$S_7 \equiv \left(\frac{p}{35}\right) - p \pmod 3. \tag{3.14}$$

We have

$$S_7 \equiv \sum_{a=0}^{p-1} K(a)\Big(\sum_{x=1}^{p-1} \zeta_p^{3(x+a/x)}\Big)^2 = \sum_{a=0}^{p-1} K(a)K(9a)^2$$
$$= \sum_{a=0}^{p-1} \sum_{x,y,z\neq 0} \zeta_p^{x+y+z+a(\frac{9}{x}+\frac{9}{y}+\frac{1}{z})} \pmod 3.$$

The rest of the proof of (3.14) proceeds as in the proof of (3.11). ∎

4. Appendix

The Sage session below shows the existence of a weight 3 newform f on $\Gamma_0(525)$ with nebentypus ψ and eigenfield $\mathbb{Q}(i, \sqrt{6}, \sqrt{14})$, such that the p-th Fourier coefficients $b(p)$ of f satisfy (1.20) for $7 < p < 100$.

The session begins by setting `G` equal to the group of 16 Dirichlet characters modulo 525 of order dividing 4. The elements of `G` are placed into a list `X`, whose last element `Y` = `X[15]` equals the quartic character ψ of conductor 105 defined in (1.19).

Let `M` denote a modular symbols space of level 525, weight 3, with character ψ. This is a vector space of dimension 160 over $\mathbb{Q}(i)$. It has a "cuspidal subspace" `S` of dimension 148, and `S` in turn has a "new subspace" `N` of dimension 92. The space `N` is decomposed into 10 further subspaces, each invariant under Hecke operators, and `D` denotes a sorted list of these 10 subspaces. For more information about these spaces, see the Sage documentation at [10].

Our desired eigenfunction f lies in the fifth invariant subspace `D[4]`, and `f` gives the first 97 terms of its q-expansion. Finally, `parent(f)` tells us that the Fourier coefficients of our eigenfunction all lie in the eigenfield $\mathbb{Q}$(`zeta4`, `a`), where `zeta4` = i and `a` is a zero of

$$x^4 + (4i+4)x^3 + 20ix^2 + (24i-24)x - 120.$$

We may take `a` = $\sqrt{7}z^7 - z(\sqrt{2}+\sqrt{3})$, where $z = \exp(2\pi i/8)$. Then the eigenfield is easily seen to be $\mathbb{Q}(i, \sqrt{6}, \sqrt{14})$. Simplifying the q-expansion `f`, we obtain the Fourier coefficients of f corresponding to primes $7 < p < 100$ given in Table 4.1.

p	11	13	17	19	23	29	31
$b(p)$	0	$2\sqrt{7}z^7$	$18\sqrt{2}z^3$	$8\sqrt{6}$	$4\sqrt{3}z^3$	$6\sqrt{14}$	$10\sqrt{6}z^2$

p	37	41	43	47	53	59	61
$b(p)$	$2\sqrt{42}z^3$	$12\sqrt{21}z^4$	$8\sqrt{42}z^5$	$12\sqrt{2}z^7$	$36\sqrt{3}z^7$	$20\sqrt{21}z^2$	$30\sqrt{6}z^2$

p	67	71	73	79	83	89	97
$b(p)$	$12\sqrt{42}z^3$	$30\sqrt{14}z^6$	$38\sqrt{7}z^3$	$50z^2$	$78\sqrt{2}z$	$20\sqrt{21}z^6$	$38\sqrt{7}z$

Comparison of Tables 2.1 and 4.1 shows that $x(p) = |b(p)|$, and so (1.20) holds for $7 < p < 100$.

SAGE SESSION

```
----------------------------------------------------------------------
| Sage Version 4.1, Release Date: 2009-07-09                         |
| Type notebook() for the GUI, and license() for information.        |
----------------------------------------------------------------------

sage: G=DirichletGroup(525,CyclotomicField(4));X=G.list();
sage: Y=X[15];Y;Y.conductor();Y.order()
[-1, -zeta4, -1]
105
4

sage: M=ModularSymbols(Y,3,sign=1);M
Modular Symbols space of dimension 160 and level 525, weight 3, character
[-1, -zeta4, -1], sign 1, over Cyclotomic Field of order 4 and degree 2

sage: S=M.cuspidal_subspace();S
Modular Symbols subspace of dimension 148 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2

sage: N=S.new_subspace();N
Modular Symbols subspace of dimension 92 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2

sage: D=N.decomposition();D
[
Modular Symbols subspace of dimension 2 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2,
Modular Symbols subspace of dimension 2 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2,
```

```
Modular Symbols subspace of dimension 4 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2,
Modular Symbols subspace of dimension 4 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2,
Modular Symbols subspace of dimension 4 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2,
Modular Symbols subspace of dimension 4 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2,
Modular Symbols subspace of dimension 8 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2,
Modular Symbols subspace of dimension 8 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2,
Modular Symbols subspace of dimension 16 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2,
Modular Symbols subspace of dimension 40 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2
]
sage: f=D[4].q_eigenform(98,"a");f
q + (1/20*zeta4*a^3 + (3/20*zeta4 - 3/20)*a^2 - 1/5*a)*q^2 + (-1/20*zeta4*a^3 +
(-3/20*zeta4 + 3/20)*a^2 + 6/5*a)*q^3 - zeta4*q^4 + ((-1/20*zeta4 + 1/20)*a^3 +
4/5*a^2 + (6/5*zeta4 + 6/5)*a + 3*zeta4)*q^6 + (-1/20*a^3 + (-13/20*zeta4 +
7/20)*a^2 + (-6/5*zeta4 + 2)*a + 2*zeta4 + 4)*q^7 + (1/4*a^3 + (3/4*zeta4 +
3/4)*a^2 + zeta4*a)*q^8 + ((1/10*zeta4 - 1/10)*a^3 - 3/5*a^2 + (-12/5*zeta4 -
12/5)*a - 9*zeta4)*q^9 + (-1/20*a^3 + (-3/20*zeta4 - 3/20)*a^2 -
6/5*zeta4*a)*q^12 + (-1/10*zeta4*a^3 + (-3/10*zeta4 + 3/10)*a^2 + 12/5*a +
2*zeta4 + 2)*q^13 + ((-3/20*zeta4 + 3/20)*a^3 + (-1/2*zeta4 + 9/10)*a^2 +
(13/5*zeta4 + 23/5)*a + 6*zeta4 + 3)*q^14 + 11*q^16 + (18*zeta4 - 18)*q^17 +
(1/4*a^3 + (-1/4*zeta4 - 1/4)*a^2 - 3*zeta4*a - 6*zeta4 + 6)*q^18 + ((-2/5*zeta4 -
2/5)*a^3 - 12/5*zeta4*a^2 + (-8/5*zeta4 + 8/5)*a)*q^19 + ((-3/10*zeta4 + 2/5)*a^3 +
(3/10*zeta4 + 11/10)*a^2 + (3/5*zeta4 - 9/5)*a - 6*zeta4 - 9)*q^21 + (1/5*a^3 +
(3/5*zeta4 + 3/5)*a^2 + 4/5*zeta4*a)*q^23 + ((-1/4*zeta4 - 1/4)*a^3 - 4*zeta4*a^2 +
(-6*zeta4 + 6)*a + 15)*q^24 + (a^2 + (2*zeta4 + 2)*a + 6*zeta4)*q^26 + (-1/20*a^3 +
(-3/20*zeta4 - 3/20)*a^2 - 6/5*zeta4*a + 18*zeta4 - 18)*q^27 + (1/20*zeta4*a^3 +
(-7/20*zeta4 - 13/20)*a^2 + (-2*zeta4 - 6/5)*a - 4*zeta4 + 2)*q^28 + ((-3/10*zeta4 +
3/10)*a^3 + 9/5*a^2 + (36/5*zeta4 + 36/5)*a + 12*zeta4)*q^29+ ((-1/2*zeta4 +
1/2)*a^3 + 3*a^2 + (2*zeta4 + 2)*a)*q^31 + (-9/20*zeta4*a^3 + (-27/20*zeta4 +
27/20)*a^2 + 9/5*a)*q^32 + ((-9/10*zeta4 - 9/10)*a^3 - 27/5*zeta4*a^2 +
(-18/5*zeta4 + 18/5)*a)*q^34 + ((1/10*zeta4 + 1/10)*a^3 + 3/5*zeta4*a^2 + (12/5*zeta4 -
12/5)*a - 9)*q^36 + ((-zeta4 + 1)*a^2 + 4*a + 6*zeta4 + 6)*q^37 + (-24*zeta4 -
24)*q^38 + ((1/10*zeta4 - 1/10)*a^3 - 3/5*a^2 + (-12/5*zeta4 - 12/5)*a -
18*zeta4)*q^39 + (6*a^2 + (12*zeta4 + 12)*a + 36*zeta4)*q^41 + ((-7/20*zeta4 -
3/10)*a^3 + (-29/20*zeta4 - 7/20)*a^2 + (-36/5*zeta4 - 3/5)*a - 30*zeta4 + 24)*q^42 +
((4*zeta4 + 4)*a^2 + 16*zeta4*a + 24*zeta4 - 24)*q^43 + 12*q^46 + (-12*zeta4 +
12)*q^47 + (-11/20*zeta4*a^3 + (-33/20*zeta4 + 33/20)*a^2 + 66/5*a)*q^48 +
((-7/10*zeta4 - 7/10)*a^3 - 21/5*zeta4*a^2 + (-14/5*zeta4 + 14/5)*a - 35*zeta4)*q^49 +
((9/10*zeta4 + 9/10)*a^3 + 27/5*zeta4*a^2 + (108/5*zeta4 - 108/5)*a)*q^51 +
(-1/10*a^3 + (-3/10*zeta4 - 3/10)*a^2 - 12/5*zeta4*a - 2*zeta4 + 2)*q^52 + (-9/5*a^3 +
```

```
(-27/5*zeta4 - 27/5)*a^2 - 36/5*zeta4*a)*q^53 + ((-19/20*zeta4 - 19/20)*a^3 -
31/5*zeta4*a^2 + (-24/5*zeta4 + 24/5)*a + 3)*q^54 + ((-3/4*zeta4 - 3/4)*a^3 +
(-9/2*zeta4 - 5/2)*a^2 + (-23*zeta4 + 13)*a - 15*zeta4 + 30)*q^56 + (4/5*zeta4*a^3 +
(32/5*zeta4 - 32/5)*a^2 - 96/5*a - 24*zeta4 - 24)*q^57 + ((3*zeta4 + 3)*a^2 +
12*zeta4*a + 18*zeta4 - 18)*q^58 + (-10*zeta4*a^2 + (-20*zeta4 + 20)*a + 60)*q^59 +
((-3/2*zeta4 + 3/2)*a^3 + 9*a^2 + (6*zeta4 + 6)*a)*q^61 + (-30*zeta4 + 30)*q^62 +
((1/4*zeta4 - 7/5)*a^3 + (-119/20*zeta4 - 149/20)*a^2 + (-78/5*zeta4 - 6)*a -
6*zeta4 + 24)*q^63 - 71*zeta4*q^64 + ((-6*zeta4 + 6)*a^2 + 24*a + 36*zeta4 + 36)*q^67 +
(18*zeta4 + 18)*q^68 + ((-1/5*zeta4 - 1/5)*a^3 - 16/5*zeta4*a^2 + (-24/5*zeta4 +
24/5)*a + 12)*q^69 + ((-3/2*zeta4 - 3/2)*a^3 - 9*zeta4*a^2 + (-36*zeta4 + 36)*a +
60)*q^71 + (-5/4*zeta4*a^3 + (5/4*zeta4 - 5/4)*a^2 - 15*a - 30*zeta4 - 30)*q^72 +
(19/10*zeta4*a^3 + (57/10*zeta4 - 57/10)*a^2 - 228/5*a - 38*zeta4 - 38)*q^73 +
((-3/10*zeta4 + 3/10)*a^3 + 9/5*a^2 + (36/5*zeta4 + 36/5)*a + 12*zeta4)*q^74 +
((2/5*zeta4 - 2/5)*a^3 - 12/5*a^2 + (-8/5*zeta4 - 8/5)*a)*q^76+ (7/10*a^3 +
(11/10*zeta4 + 11/10)*a^2 - 6/5*zeta4*a - 6*zeta4 + 6)*q^78 + 50*zeta4*q^79 +
((zeta4 + 1)*a^3 + 6*zeta4*a^2 + (24*zeta4 - 24)*a - 9)*q^81 + (9/5*a^3 + (27/5*zeta4 +
27/5)*a^2 + 216/5*zeta4*a + 36*zeta4 - 36)*q^82 + (78*zeta4 + 78)*q^83 + ((-2/5*zeta4 -
3/10)*a^3 + (-11/10*zeta4 + 3/10)*a^2 + (9/5*zeta4 + 3/5)*a + 9*zeta4 - 6)*q^84 +
((6/5*zeta4 + 6/5)*a^3 + 36/5*zeta4*a^2 + (144/5*zeta4 - 144/5)*a - 48)*q^86 +
(-3/5*a^3 + (-9/5*zeta4 - 9/5)*a^2 - 72/5*zeta4*a - 54*zeta4 + 54)*q^87 +
(10*zeta4*a^2 + (20*zeta4 - 20)*a - 60)*q^89 + ((-7/10*zeta4 + 7/10)*a^3 + 21/5*a^2 +
(14/5*zeta4 + 14/5)*a - 14)*q^91 + (-1/5*zeta4*a^3 + (-3/5*zeta4 + 3/5)*a^2 +
4/5*a)*q^92 + (-a^3 + (-8*zeta4 - 8)*a^2 - 24*zeta4*a - 30*zeta4 + 30)*q^93 +
((3/5*zeta4 + 3/5)*a^3 + 18/5*zeta4*a^2 + (12/5*zeta4 - 12/5)*a)*q^94 + ((9/20*zeta4 -
9/20)*a^3 - 36/5*a^2 + (-54/5*zeta4 - 54/5)*a - 27*zeta4)*q^96 + (19/10*a^3 +
(57/10*zeta4 + 57/10)*a^2 + 228/5*zeta4*a + 38*zeta4 - 38)*q^97 + O(q^98)

sage: parent(f)
Power Series Ring in q over Number Field in a with defining polynomial
x^4 + (4*zeta4 + 4)*x^3 + 20*zeta4*x^2 + (24*zeta4 - 24)*x - 120 over its base field
```

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