



# Several complementary inequalities to inequalities of Hermite-Hadamard type for $s$ -convex functions

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## Abstract

In this paper, we establish some new Hermite-Hadamard inequalities for  $s$ -convex functions via fractional integrals. Some Hermite-Hadamard type inequalities for products of two convex and  $s$ -convex functions via Riemann-Liouville integrals are also established. ©2016 All rights reserved.

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## 1. Introduction and preliminaries

If  $f : I \rightarrow R$  is a convex function on the interval  $I$ , then for any  $a, b \in I$  with  $a \neq b$  we have the following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality.

**Definition 1.1** ([7]).  $f : I \subset [0, \infty) \rightarrow [0, \infty)$  is said to be  $s$ -convex in the second sense, or that  $f$  belongs to the class  $K_s^2$ , if the inequality

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y) \quad (1.2)$$

holds for all  $x, y \in I$ ,  $\alpha \in [0, 1]$  and for some fixed  $s \in (0, 1]$ .

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It can be easily seen that for  $s = 1$ ,  $s$ -convexity reduces to ordinary convexity of functions defined on  $[0, \infty)$ .

In [6], Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the  $s$ -convex functions.

**Theorem 1.2** ([6]). *Suppose that  $f : [0, \infty) \rightarrow [0, \infty)$  is an  $s$ -convex function in the second sense, where  $s \in (0, 1)$  and let  $a, b \in [0, \infty)$ ,  $a < b$ . If  $f \in L[a, b]$ , then the following inequality holds*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{s+1}. \tag{1.3}$$

In [8], İşcan gave definition of harmonically convexity as follows:

**Definition 1.3** ([8]). Let  $I \subseteq \mathbb{R} \setminus \{0\}$  be a real interval. A function  $f : I \rightarrow \mathbb{R}$  is said to be harmonically convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.4}$$

for all  $x, y \in I$  and  $t \in [0, 1]$ . If the inequality in (1.4) is reversed, then  $f$  is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds.

**Theorem 1.4** ([8]). *Let  $f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  be a harmonically convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L(a, b)$  then we have*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}. \tag{1.5}$$

The definition of harmonically  $s$ -convex functions is proposed by İşcan in [9].

**Definition 1.5** ([9]). Let  $I \subseteq \mathbb{R} \setminus \{0\}$  be a real interval. A function  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  is said to be harmonically  $s$ -convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq t^s f(y) + (1-t)^s f(x) \tag{1.6}$$

for all  $x, y \in I$ ,  $t \in [0, 1]$  and for some fixed  $s \in (0, 1]$ . If the inequality in (1.6) is reversed, then  $f$  is said to be harmonically  $s$ -concave.

The following Hermite-Hadamard inequality for harmonically  $s$ -convex functions holds.

**Theorem 1.6** ([9]). *Let  $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$  be a harmonically  $s$ -convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L(a, b)$  then we have*

$$2^{s-1} f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{s+1}. \tag{1.7}$$

In [4], Chen and Wu discussed Fejér and Hermite-Hadamard type inequalities for Harmonically convex functions and presented the following inequality:

**Theorem 1.7** ([4]). *Let  $f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  be a harmonically convex function and  $a, b \in I$  with  $a < b$ . If  $f \in L(a, b)$  then we have*

$$f\left(\frac{2ab}{a+b}\right) \int_a^b \frac{p(x)}{x^2} dx \leq \int_a^b \frac{f(x)}{x^2} p(x) dx \leq \frac{f(a) + f(b)}{2} \int_a^b \frac{p(x)}{x^2} dx, \tag{1.8}$$

where  $p : [a, b] \rightarrow \mathbb{R}$  is nonnegative, integrable and satisfies

$$p\left(\frac{ab}{x}\right) = p\left(\frac{ab}{a+b-x}\right). \tag{1.9}$$

Some refinements of the Hermite-Hadamard inequality for convex functions have been extensively investigated by a number of authors (e.g., [1], [2], [3], [4] and [5]).

In [11], Pachpatte established two new Hermite-Hadamard type inequalities for products of convex functions as follows.

**Theorem 1.8.** *Let  $f$  and  $g$  be real-valued, nonnegative and convex functions on  $[a, b]$ . Then*

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a, b) + \frac{1}{6}N(a, b), \tag{1.10}$$

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{6}M(a, b) + \frac{1}{3}N(a, b), \tag{1.11}$$

where  $M(a, b) = f(a)g(a) + f(b)g(b)$  and  $N(a, b) = f(a)g(b) + f(b)g(a)$ .

Some Hermite-Hadamard type inequalities for products of two convex and  $s$ -convex functions are proposed by Kirmaci *et al.* in [10].

**Theorem 1.9** ([10]). *Let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $a, b \in [0, \infty)$ ,  $a < b$ , be functions such that  $g, fg \in L[a, b]$ . If  $f$  is convex and nonnegative on  $[a, b]$  and  $g$  is  $s$ -convex on  $[a, b]$  for some fixed  $s \in (0, 1]$ , then*

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{s+2}M(a, b) + \frac{1}{(s+1)(s+2)}N(a, b), \tag{1.12}$$

where  $M(a, b) = f(a)g(a) + f(b)g(b)$ ,  $N(a, b) = f(a)g(b) + f(b)g(a)$ .

**Theorem 1.10** ([10]). *Let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $a, b \in [0, \infty)$ ,  $a < b$ , be functions such that  $f, g$  and  $fg \in L[a, b]$ . If  $f$  is  $s_1$ -convex and  $g$  is  $s_2$ -convex on  $[a, b]$  for some fixed  $s_1, s_2 \in (0, 1]$ , then*

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{s_1+s_2+1}M(a, b) + \beta(s_2+1, s_1+1)N(a, b), \tag{1.13}$$

where  $M(a, b) = f(a)g(a) + f(b)g(b)$ ,  $N(a, b) = f(a)g(b) + f(b)g(a)$ .

**Theorem 1.11** ([10]). *Let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $a, b \in [0, \infty)$ ,  $a < b$ , be functions such that  $fg \in L[a, b]$ . If  $f$  is convex and nonnegative on  $[a, b]$  and  $g$  is  $s$ -convex on  $[a, b]$  for some fixed  $s \in (0, 1]$ , then*

$$2^s f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{(s+1)(s+2)}M(a, b) + \frac{1}{s+2}N(a, b), \tag{1.14}$$

where  $M(a, b) = f(a)g(a) + f(b)g(b)$ ,  $N(a, b) = f(a)g(b) + f(b)g(a)$ .

In [5], Chen and Wu discussed Hermite-Hadamard type inequalities for Harmonically  $s$ -convex functions and obtained the following result:

**Theorem 1.12** ([5]). *Let  $f, g : [a, b] \rightarrow [0, \infty)$ ,  $a, b \in (0, \infty)$ ,  $a < b$ , be functions such that  $f, g, fg \in L[a, b]$ . If  $f$  is harmonically  $s_1$ -convex and  $g$  is harmonically  $s_2$ -convex on  $[a, b]$  for some fixed  $s_1, s_2 \in (0, 1]$ , then*

$$\frac{ab}{b-a} \int_a^b \frac{f(x)g(x)}{x^2} dx \leq \frac{1}{1+s_1+s_2}M(a, b) + \frac{\Gamma(1+s_1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)}N(a, b),$$

where  $M(a, b) = f(a)g(a) + f(b)g(b)$ ,  $N(a, b) = f(a)g(b) + f(b)g(a)$ .

**Theorem 1.13** ([5]). *Let  $f, g : [a, b] \rightarrow [0, \infty)$ ,  $a, b \in (0, \infty)$ ,  $a < b$ , be functions such that  $f, g, fg \in L[a, b]$ . If  $f$  is harmonically  $s_1$ -convex and  $g$  is harmonically  $s_2$ -convex on  $[a, b]$  for some fixed  $s_1, s_2 \in (0, 1]$ , then*

$$2^{s_1+s_2-1} f\left(\frac{2ab}{a+b}\right)g\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)g(x)}{x^2} dx + M(a,b) \frac{\Gamma(1+s_1)\Gamma(s_2+1)}{\Gamma(s_1+s_2+2)} + \frac{1}{s_2+s_1+1} N(a,b),$$

where  $M(a,b) = f(a)g(a) + f(b)g(b)$ ,  $N(a,b) = f(a)g(b) + f(b)g(a)$ .

Sarikaya *et al.* [12] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

**Theorem 1.14** ([12]). *Let  $f : [a, b] \rightarrow R$  be a positive function with  $a < b$  and  $f \in L[a, b]$ . If  $f$  is a convex function on  $[a, b]$ , then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2}, \tag{1.15}$$

with  $\alpha > 0$ .

We remark that the symbol  $J_{a^+}^\alpha$  and  $J_{b^-}^\alpha f$  denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order  $\alpha \geq 0$  with  $a \geq 0$  which are defined by

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a,$$

and

$$J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b,$$

respectively. Here,  $\Gamma(\alpha)$  is the Gamma function defined by  $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ .

Chen and Wu [3] investigated the Hermite-Hadamard type inequalities for products of two  $h$ -convex functions and established the following inequality:

**Theorem 1.15.** *Let  $f \in SX(h_1, I)$ ,  $g \in SX(h_2, I)$ ,  $a, b \in I$ ,  $a < b$ , be functions such that  $fg \in L[a, b]$ , and  $h_1 h_2 \in L_1[0, 1]$ , then the following inequality for fractional integrals holds:*

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a^+}^\alpha f(b)g(b) + J_{b^-}^\alpha f(a)g(a)] \leq M(a,b) \int_0^1 t^{\alpha-1} [h_1(t)h_2(t) + h_1(1-t)h_2(1-t)] dt + N(a,b) \int_0^1 t^{\alpha-1} [h_1(t)h_2(1-t) + h_1(1-t)h_2(t)] dt,$$

where  $M(a,b) = f(a)g(a) + f(b)g(b)$ ,  $N(a,b) = f(a)g(b) + f(b)g(a)$ .

**Theorem 1.16.** *Let  $f \in SX(h_1, I)$ ,  $g \in SX(h_2, I)$ ,  $a, b \in I$ ,  $a < b$ , be functions such that  $fg \in L_1[a, b]$ , and  $h_1 h_2 \in L[0, 1]$ , then the following inequality for fractional integrals holds:*

$$\frac{1}{\alpha h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a^+}^\alpha f(b)g(b) + J_{b^-}^\alpha f(a)g(a)] + M(a,b) \int_0^1 t^{\alpha-1} [h_1(t)h_2(1-t) + h_1(1-t)h_2(t)] dt + N(a,b) \int_0^1 t^{\alpha-1} [h_1(t)h_2(t) + h_1(1-t)h_2(1-t)] dt,$$

where  $M(a,b) = f(a)g(a) + f(b)g(b)$ ,  $N(a,b) = f(a)g(b) + f(b)g(a)$ .

In [13], Set *et al.* established the following Hermite-Hadamard inequalities for  $s$ -convex functions in the second sense via fractional integrals.

**Theorem 1.17** ([13]). *Let  $f : [a, b] \rightarrow R$  be a positive function with  $0 \leq a < b$  and  $f \in L[a, b]$ . If  $f$  is  $s$ -convex in the second sense on  $[a, b]$  for some fixed  $s \in (0, 1]$ , then the following inequalities for fractional integrals with  $\alpha > 0$  hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \leq \left[\frac{1}{\alpha+s} + \beta(\alpha, s+1)\right] \frac{f(a) + f(b)}{2}, \tag{1.16}$$

where  $\beta$  is Euler Beta function.

In this paper, we obtain some new Hermite-Hadamard type inequalities for  $s$ -convex functions via Riemann-Liouville fractional integrals. Several Hermite-Hadamard type inequalities for products of two convex and  $s$ -convex functions are also established.

## 2. Main results

In order to prove our main theorems, we need the following fundamental integral identity including the second order derivatives of a given function via Riemann-Liouville integrals which was established by Wang *et al.* in [14].

**Lemma 2.1** ([14]). *Let  $f : [a, b] \rightarrow R$  be a twice differentiable mapping on  $(a, b)$  with  $a < b$ . If  $f'' \in L[a, b]$ , then the following equality for fractional integrals hold:*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a) + f(b)}{2} = \frac{(b-a)^2}{2} \int_0^1 \frac{(1-t)^{\alpha+1} + t^{\alpha+1} - 1}{\alpha+1} f''(ta + (1-t)b) dt. \tag{2.1}$$

**Theorem 2.2.** *Let  $f : [a, b] \rightarrow R$  be a twice differentiable mapping on  $(a, b)$  with  $a < b$  such that  $f'' \in L[a, b]$ . If  $|f''|$  is  $s$ -convex on  $[a, b]$ , then the following inequality holds:*

$$\begin{aligned} & \left| \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a) + f(b)}{2} \right| \\ & \leq \frac{(b-a)^2}{2} \left( \frac{1}{s+1} - \frac{1}{\alpha+s+2} - \beta(\alpha+2, s+1) \right) \frac{|f''(a)| + |f''(b)|}{\alpha+1}. \end{aligned}$$

*Proof.* From lemma 2.1, we get

$$\begin{aligned} & \left| \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a) + f(b)}{2} \right| \\ & \leq \frac{(b-a)^2}{2} \int_0^1 \left| \frac{(1-t)^{\alpha+1} + t^{\alpha+1} - 1}{\alpha+1} \right| |f''(ta + (1-t)b)| dt. \end{aligned} \tag{2.2}$$

Because  $(1-t)^{\alpha+1} + t^{\alpha+1} \leq 1$  for any  $t \in [0, 1]$  and  $|f''|$  is  $s$ -convex on  $[a, b]$ , we get

$$\begin{aligned} & \int_0^1 \left| \frac{(1-t)^{\alpha+1} + t^{\alpha+1} - 1}{\alpha+1} \right| |f''(ta + (1-t)b)| dt \\ & \leq \int_0^1 \frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \left( t^s |f''(a)| + (1-t)^s |f''(b)| \right) dt \\ & = \left( \frac{1}{s+1} - \frac{1}{\alpha+s+2} - \beta(\alpha+2, s+1) \right) \frac{|f''(a)| + |f''(b)|}{\alpha+1}. \end{aligned} \tag{2.3}$$

Now by (2.2), (2.3), we can obtain the desired result. □

**Theorem 2.3.** *Let  $f : [a, b] \rightarrow R$  be a twice differentiable mapping on  $(a, b)$  with  $a < b$  such that  $f'' \in L[a, b]$ . If  $|f''|^q$  is  $s$ -convex on  $[a, b]$  with  $q \geq 1$ , then the following inequality holds:*

$$\begin{aligned} & \left| \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a) + f(b)}{2} \right| \\ & \leq \frac{(b-a)^2}{2(\alpha+1)} \left( \frac{\alpha}{\alpha+2} \right)^{1-\frac{1}{q}} \times \left( \frac{1}{s+1} - \beta(\alpha+2, s+1) - \frac{1}{\alpha+s+2} \right)^{\frac{1}{q}} \left( |f''(a)|^q + |f''(b)|^q \right)^{\frac{1}{q}}. \end{aligned}$$

*Proof.* From lemma 2.1 and the power mean inequality for  $q$ , we have

$$\begin{aligned} & \left| \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a) + f(b)}{2} \right| \\ & \leq \frac{(b - a)^2}{2(\alpha + 1)} \left[ \int_0^1 (1 - (1 - t)^{\alpha+1} - t^{\alpha+1}) dt \right]^{1 - \frac{1}{q}} \\ & \quad \times \left( \int_0^1 (1 - (1 - t)^{\alpha+1} - t^{\alpha+1}) |f''(ta + (1 - t)b)|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{(b - a)^2}{2(\alpha + 1)} \left[ \int_0^1 (1 - (1 - t)^{\alpha+1} - t^{\alpha+1}) dt \right]^{1 - \frac{1}{q}} \\ & \quad \times \left( \int_0^1 (1 - (1 - t)^{\alpha+1} - t^{\alpha+1}) (t^s |f''(a)|^q + (1 - t)^s |f''(b)|^q) dt \right)^{\frac{1}{q}} \\ & = \frac{(b - a)^2}{2(\alpha + 1)} \left( \frac{\alpha}{\alpha + 2} \right)^{1 - \frac{1}{q}} \left( \frac{1}{s + 1} - \beta(\alpha + 2, s + 1) - \frac{1}{\alpha + s + 2} \right)^{\frac{1}{q}} \left( |f''(a)|^q + |f''(b)|^q \right)^{\frac{1}{q}}, \end{aligned}$$

which completes the proof. □

**Theorem 2.4.** Let  $f : [a, b] \rightarrow R$  be a twice differentiable mapping on  $(a, b)$  with  $a < b$  such that  $f'' \in L[a, b]$ . If  $|f''|^q$  is  $s$ -convex on  $[a, b]$  with  $q > 1$ , then the following inequality holds:

$$\left| \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b - a)^2}{2(\alpha + 1)} \left( 1 - \frac{2}{p(\alpha + 1) + 1} \right)^{\frac{1}{p}} \left( \frac{|f''(a)|^q + |f''(b)|^q}{s + 1} \right)^{\frac{1}{q}},$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof.* From lemma 2.1 and using the Hölder inequality, we have

$$\begin{aligned} & \left| \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - \frac{f(a) + f(b)}{2} \right| \\ & \leq \frac{(b - a)^2}{2} \int_0^1 \left( \frac{1 - (1 - t)^{\alpha+1} - t^{\alpha+1}}{\alpha + 1} \right) |f''(ta + (1 - t)b)| dt \\ & \leq \frac{(b - a)^2}{2(\alpha + 1)} \left( \int_0^1 (1 - (1 - t)^{\alpha+1} - t^{\alpha+1})^p dt \right)^{\frac{1}{p}} \left( \int_0^1 |f''(ta + (1 - t)b)|^q dt \right)^{\frac{1}{q}} \tag{2.4} \\ & \leq \frac{(b - a)^2}{2(\alpha + 1)} \left( \int_0^1 (1 - (1 - t)^{p(\alpha+1)} - t^{p(\alpha+1)}) dt \right)^{\frac{1}{p}} \left( \int_0^1 |f''(ta + (1 - t)b)|^q dt \right)^{\frac{1}{q}} \\ & = \frac{(b - a)^2}{2(\alpha + 1)} \left( 1 - \frac{2}{p(\alpha + 1) + 1} \right)^{\frac{1}{p}} \left( \int_0^1 |f''(ta + (1 - t)b)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Because  $|f''|^q$  is  $s$ -convex, we have

$$\int_0^1 |f''(ta + (1 - t)b)|^q dt \leq \frac{|f''(a)|^q + |f''(b)|^q}{s + 1}. \tag{2.5}$$

Here we use

$$(1 - (1 - t)^{\alpha+1} - t^{\alpha+1})^p \leq 1 - (1 - t)^{p(\alpha+1)} - t^{p(\alpha+1)}$$

for any  $t \in [0, 1]$ , which follows from

$$(A - B)^p \leq A^p - B^p$$

for any  $A > B \geq 0$  and  $p \geq 1$ . From (2.4) and (2.5), we complete the proof. □

We note that the Beta and Gamma functions are defined, respectively, as follows

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad x > 0,$$

$$\beta(x, y) = \int_0^1 (1-t)^{y-1}t^{x-1}dt, \quad x > 0, \quad y > 0,$$

and

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

**Theorem 2.5.** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $a, b \in [0, \infty)$ ,  $a < b$ , be functions such that  $g, fg \in L[a, b]$ . If  $f$  is convex and nonnegative and  $g$  is  $s$ -convex on  $[a, b]$  for some fixed  $s \in (0, 1]$ , then the following inequality for fractional integrals holds:*

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a^+}^\alpha f(b)g(b) + J_{b^-}^\alpha f(a)g(a)]$$

$$\leq \left( \frac{1}{\alpha + s + 1} + \beta(\alpha, s + 2) \right) M(a, b) + \left( \beta(\alpha + 1, s + 1) + \frac{1}{(\alpha + s)(\alpha + s + 1)} \right) N(a, b),$$

where  $M(a, b) = f(a)g(a) + f(b)g(b)$ ,  $N(a, b) = f(a)g(b) + f(b)g(a)$ .

*Proof.* Since  $f$  is convex and  $g$  is  $s$ -convex on  $[a, b]$ , then for  $t \in [0, 1]$  we get

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b), \tag{2.6}$$

and

$$g(ta + (1-t)b) \leq t^s g(a) + (1-t)^s g(b). \tag{2.7}$$

From (2.6) and (2.7), we get

$$f(ta + (1-t)b)g(ta + (1-t)b) \leq t^{s+1} f(a)g(a) + (1-t)^{s+1} f(b)g(b) + t(1-t)^s f(a)g(b) + (1-t)t^s f(b)g(a).$$

Similarly, we have

$$f((1-t)a + tb)g((1-t)a + tb) \leq (1-t)^{s+1} f(a)g(a) + t^{s+1} f(b)g(b) + (1-t)t^s f(a)g(b) + t(1-t)^s f(b)g(a).$$

So

$$f(ta + (1-t)b)g(ta + (1-t)b) + f((1-t)a + tb)g((1-t)a + tb)$$

$$\leq (t^{s+1} + (1-t)^{s+1})[f(a)g(a) + f(b)g(b)]$$

$$+ (t(1-t)^s + (1-t)t^s)[f(a)g(b) + f(b)g(a)].$$

Multiplying both sides of above inequality by  $t^{\alpha-1}$ , then integrating the resulting inequality with respect to  $t$  over  $[0, 1]$ , we obtain

$$\int_0^1 t^{\alpha-1} f(ta + (1-t)b)g(ta + (1-t)b)dt + \int_0^1 t^{\alpha-1} f((1-t)a + tb)g((1-t)a + tb)dt$$

$$= \int_b^a \left( \frac{b-u}{b-a} \right)^{\alpha-1} f(u)g(u) \frac{du}{a-b} + \int_a^b \left( \frac{v-a}{b-a} \right)^{\alpha-1} f(v)g(v) \frac{dv}{b-a}$$

$$= \frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a^+}^\alpha f(b)g(b) + J_{b^-}^\alpha f(a)g(a)]$$

$$\leq [f(a)g(a) + f(b)g(b)] \int_0^1 t^{\alpha-1} (t^{s+1} + (1-t)^{s+1}) dt$$

$$\begin{aligned}
 &+ [f(a)g(b) + f(b)g(a)] \int_0^1 t^{\alpha-1}(t(1-t)^s + (1-t)t^s) dt \\
 &= \left( \frac{1}{\alpha + s + 1} + \beta(\alpha, s + 2) \right) [f(a)g(a) + f(b)g(b)] \\
 &\quad + \left( \beta(\alpha + 1, s + 1) + \frac{1}{(\alpha + s)(\alpha + s + 1)} \right) [f(a)g(b) + f(b)g(a)] \\
 &= \left( \frac{1}{\alpha + s + 1} + \beta(\alpha, s + 2) \right) M(a, b) + \left( \beta(\alpha + 1, s + 1) + \frac{1}{(\alpha + s)(\alpha + s + 1)} \right) N(a, b).
 \end{aligned}$$

So

$$\begin{aligned}
 &\frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a^+}^\alpha f(b)g(b) + J_{b^-}^\alpha f(a)g(a)] \\
 &\leq \left( \frac{1}{\alpha + s + 1} + \beta(\alpha, s + 2) \right) M(a, b) + \left( \beta(\alpha + 1, s + 1) + \frac{1}{(\alpha + s)(\alpha + s + 1)} \right) N(a, b),
 \end{aligned}$$

which completes the proof. □

*Remark 2.6.* Taking  $\alpha = 1$  in Theorem 2.5, we have

$$\begin{aligned}
 \frac{1}{b-a} \int_a^b f(x)g(x) dx &\leq \left( \frac{1}{s+2} + \beta(1, s+2) \right) \frac{M(a, b)}{2} + \left( \beta(2, s+1) + \frac{1}{(1+s)(s+2)} \right) \frac{N(a, b)}{2} \\
 &= \frac{1}{s+2} M(a, b) + \frac{1}{(1+s)(s+2)} N(a, b)
 \end{aligned}$$

which is the result of (1.12).

*Remark 2.7.* Choosing  $f(x) = 1$  for all  $x \in [a, b]$  in Theorem 2.5 gives

$$\begin{aligned}
 \frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)] &\leq \left( \frac{1}{\alpha + s + 1} + \beta(\alpha, s + 2) + \beta(\alpha + 1, s + 1) \right. \\
 &\quad \left. + \frac{1}{(\alpha + s)(\alpha + s + 1)} \right) [g(a) + g(b)] \\
 &= \left( \frac{1}{\alpha + s} + \beta(\alpha, s + 1) \right) [g(a) + g(b)],
 \end{aligned}$$

which is the right hand side of (1.16).

**Theorem 2.8.** Let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $a, b \in [0, \infty)$ ,  $a < b$ , be functions such that  $f, g, fg \in L[a, b]$ . If  $f$  is  $s_1$ -convex and  $g$  is  $s_2$ -convex on  $[a, b]$  for some fixed  $s_1, s_2 \in (0, 1]$ , then the following inequality for fractional integrals holds:

$$\begin{aligned}
 \frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a^+}^\alpha f(b)g(b) + J_{b^-}^\alpha f(a)g(a)] &\leq \left( \frac{1}{\alpha + s_1 + s_2} + \beta(\alpha, s_1 + s_2 + 1) \right) M(a, b) \\
 &\quad + \left( \beta(\alpha + s_1, s_2 + 1) + \beta(\alpha + s_2, s_1 + 1) \right) N(a, b),
 \end{aligned}$$

where  $M(a, b) = f(a)g(a) + f(b)g(b)$ ,  $N(a, b) = f(a)g(b) + f(b)g(a)$ .

*Proof.* Since  $f$  is  $s_1$ -convex and  $g$  is  $s_2$ -convex on  $[a, b]$ , then for  $t \in [0, 1]$  we get

$$f(ta + (1-t)b) \leq t^{s_1} f(a) + (1-t)^{s_1} f(b), \tag{2.8}$$

and

$$g(ta + (1-t)b) \leq t^{s_2} g(a) + (1-t)^{s_2} g(b). \tag{2.9}$$



From (2.8) and (2.9), we get

$$f(ta+(1-t)b)g(ta+(1-t)b) \leq t^{s_1+s_2} f(a)g(a) + (1-t)^{s_1+s_2} f(b)g(b) + t^{s_1}(1-t)^{s_2} f(a)g(b) + (1-t)^{s_1}t^{s_2} f(b)g(a).$$

Similarly, we have

$$f((1-t)a+tb)g((1-t)a+tb) \leq (1-t)^{s_1+s_2} f(a)g(a) + t^{s_1+s_2} f(b)g(b) + (1-t)^{s_1}t^{s_2} f(a)g(b) + t^{s_1}(1-t)^{s_2} f(b)g(a).$$

So

$$\begin{aligned} & f(ta + (1 - t)b)g(ta + (1 - t)b) + f((1 - t)a + tb)g((1 - t)a + tb) \\ & \leq (t^{s_1+s_2} + (1 - t)^{s_1+s_2})[f(a)g(a) + f(b)g(b)] \\ & \quad + (t^{s_1}(1 - t)^{s_2} + (1 - t)^{s_1}t^{s_2})[f(a)g(b) + f(b)g(a)]. \end{aligned}$$

Multiplying both sides of above inequality by  $t^{\alpha-1}$ , then integrating the resulting inequality with respect to  $t$  over  $[0, 1]$ , we obtain

$$\begin{aligned} & \int_0^1 t^{\alpha-1} f(ta + (1 - t)b)g(ta + (1 - t)b)dt + \int_0^1 t^{\alpha-1} f((1 - t)a + tb)g((1 - t)a + tb)dt \\ & = \int_b^a \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)g(u) \frac{du}{a-b} + \int_a^b \left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v)g(v) \frac{dv}{b-a} \\ & = \frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a^+}^\alpha f(b)g(b) + J_{b^-}^\alpha f(a)g(a)] \\ & \leq [f(a)g(a) + f(b)g(b)] \int_0^1 t^{\alpha-1}(t^{s_1+s_2} + (1-t)^{s_1+s_2})dt \\ & \quad + [f(a)g(b) + f(b)g(a)] \int_0^1 t^{\alpha-1}(t^{s_1}(1-t)^{s_2} + (1-t)^{s_1}t^{s_2})dt \\ & = \left(\frac{1}{\alpha + s_1 + s_2} + \beta(\alpha, s_1 + s_2 + 1)\right)[f(a)g(a) + f(b)g(b)] \\ & \quad + \left(\beta(\alpha + s_1, s_2 + 1) + \beta(\alpha + s_2, s_1 + 1)\right)[f(a)g(b) + f(b)g(a)] \\ & = \left(\frac{1}{\alpha + s_1 + s_2} + \beta(\alpha, s_1 + s_2 + 1)\right)M(a, b) + \left(\beta(\alpha + s_1, s_2 + 1) + \beta(\alpha + s_2, s_1 + 1)\right)N(a, b). \end{aligned}$$

So

$$\begin{aligned} \frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a^+}^\alpha f(b)g(b) + J_{b^-}^\alpha f(a)g(a)] & \leq \left(\frac{1}{\alpha + s_1 + s_2} + \beta(\alpha, s_1 + s_2 + 1)\right)M(a, b) \\ & \quad + \left(\beta(\alpha + s_1, s_2 + 1) + \beta(\alpha + s_2, s_1 + 1)\right)N(a, b), \end{aligned}$$

which completes the proof. □

*Remark 2.9.* Putting  $\alpha = 1$  in Theorem 2.8 leads to

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x)g(x)dx & \leq \left(\frac{1}{s_1 + s_2 + 1} + \beta(1, s_1 + s_2 + 1)\right) \frac{M(a, b)}{2} \\ & \quad + \left(\beta(1 + s_1, s_2 + 1) + \beta(1 + s_2, s_1 + 1)\right) \frac{N(a, b)}{2} \\ & = \frac{1}{s_1 + s_2 + 1} M(a, b) + \beta(s_2 + 1, s_1 + 1)N(a, b), \end{aligned}$$

which is the result of (1.13).

**Theorem 2.10.** *Let  $f, g : [a, b] \rightarrow \mathbb{R}$ ,  $a, b \in [0, \infty)$ ,  $a < b$ , be functions such that  $fg \in L[a, b]$ . If  $f$  is convex and nonnegative on  $[a, b]$  and  $g$  is  $s$ -convex on  $[a, b]$  for some fixed  $s \in (0, 1]$ , then*

$$\begin{aligned} 2^s f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b)g(b) + J_{b^-}^\alpha f(a)g(a)] \\ &\quad + \frac{1}{2}M(a,b)\left(\beta(\alpha+1, s+1) + \frac{1}{(\alpha+s)(\alpha+s+1)}\right) \\ &\quad + \frac{1}{2}N(a,b)\left(\beta(\alpha, s+2) + \frac{1}{\alpha+s+1}\right), \end{aligned}$$

where  $M(a, b) = f(a)g(a) + f(b)g(b)$ ,  $N(a, b) = f(a)g(b) + f(b)g(a)$ .

*Proof.* We can write

$$\frac{a+b}{2} = \frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2},$$

so

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\ &= f\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right)g\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \\ &\leq \frac{1}{2^{s+1}} \left[ f(ta+(1-t)b) + f((1-t)a+tb) \right] \left[ g(ta+(1-t)b) + g((1-t)a+tb) \right] \\ &= \frac{1}{2^{s+1}} \left[ f(ta+(1-t)b)g(ta+(1-t)b) + f((1-t)a+tb)g((1-t)a+tb) \right. \\ &\quad \left. + f(ta+(1-t)b)g((1-t)a+tb) + f((1-t)a+tb)g(ta+(1-t)b) \right] \\ &\leq \frac{1}{2^{s+1}} \left[ f(ta+(1-t)b)g(ta+(1-t)b) + f((1-t)a+tb)g((1-t)a+tb) \right] \\ &\quad + \frac{1}{2^{s+1}} \left\{ \left[ tf(a) + (1-t)f(b) \right] \left[ (1-t)^s g(a) + t^s g(b) \right] \right. \\ &\quad \left. + \left[ (1-t)f(a) + tf(b) \right] \left[ t^s g(a) + (1-t)^s g(b) \right] \right\} \\ &= \frac{1}{2^{s+1}} \left[ f(ta+(1-t)b)g(ta+(1-t)b) + f((1-t)a+tb)g((1-t)a+tb) \right] \\ &\quad + \frac{1}{2^{s+1}} \left\{ \left[ t(1-t)^s + (1-t)t^s \right] M(a,b) + \left[ (1-t)^{s+1} + t^{s+1} \right] N(a,b) \right\}. \end{aligned}$$

Multiplying both sides of above inequality by  $t^{\alpha-1}$ , then integrating the resulting inequality with respect to  $t$  over  $[0, 1]$ , we get

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \int_0^1 t^{\alpha-1} dt &\leq \frac{1}{2^{s+1}} \left[ \int_0^1 t^{\alpha-1} f(ta+(1-t)b)g(ta+(1-t)b) dt \right. \\ &\quad \left. + \int_0^1 t^{\alpha-1} f((1-t)a+tb)g((1-t)a+tb) dt \right] \\ &\quad + \frac{1}{2^{s+1}} \left\{ M(a,b) \int_0^1 t^{\alpha-1} [t(1-t)^s + (1-t)t^s] dt \right. \\ &\quad \left. + N(a,b) \int_0^1 t^{\alpha-1} [(1-t)^{s+1} + t^{s+1}] dt \right\}. \end{aligned}$$

That is

$$\begin{aligned} \frac{1}{\alpha} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) &\leq \frac{1}{2^{s+1}} \left[ \frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a^+}^\alpha f(b)g(b) + J_{b^-}^\alpha f(a)g(a)] \right] \\ &\quad + \frac{1}{2^{s+1}} \left\{ M(a, b) \int_0^1 t^{\alpha-1} [t(1-t)^s + (1-t)t^s] dt \right. \\ &\quad \left. + N(a, b) \int_0^1 t^{\alpha-1} [(1-t)^{s+1} + t^{s+1}] dt \right\}. \end{aligned}$$

From

$$\int_0^1 t^{\alpha-1} [t(1-t)^s + (1-t)t^s] dt = \beta(\alpha + 1, s + 1) + \frac{1}{(\alpha + s)(\alpha + s + 1)}$$

and

$$\int_0^1 t^{\alpha-1} [(1-t)^{s+1} + t^{s+1}] dt = \beta(\alpha, s + 2) + \frac{1}{\alpha + s + 1},$$

we get

$$\begin{aligned} 2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b)g(b) + J_{b^-}^\alpha f(a)g(a)] \\ &\quad + \frac{1}{2} M(a, b) \left( \beta(\alpha + 1, s + 1) + \frac{1}{(\alpha + s)(\alpha + s + 1)} \right) \\ &\quad + \frac{1}{2} N(a, b) \left( \beta(\alpha, s + 2) + \frac{1}{\alpha + s + 1} \right), \end{aligned}$$

which completes the proof. □

*Remark 2.11.* Setting  $\alpha = 1$  in Theorem 2.10, then

$$\begin{aligned} 2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(x)g(x)dx \\ &\quad + \frac{1}{2} M(a, b) \left( \beta(2, s + 1) + \frac{1}{(s + 1)(s + 2)} \right) \\ &\quad + \frac{1}{2} N(a, b) \left( \beta(1, s + 2) + \frac{1}{s + 2} \right) \\ &= \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{(s + 1)(s + 2)} M(a, b) + \frac{1}{s + 2} N(a, b), \end{aligned}$$

which is the result of (1.14).

*Remark 2.12.* Choosing  $f(x) = 1$  for all  $x \in [a, b]$  in Theorem 2.10, we have

$$2^s g\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)] + \left( \beta(\alpha, s + 1) + \frac{1}{\alpha + s} \right) \frac{g(a) + g(b)}{2}.$$

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