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Some Differentiable Formulas of Special Functions

Jianbing Cao
Qingdao University of Science and Technology
China

Fahui Zhai
Qingdao University of Science and Technology
China

Xiquan Liang
Qingdao University of Science and Technology
China

Summary. This article contains some differentiable formulas of special functions.

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The terminology and notation used in this paper are introduced in the following papers: [13], [15], [16], [2], [4], [10], [12], [3], [1], [6], [9], [7], [8], [11], [17], [5], and [14].

For simplicity, we use the following convention: x, a, b are real numbers, n is a natural number, Z is an open subset of $\mathbb{R}$, and f, f_1, f_2, g are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Next we state a number of propositions:

(1) Suppose $Z \subseteq \mathrm{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = a + x$ and $f_2(x) = a - x$ and $f_2(x) \neq 0$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{2 \cdot a}{(a-x)^{\mathbf{2}}}$.

(2) Suppose $Z \subseteq \mathrm{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_2(x) = x + a$ and $f_2(x) \neq 0$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{2 \cdot a}{(x+a)^{\mathbf{2}}}$.

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(3) Suppose $Z \subseteq \mathrm{dom}(\frac{f_1}{f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = x - a$ and $f_2(x) = x - b$ and $f_2(x) \neq 0$. Then $\frac{f_1}{f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1}{f_2})'_{\restriction Z}(x) = \frac{a-b}{(x-b)^{\mathbf{2}}}$.

(4) Suppose $Z \subseteq \mathrm{dom}\, f$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then $\frac{1}{f}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x^{\mathbf{2}}}$.

(5) Suppose $Z \subseteq \mathrm{dom}((\text{the function sin}) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) (the function sin) $\cdot \frac{1}{f}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function sin}) \cdot \frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x^{\mathbf{2}}} \cdot (\text{the function cos})(\frac{1}{x})$.

(6) Suppose $Z \subseteq \mathrm{dom}((\text{the function cos}) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) (the function cos) $\cdot \frac{1}{f}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function cos}) \cdot \frac{1}{f})'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}}} \cdot (\text{the function sin})(\frac{1}{x})$.

(7) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\, ((\text{the function sin}) \cdot \frac{1}{f}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\mathrm{id}_Z\, ((\text{the function sin}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\, ((\text{the function sin}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = (\text{the function sin})(\frac{1}{x}) - \frac{1}{x} \cdot (\text{the function cos})(\frac{1}{x})$.

(8) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\, ((\text{the function cos}) \cdot \frac{1}{f}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\mathrm{id}_Z\, ((\text{the function cos}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\, ((\text{the function cos}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = (\text{the function cos})(\frac{1}{x}) + \frac{1}{x} \cdot (\text{the function sin})(\frac{1}{x})$.

(9) Suppose $Z \subseteq \mathrm{dom}(((\text{the function sin}) \cdot \frac{1}{f})\, ((\text{the function cos}) \cdot \frac{1}{f}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $((\text{the function sin}) \cdot \frac{1}{f})\, ((\text{the function cos}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function sin}) \cdot \frac{1}{f})\, ((\text{the function cos}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}}} \cdot ((\text{the function sin})(\frac{1}{x})^{\mathbf{2}} - (\text{the function cos})(\frac{1}{x})^{\mathbf{2}})$.

(10) Suppose $Z \subseteq \mathrm{dom}(((\text{the function sin}) \cdot f)\, ((^{n}_{\mathbb{Z}}) \cdot (\text{the function sin})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function sin}) \cdot f)\, ((^{n}_{\mathbb{Z}}) \cdot (\text{the function sin}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function sin}) \cdot f)\, ((^{n}_{\mathbb{Z}}) \cdot (\text{the function sin})))'_{\restriction Z}(x) = n \cdot (\text{the function sin})(x)^{n-1}_{\mathbb{Z}} \cdot (\text{the function sin})((n+1) \cdot x)$.

(11) Suppose $Z \subseteq \text{dom}(((\text{the function cos}) \cdot f)\,((^{n}_{\mathbb{Z}}) \cdot (\text{the function sin})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function cos}) \cdot f)\,((^{n}_{\mathbb{Z}}) \cdot (\text{the function sin}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function cos}) \cdot f)\,((^{n}_{\mathbb{Z}}) \cdot (\text{the function sin})))'_{\restriction Z}(x) = n \cdot (\text{the function sin})(x)^{n-1}_{\mathbb{Z}} \cdot (\text{the function cos})((n+1) \cdot x)$.

(12) Suppose $Z \subseteq \text{dom}(((\text{the function cos}) \cdot f)\,((^{n}_{\mathbb{Z}}) \cdot (\text{the function cos})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function cos}) \cdot f)\,((^{n}_{\mathbb{Z}}) \cdot (\text{the function cos}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function cos}) \cdot f)\,((^{n}_{\mathbb{Z}}) \cdot (\text{the function cos})))'_{\restriction Z}(x) = -n \cdot (\text{the function cos})(x)^{n-1}_{\mathbb{Z}} \cdot (\text{the function sin})((n+1) \cdot x)$.

(13) Suppose $Z \subseteq \text{dom}(((\text{the function sin}) \cdot f)\,((^{n}_{\mathbb{Z}}) \cdot (\text{the function cos})))$ and $n \geq 1$ and for every x such that $x \in Z$ holds $f(x) = n \cdot x$. Then

(i) $((\text{the function sin}) \cdot f)\,((^{n}_{\mathbb{Z}}) \cdot (\text{the function cos}))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(((\text{the function sin}) \cdot f)\,((^{n}_{\mathbb{Z}}) \cdot (\text{the function cos})))'_{\restriction Z}(x) = n \cdot (\text{the function cos})(x)^{n-1}_{\mathbb{Z}} \cdot (\text{the function cos})((n+1) \cdot x)$.

(14) Suppose $Z \subseteq \text{dom}(\frac{1}{f}\,(\text{the function sin}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\frac{1}{f}\,(\text{the function sin})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{f}\,(\text{the function sin}))'_{\restriction Z}(x) = \frac{1}{x} \cdot (\text{the function cos})(x) - \frac{1}{x^{2}} \cdot (\text{the function sin})(x)$.

(15) Suppose $Z \subseteq \text{dom}(\frac{1}{f}\,(\text{the function cos}))$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $\frac{1}{f}\,(\text{the function cos})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{f}\,(\text{the function cos}))'_{\restriction Z}(x) = -\frac{1}{x} \cdot (\text{the function sin})(x) - \frac{1}{x^{2}} \cdot (\text{the function cos})(x)$.

(16) Suppose $Z \subseteq \text{dom}((\text{the function sin}) + (^{\frac{1}{2}}_{\mathbb{R}}) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) > 0$. Then

(i) $(\text{the function sin}) + (^{\frac{1}{2}}_{\mathbb{R}}) \cdot f$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function sin}) + (^{\frac{1}{2}}_{\mathbb{R}}) \cdot f)'_{\restriction Z}(x) = (\text{the function cos})(x) + \frac{1}{2} \cdot x_{\mathbb{R}}^{-\frac{1}{2}}$.

(17) Suppose $Z \subseteq \text{dom}(g\,((\text{the function sin}) \cdot \frac{1}{f}))$ and $g = {}^{2}_{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $g\,((\text{the function sin}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(g\,((\text{the function sin}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = 2 \cdot x \cdot (\text{the function sin})(\frac{1}{x}) - (\text{the function cos})(\frac{1}{x})$.

(18) Suppose $Z \subseteq \text{dom}(g\,((\text{the function cos}) \cdot \frac{1}{f}))$ and $g = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) \neq 0$. Then

(i) $g\,((\text{the function cos}) \cdot \frac{1}{f})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(g\,((\text{the function cos}) \cdot \frac{1}{f}))'_{\restriction Z}(x) = 2 \cdot x \cdot (\text{the function cos})(\frac{1}{x}) + (\text{the function sin})(\frac{1}{x})$.

(19) Suppose $Z \subseteq \text{dom}(\log\_(e) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = x$ and $f(x) > 0$. Then $\log\_(e) \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot f)'_{\restriction Z}(x) = \frac{1}{x}$.

(20) Suppose $Z \subseteq \text{dom}(\text{id}_Z\, f)$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$. Then $\text{id}_Z\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\text{id}_Z\, f)'_{\restriction Z}(x) = 1 + (\log\_(e))(x)$.

(21) Suppose $Z \subseteq \text{dom}(g\, f)$ and $g = \frac{2}{\mathbb{Z}}$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$. Then $g\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(g\, f)'_{\restriction Z}(x) = x + 2 \cdot x \cdot (\log\_(e))(x)$.

(22) Suppose $Z \subseteq \text{dom}(\frac{f_1+f_2}{f_1-f_2})$ and for every x such that $x \in Z$ holds $f_1(x) = a$ and $f_2 = \frac{2}{\mathbb{Z}}$ and for every x such that $x \in Z$ holds $(f_1 - f_2)(x) > 0$. Then $\frac{f_1+f_2}{f_1-f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{f_1+f_2}{f_1-f_2})'_{\restriction Z}(x) = \frac{4 \cdot a \cdot x}{(a-x^2)^2}$.

(23) Suppose that

(i) $Z \subseteq \text{dom}(\log\_(e) \cdot \frac{f_1+f_2}{f_1-f_2})$,

(ii) for every x such that $x \in Z$ holds $f_1(x) = a$,

(iii) $f_2 = \frac{2}{\mathbb{Z}}$,

(iv) for every x such that $x \in Z$ holds $(f_1 - f_2)(x) > 0$, and

(v) for every x such that $x \in Z$ holds $(f_1 + f_2)(x) > 0$.

Then $\log\_(e) \cdot \frac{f_1+f_2}{f_1-f_2}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\log\_(e) \cdot \frac{f_1+f_2}{f_1-f_2})'_{\restriction Z}(x) = \frac{4 \cdot a \cdot x}{a^2-x^4}$.

(24) Suppose $Z \subseteq \text{dom}(\frac{1}{f}\, g)$ and for every x such that $x \in Z$ holds $f(x) = x$ and $g = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$. Then $\frac{1}{f}\, g$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f}\, g)'_{\restriction Z}(x) = \frac{1}{x^{\mathbf{2}}} \cdot (1 - (\log\_(e))(x))$.

(25) Suppose $Z \subseteq \text{dom}(\frac{1}{f})$ and $f = \log\_(e) \cdot f_1$ and for every x such that $x \in Z$ holds $f_1(x) = x$ and $f_1(x) > 0$ and for every x such that $x \in Z$ holds $f(x) \neq 0$. Then $\frac{1}{f}$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{x \cdot (\log\_(e))(x)^{\mathbf{2}}}$.

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