Research Article

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Several explicit formulas for (degenerate) Narumi and Cauchy polynomials and numbers

https://doi.org/10.1515/math-2021-0079
received May 17, 2021; accepted July 21, 2021

Abstract: In this paper, with the aid of the Faà di Bruno formula and by virtue of properties of the Bell polynomials of the second kind, the authors define a kind of notion of degenerate Narumi numbers and polynomials, establish explicit formulas for degenerate Narumi numbers and polynomials, and derive explicit formulas for the Narumi numbers and polynomials and for (degenerate) Cauchy numbers.

Keywords: Narumi number, Narumi polynomial, Cauchy number, degenerate Cauchy number, explicit formula, Bell polynomial of the second kind, Faà di Bruno formula, generating function

MSC 2020: 11B83, 11C08, 33B10

Dedicated to Professor Ce-Wen Cao at School of Mathematics and Statistics, Zhengzhou University, China.

1 Preliminaries and motivations

In this section, we first recall several known notions and mention our motivations.

1.1 Stirling numbers

The Stirling numbers of the first kind $s(n,k)$ for $n \ge k \ge 0$ can be generated [1,2] by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}, \quad |x| < 1$$

and can be computed by

$$|s(n+1,k+1)| = n!\sum_{\ell_1=k}^{n}\frac{1}{\ell_1}\sum_{\ell_2=k-1}^{\ell_1-1}\frac{1}{\ell_2}\cdots\sum_{\ell_{k-1}=2}^{\ell_{k-2}-1}\frac{1}{\ell_{k-1}}\sum_{\ell_k=1}^{\ell_{k-1}-1}\frac{1}{\ell_k}, \quad n \ge k \ge 1,$$

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which was derived in [3, Corollary 2.3] and can be reformulated as

$$\frac{|s(n+1,k+1)|}{n!}=\sum_{m=k}^{n}\frac{|s(m,k)|}{m!},\quad n\ge k\ge 1.$$

The Stirling numbers of the second kind $S(n,k)$ for $n\ge k\ge 0$ can be generated [1,2] by

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!} \tag{1}$$

and can be computed [4] by

$$S(n,k)=\frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\ell^n.$$

In [5, p. 303, equation (1.2)], see also [6–8], the r-associate Stirling numbers of the second kind, denoted by $S(n,k;r)$, were defined by

$$\left(e^x-\sum_{i=0}^{r}\frac{x^i}{i!}\right)^k=\left(\sum_{i=r+1}^{\infty}\frac{x^i}{i!}\right)^k=k!\sum_{n=(r+1)k}^{\infty}S(n,k;r)\frac{x^n}{n!}. \tag{2}$$

It is clear that $S(n,k;0)=S(n,k)$.

Proposition 1.1. ([5, p. 306, (3.11)] and [6, Theorem 3.1]) *For $k\ge 1$, the 1-associate Stirling numbers of the second kind $S(n,k;1)$ satisfy $S(0,0;1)=1$, $S(n,0;1)=0$ for $n\ge 1$, and*

$$S(n,k;1)=\begin{cases}\dfrac{1}{k!}\displaystyle\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{m=0}^{j}\binom{j}{m}\langle n\rangle_m(k-j)^{n-m}, & n\ge 2k\ge 2;\\ 0, & 0\le n<2k,\end{cases}$$

where the falling factorial

$$\langle x\rangle_n=\prod_{k=0}^{n-1}(x-k)=\begin{cases}x(x-1)\cdots(x-n+1), & n\ge 1;\\ 1, & n=0.\end{cases}$$

1.2 Degenerate Narumi and Cauchy numbers and polynomials

The Narumi polynomials $N_n^{(\alpha)}(x)$ were defined by means of the generating function

$$\left[\frac{t}{\ln(1+t)}\right]^{\alpha}(1+t)^x=\sum_{n=0}^{\infty}N_n^{(\alpha)}(x)\frac{t^n}{n!} \tag{3}$$

in [9, p. 127]. In particular, the quantities $N_n^{(\alpha)}(0)=N_n^{(\alpha)}$ are called the Narumi numbers. In [10], several Sheffer sequences and many relations of several polynomials arising from umbral calculus were dealt with. In [11], the Narumi polynomials of the Barnes type were defined and many interesting identities in the light of umbral calculus were established.

When letting $\alpha=1$ and $x=0$ in (3), the quantities $N_n^{(\alpha)}(x)$ become the Cauchy numbers of the first kind C_n, which have been investigated in [3,12–14] and closely related references therein.

In [15], Kim and two coauthors defined degenerate Cauchy numbers $C_n(\lambda)$ by

$$\frac{\lambda[e^{[(1+t)^{\lambda}-1]/\lambda}-1]}{(1+t)^{\lambda}-1}=\sum_{n=0}^{\infty}C_n(\lambda)\frac{t^n}{n!}$$

and showed that the family of nonlinear differential equations

$$(1+t)^n[(1+t)^\lambda-1]^n F_\lambda^{(n)}(t)=F_\lambda(t)\sum_{i=1}^{2n}a_i(n,\lambda)(1+t)^{i\lambda}+\sum_{i=1}^{2n-1}b_i(n,\lambda)(1+t)^{i\lambda}$$

for $n\in\mathbb{N}$ has the same solutions

$$F_\lambda(t)=\frac{e^{[(1+t)^\lambda-1]/\lambda}-1}{(1+t)^\lambda-1}, \tag{4}$$

where $a_i(n,\lambda)$ for $1\le i\le 2n$ and $b_i(n,\lambda)$ for $1\le i\le 2n-1$ are uniquely determined by

$$\begin{aligned} a_1(n,\lambda)&=-\frac{1}{\lambda}\langle n-1-\lambda\rangle_{n+1},\\ a_2(n,\lambda)&=\langle n-1-2\lambda\rangle_{n-1}-\frac{1}{\lambda}\sum_{i=0}^{n-2}[\lambda-(\lambda+1)(n-i)]\langle n-i-2-\lambda\rangle_{n-i}\langle n-1-2\lambda\rangle_i,\\ a_i(n,\lambda)&=[(i-1)\lambda-(\lambda+1)n]a_i(n-1,\lambda)+a_{i-2}(n-1,\lambda)+(n-1-i\lambda)a_i(n-1,\lambda)\end{aligned} \tag{5}$$

for $3\le i\le 2n-2$,

$$\begin{aligned} a_{2n-1}(n,\lambda)&=\frac{1}{2}n[(\lambda-1)(n-1)-2(\lambda+1)],\quad a_{2n}(n,\lambda)=1,\quad b_1(n,\lambda)=\langle n-1-\lambda\rangle_{n-1},\\ b_i(n,\lambda)&=[(i-1)\lambda-(\lambda+1)(n-1)]b_{i-1}(n-1,\lambda)+a_{i-1}(n-1,\lambda)+(n-1-i\lambda)b_i(n-1,\lambda)\end{aligned} \tag{6}$$

for $2\le i\le 2n-3$, and

$$b_{2n-2}(n,\lambda)=(\lambda-1)\binom{n-1}{2}-2(n-1)-\lambda,\quad b_{2n-1}(n,\lambda)=1. \tag{7}$$

Since

$$\lim_{\lambda\to 0}\frac{(1+t)^\lambda-1}{\lambda}=\ln(1+t),$$

we have

$$\lim_{\lambda\to 0}C_n(\lambda)=C_n,\quad n\ge 0.$$

This should be what the word “degenerate” means.

In previous study [16], for simplifying and signifying those expressions in (5), (6), and (7), by virtue of the Faà di Bruno formula (12), the identity (13), and the closed-form expression (15), Qi and his two coauthors established the following two conclusions.

1. For $n\ge 0$, degenerate Cauchy numbers $C_n(\lambda)$ and the Cauchy numbers C_n can be explicitly computed by

$$C_n(\lambda)=\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)!}\frac{1}{\lambda^k}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle\ell\lambda\rangle_n \tag{8}$$

   and

$$C_n=\sum_{k=0}^{n}\frac{s(n,k)}{k+1}, \tag{9}$$

   respectively. See [16, Theorem 1].

2. For $n\in\mathbb{N}$, the function $F_\lambda(t)$ in (4) and its derivatives satisfy

$$F_\lambda^{(n)}(t)=\frac{F_\lambda(t)}{(1+t)^n[1-(1+t)^\lambda]^n}\sum_{i=1}^{2n}\alpha_i(n,\lambda)(1+t)^{i\lambda}+\sum_{i=1}^{2n-1}\beta_i(n,\lambda)(1+t)^{i\lambda}$$

with

$$\alpha_i(n,\lambda)=\sum_{\substack{k+m=i\\ 1\le k\le n\\ 0\le m\le n}}(-1)^m A_k(n,\lambda)\sum_{\ell=0}^{\min\{n-m,k\}}\frac{\lambda^\ell}{(k-\ell)!}\binom{n-\ell}{m}$$

for $1\le i\le 2n$ and

$$\beta_i(n,\lambda)=\sum_{\substack{k+m=i\\ 1\le k\le n\\ 0\le m\le n-1}}(-1)^{m+1} A_k(n,\lambda)\sum_{\ell=0}^{\min\{k-1,n-m-1\}}\frac{\lambda^\ell}{(k-\ell)!}\binom{n-\ell-1}{m}$$

for $1\le i\le 2n-1$, where

$$A_k(n,\lambda)=\frac{(-1)^k}{\lambda^k}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell\lambda-q).$$

See [16, Theorem 2].

From the explicit formula (8), we obtain the first seven values of degenerate Cauchy numbers $C_n(\lambda)$ for $0\le n\le 6$ as follows:

$$\begin{aligned}
&C_0(\lambda)=1,\quad C_1(\lambda)=\frac{1}{2},\quad C_2(\lambda)=\frac{1}{6}(3\lambda-1),\quad C_3(\lambda)=\frac{1}{4}(2\lambda^2-2\lambda+1),\\
&C_4(\lambda)=\frac{\lambda^3}{2}-\frac{2\lambda^2}{3}+\lambda-\frac{19}{30},\quad C_5(\lambda)=\frac{1}{12}(6\lambda^4+5\lambda^2-36\lambda+27),\\
&C_6(\lambda)=\frac{\lambda^5}{2}+\frac{17\lambda^4}{6}-10\lambda^3+\frac{61\lambda^2}{12}+12\lambda-\frac{863}{84}.
\end{aligned}$$

In recent years, the authors also investigated other sequences of special numbers and polynomials in [17–28] and closely related references therein.

1.3 Motivations

In this paper, we introduce degenerate Narumi polynomials $N_n^{(\alpha)}(x,\lambda)$ by

$$\left(\frac{\lambda[e^{[(1+t)^\lambda-1]/\lambda}-1]}{(1+t)^\lambda-1}\right)^\alpha e^{x[(1+t)^\lambda-1]/\lambda}=\sum_{n=0}^{\infty}N_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!}. \tag{10}$$

It is clear that

$$\lim_{\lambda\to0}N_n^{(\alpha)}(x,\lambda)=N_n^{(\alpha)}(x),\quad N_n^{(1)}(0,\lambda)=C_n(\lambda).$$

When $x=0$, we call $N_n^{(\alpha)}(0,\lambda)$ degenerate Narumi numbers and denote them by the notation $\mathcal{N}_n^{(\alpha)}(\lambda)$, that is, taking $x=0$ in (10),

$$\left(\frac{\lambda[e^{[(1+t)^\lambda-1]/\lambda}-1]}{(1+t)^\lambda-1}\right)^\alpha=\sum_{n=0}^{\infty}\mathcal{N}_n^{(\alpha)}(\lambda)\frac{t^n}{n!}. \tag{11}$$

It is easy to see that $N_0^{(\alpha)}(x,\lambda)=1$ and $\mathcal{N}_0^{(\alpha)}(\lambda)=1$.

In this paper, we mainly aim at establishing explicit formulas for degenerate Narumi numbers $\mathcal{N}_n^{(\alpha)}(\lambda)$ and degenerate Narumi polynomials $N_n^{(\alpha)}(x,\lambda)$. Consequently, we derive explicit formulas for the Cauchy numbers C_n and degenerate Cauchy numbers $C_n(\lambda)$.

2 Properties of second kind Bell polynomials

The Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \ge k \ge 0$ were defined in [2, p. 134] by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1} i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}} \frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

For $n \in \mathbb{N}$, the Faà di Bruno formula is described in [2, p. 139] in terms of the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} f\circ h(t) = \sum_{k=1}^{n} f^{(k)}(h(t))B_{n,k}(h'(t), h''(t), \dots, h^{(n-k+1)}(t)). \tag{12}$$

In [2, p. 135], there is an identity

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}), \tag{13}$$

where $n \ge k \ge 0$ and a, b, λ, α are any complex numbers. At the end of [2, p. 133], there is the formula

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1})\frac{t^n}{n!}, \quad k \ge 0. \tag{14}$$

In [29, Remark 1], there existed the formula

$$B_{n,k}\left(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\right) = \frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell-q\lambda). \tag{15}$$

The explicit formula (15) has been applied and reviewed in [16, Lemma 3], [30, Lemma 2.6], [31, Section 2], [32, First proof of Theorem 2], [33, Lemma 2.2], [34, Remark 6.1], [35, Lemma 4], and [36, Section 1.3]. The explicit formula (15) is equivalent to

$$B_{n,k}(\langle\lambda\rangle_1, \langle\lambda\rangle_2, \dots, \langle\lambda\rangle_{n-k+1}) = \frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\langle\lambda\ell\rangle_n, \tag{16}$$

which was presented in [37, Theorems 2.1 and 4.1]. In [38, Remark 7.5], the explicit formulas (15) and (16) were rearranged as

$$B_{n,k}\left(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\right) = (-1)^k\frac{\lambda^{n-1}(n-1)!}{k!}\sum_{\ell=1}^{k}(-1)^{\ell}\ell\binom{k}{\ell}\binom{\ell/\lambda-1}{n-1}$$

for $\lambda \ne 0$ and

$$B_{n,k}(\langle\lambda\rangle_1, \langle\lambda\rangle_2, \dots, \langle\lambda\rangle_{n-k+1}) = (-1)^k\lambda\frac{(n-1)!}{k!}\sum_{\ell=1}^{k}(-1)^{\ell}\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}, \tag{17}$$

where $n \in \mathbb{N}$ and generalized binomial coefficient $\binom{z}{w}$ is defined by

$$\binom{z}{w} = \begin{cases} \dfrac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z, w, z-w \in \mathbb{C}\setminus\{-1,-2,\dots\};\\ 0, \quad z \in \mathbb{C}\setminus\{-1,-2,\dots\}, & w, z-w \in \{-1,-2,\dots\}.\end{cases}$$

For establishing explicit formulas for degenerate Narumi polynomials $N_n^{(\alpha)}(x,\lambda)$ and for the Narumi polynomials $N_n^{(\alpha)}(x)$, we derive two explicit formulas for special values of the Bell polynomials of the second kind

$$B_{n,k}\left(\frac{b^2-a^2}{2},\frac{b^3-a^3}{3},\dots,\frac{b^{n-k+2}-a^{n-k+2}}{n-k+2}\right). \tag{18}$$

Lemma 2.1. *Let $a, b \in \mathbb{C}$.*

1. *When $n \ge k \ge 0$ and $n \in \mathbb{N}$, the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ satisfy*

$$\begin{aligned}
&B_{n,k}\left(\frac{b^2-a^2}{2},\frac{b^3-a^3}{3},\dots,\frac{b^{n-k+2}-a^{n-k+2}}{n-k+2}\right)\\
&=\frac{(-1)^k}{k!}\frac{n!}{(n+k)!}\Bigg\{\sum_{\ell=1}^{k-1}(-1)^\ell\binom{k}{\ell}\sum_{q=0}^{n-k}\binom{n+k}{2\ell+q}b^{2\ell+q}a^{n+k-(2\ell+q)}\\
&\times\left[\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\sum_{m=0}^{j}\binom{j}{m}(\ell-j)^{2\ell+q-m}\langle 2\ell+q\rangle_m\right]\\
&\times\left[\sum_{j=0}^{k-\ell}(-1)^j\binom{k-\ell}{j}\sum_{m=0}^{j}\binom{j}{m}(k-\ell-j)^{n+k-(2\ell+q)-m}\langle n+k-(2\ell+q)\rangle_m\right]\\
&+[(-1)^kb^{n+k}+a^{n+k}]\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{m=0}^{j}\binom{j}{m}(k-j)^{n+k-m}\langle n+k\rangle_m\Bigg\}.
\end{aligned} \tag{19}$$

2. *When $n \ge k \ge 0$, the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ satisfy*

$$\begin{aligned}
&B_{n,k}\left(\frac{b^2-a^2}{2},\frac{b^3-a^3}{3},\dots,\frac{b^{n-k+2}-a^{n-k+2}}{n-k+2}\right)\\
&=\frac{1}{k!}\sum_{r+s=k}\sum_{\ell+m=n}(-1)^s\binom{k}{r}\binom{n}{\ell}\frac{b^{r+\ell}}{\binom{r+\ell}{r}}\frac{a^{s+m}}{\binom{s+m}{s}}\\
&\times\left[\sum_{j=0}^{r}(-1)^{r-j}\binom{\ell+r}{r-j}S(\ell+j,j)\right]\left[\sum_{j=0}^{s}(-1)^{s-j}\binom{m+s}{s-j}S(m+j,j)\right].
\end{aligned} \tag{20}$$

Proof. For $b > a > 0$, let

$$G_{a,b}(u)=\begin{cases}\dfrac{e^{bu}-e^{au}}{u}, & u\ne 0;\\ b-a, & u=0.\end{cases} \tag{21}$$

It is easy to see that

$$G_{a,b}^{(k)}(u)=\int_a^b t^k e^{tu}\,\mathrm{d}t, \quad k\in\{0\}\cup\mathbb{N} \tag{22}$$

and

$$G_{a,b}^{(k)}(0)=\frac{b^{k+1}-a^{k+1}}{k+1}, \quad k\in\{0\}\cup\mathbb{N}. \tag{23}$$

Employing the formula

$$B_{n,k}\left(\frac{x_2}{2},\frac{x_3}{3},\dots,\frac{x_{n-k+2}}{n-k+2}\right)=\frac{n!}{(n+k)!}B_{n+k,k}(0,x_2,x_3,\dots,x_{n+1})$$

in [2, p. 136], we acquire

$$B_{n,k}(G'_{a,b}(0), G''_{a,b}(0), \dots, G^{(n-k+1)}_{a,b}(0)) = B_{n,k}\left(\frac{b^2-a^2}{2}, \frac{b^3-a^3}{3}, \dots, \frac{b^{n-k+2}-a^{n-k+2}}{n-k+2}\right)$$
$$= \frac{n!}{(n+k)!} B_{n+k,k}(0, b^2-a^2, b^3-a^3, \dots, b^{n+1}-a^{n+1}).$$

Making use of the formula (14) yields

$$\sum_{n=0}^{\infty} B_{n+k,k}(x_1, x_2, \dots, x_{n+1}) \frac{k!n!}{(n+k)!} \frac{t^{n+k}}{n!} = \left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k,$$
$$\sum_{n=0}^{\infty} \frac{B_{n+k,k}(x_1, x_2, \dots, x_{n+1})}{\binom{n+k}{k}} \frac{t^{n+k}}{n!} = \left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k,$$
$$B_{n+k,k}(x_1, x_2, \dots, x_{n+1}) = \binom{n+k}{k} \lim_{t\to 0} \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\sum_{m=0}^{\infty} x_{m+1} \frac{t^m}{(m+1)!}\right]^k.$$

Accordingly, considering (2), we obtain

$$\begin{aligned}
&B_{n+k,k}(0, b^2-a^2, b^3-a^3, \dots, b^{n+1}-a^{n+1}) \\
&= \binom{n+k}{k} \lim_{t\to 0} \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\sum_{m=1}^{\infty} (b^{m+1}-a^{m+1}) \frac{t^m}{(m+1)!}\right]^k \\
&= \binom{n+k}{k} \lim_{t\to 0} \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\frac{(e^{bt}-bt-1)-(e^{at}-at-1)}{t}\right]^k \\
&= \binom{n+k}{k} \lim_{t\to 0} \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\frac{(-1)^k}{t^k} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} (e^{bt}-bt-1)^\ell (e^{at}-at-1)^{k-\ell}\right] \\
&= \binom{n+k}{k} \lim_{t\to 0} \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\frac{(-1)^k}{t^k} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell}\left[\ell! \sum_{q=2\ell}^{\infty} S(q,\ell;1) b^q \frac{t^q}{q!}\right]\left[(k-\ell)! \sum_{p=2(k-\ell)}^{\infty} S(p,k-\ell;1) a^p \frac{t^p}{p!}\right]\right) \\
&= (-1)^k \frac{(n+k)!}{n!} \lim_{t\to 0} \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(t^k \sum_{\ell=0}^{k} (-1)^\ell \left[\sum_{q=0}^{\infty} \frac{S(2\ell+q,\ell;1) b^{2\ell+q}}{(2\ell+q)!} t^q\right]\left[\sum_{p=0}^{\infty} \frac{S(2(k-\ell)+p,k-\ell;1) a^{2(k-\ell)+p}}{[2(k-\ell)+p]!} t^p\right]\right) \\
&= (-1)^k \frac{(n+k)!}{n!} \lim_{t\to 0} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \sum_{\ell=0}^{k} (-1)^\ell \sum_{m=0}^{\infty}\left[\sum_{q=0}^{m} \frac{S(2\ell+q,\ell;1) b^{2\ell+q}}{(2\ell+q)!} \frac{S(2(k-\ell)+m-q,k-\ell;1) a^{2(k-\ell)+m-q}}{[2(k-\ell)+m-q]!}\right] t^{m+k} \\
&= (-1)^k (n+k)! \sum_{\ell=0}^{k} (-1)^\ell \left[\sum_{q=0}^{n-k} \frac{S(2\ell+q,\ell;1) b^{2\ell+q}}{(2\ell+q)!} \frac{S(n+k-2\ell-q,k-\ell;1) a^{n+k-2\ell-q}}{(n+k-2\ell-q)!}\right].
\end{aligned}$$

Consequently, we acquire

$$\begin{aligned}
&B_{n,k}\left(\frac{b^2-a^2}{2}, \frac{b^3-a^3}{3}, \dots, \frac{b^{n-k+2}-a^{n-k+2}}{n-k+2}\right) \\
&= (-1)^k n! \sum_{\ell=0}^{k} (-1)^\ell \sum_{q=0}^{n-k} \frac{S(2\ell+q,\ell;1) b^{2\ell+q}}{(2\ell+q)!} \frac{S(n+k-2\ell-q,k-\ell;1) a^{n+k-2\ell-q}}{(n+k-2\ell-q)!}.
\end{aligned} \tag{24}$$

Further applying Proposition 1.1 to (24) yields

$$\begin{aligned}
&B_{n,k}\left(\frac{b^2-a^2}{2}, \frac{b^3-a^3}{3}, \dots, \frac{b^{n-k+2}-a^{n-k+2}}{n-k+2}\right) \\
&= (-1)^k n! \sum_{\ell=1}^{k-1} (-1)^\ell \sum_{q=0}^{n-k} \frac{S(2\ell+q,\ell;1) b^{2\ell+q}}{(2\ell+q)!} \frac{S(n+k-2\ell-q,k-\ell;1) a^{n+k-2\ell-q}}{(n+k-2\ell-q)!}
\end{aligned}$$

$$
\begin{aligned}
&+(-1)^k n!\sum_{q=1}^{n-k}\frac{S(q,0;1)b^q}{q!}\frac{S(n+k-q,k;1)a^{n+k-q}}{(n+k-q)!}+(-1)^k n!\frac{S(0,0;1)b^0}{0!}\frac{S(n+k,k;1)a^{n+k}}{(n+k)!}\\
&+(-1)^k n!(-1)^k\sum_{q=0}^{n-k-1}\frac{S(2k+q,k;1)b^{2k+q}}{(2k+q)!}\frac{S(n-k-q,0;1)a^{n-k-q}}{(n-k-q)!}\\
&+(-1)^k n!(-1)^k\frac{S(n+k,k;1)b^{n+k}}{(n+k)!}\frac{S(0,0;1)a^0}{0!}\\
=\ &(-1)^k n!\sum_{\ell=1}^{k-1}(-1)^\ell\sum_{q=0}^{n-k}\frac{S(2\ell+q,\ell;1)b^{2\ell+q}}{(2\ell+q)!}\frac{S(n+k-2\ell-q,k-\ell;1)a^{n+k-2\ell-q}}{(n+k-2\ell-q)!}\\
&+(-1)^k n!\frac{S(n+k,k;1)a^{n+k}}{(n+k)!}+n!\frac{S(n+k,k;1)b^{n+k}}{(n+k)!}\\
=\ &(-1)^k n!\sum_{\ell=1}^{k-1}(-1)^\ell\frac{1}{\ell!}\frac{1}{(k-\ell)!}\sum_{q=0}^{n-k}\frac{b^{2\ell+q}}{(2\ell+q)!}\frac{a^{n+k-2\ell-q}}{(n+k-2\ell-q)!}\left[\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\sum_{m=0}^{j}\binom{j}{m}\langle 2\ell+q\rangle_m(\ell-j)^{2\ell+q-m}\right]\\
&\times\left[\sum_{j=0}^{k-\ell}(-1)^j\binom{k-\ell}{j}\sum_{m=0}^{j}\binom{j}{m}\langle n+k-2\ell-q\rangle_m(k-\ell-j)^{n+k-2\ell-q-m}\right]\\
&+[b^{n+k}+(-1)^k a^{n+k}]\frac{n!}{(n+k)!}\frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{m=0}^{j}\binom{j}{m}\langle n+k\rangle_m(k-j)^{n+k-m},
\end{aligned}
$$

which can be rearranged as the explicit formula (19).

In [36, Section 1.8], it was given for $n\ge k\ge 0$ that

$$
B_{n,k}\left(\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n-k+2}\right)=\frac{n!}{(n+k)!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{n+k}{k-\ell}S(n+\ell,\ell). \tag{25}
$$

In [2, p. 136, equation [3n]], it was given that the Bell polynomials of the second kind $B_{n,k}$ satisfy

$$
B_{n,k}(x_1+y_1,x_2+y_2,\dots,x_{n-k+1}+y_{n-k+1})=\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}B_{\ell,r}(x_1,x_2,\dots,x_{\ell-r+1})B_{m,s}(y_1,y_2,\dots,y_{m-s+1}). \tag{26}
$$

Therefore, by virtue of the identities (26), (13), and (25) in sequence, we obtain

$$
\begin{aligned}
&B_{n,k}\left(\frac{b^2-a^2}{2},\frac{b^3-a^3}{3},\dots,\frac{b^{n-k+2}-a^{n-k+2}}{n-k+2}\right)\\
&=\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}B_{\ell,r}\left(\frac{b^2}{2},\frac{b^3}{3},\dots,\frac{b^{\ell-r+2}}{\ell-r+2}\right)B_{m,s}\left(\frac{-a^2}{2},\frac{-a^3}{3},\dots,\frac{-a^{m-s+2}}{m-s+2}\right)\\
&=\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}b^{\ell+r}B_{\ell,r}\left(\frac{1}{2},\frac{1}{3},\dots,\frac{1}{\ell-r+2}\right)(-1)^s a^{m+s}B_{m,s}\left(\frac{1}{2},\frac{1}{3},\dots,\frac{1}{m-s+2}\right)\\
&=\sum_{r+s=k}\sum_{\ell+m=n}(-1)^s\binom{n}{\ell}a^{m+s}b^{\ell+r}\left[\frac{\ell!}{(\ell+r)!}\sum_{j=0}^{r}(-1)^{r-j}\binom{\ell+r}{r-j}S(\ell+j,j)\right]\\
&\quad\times\left[\frac{m!}{(m+s)!}\sum_{j=0}^{s}(-1)^{s-j}\binom{m+s}{s-j}S(m+j,j)\right]\\
&=\frac{1}{k!}\sum_{r+s=k}\sum_{\ell+m=n}(-1)^s\binom{k}{r}\binom{n}{\ell}\frac{b^{r+\ell}}{\binom{r+\ell}{r}}\frac{a^{s+m}}{\binom{s+m}{s}}\left[\sum_{j=0}^{r}(-1)^{r-j}\binom{\ell+r}{r-j}S(\ell+j,j)\right]\\
&\quad\times\left[\sum_{j=0}^{s}(-1)^{s-j}\binom{m+s}{s-j}S(m+j,j)\right].
\end{aligned}
$$

The explicit formula (20) is thus proved. The proof of Lemma 2.1 is complete. □

Table 1: The first few values of the Bell polynomials of the second kind in (18) for $1 \le n, k \le 4$

$B_{n,k}$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$k = 1$	$\frac{1}{2}(b^2 - a^2)$	$\frac{1}{3}(b^3 - a^3)$	$\frac{1}{4}(b^4 - a^4)$	$\frac{1}{5}(b^5 - a^5)$
$k = 2$	0	$\frac{1}{4}(a^2 - b^2)^2$	$\frac{1}{2}(a^5 - a^3b^2 - a^2b^3 + b^5)$	$\frac{1}{6}(5a^6 - 3a^4b^2 - 4a^3b^3 - 3a^2b^4 + 5b^6)$
$k = 3$	0	0	$\frac{1}{8}(b^2 - a^2)^3$	$-\frac{1}{2}(a - b)^3(a + b)^2(a^2 + ab + b^2)$
$k = 4$	0	0	0	$\frac{1}{16}(a^2 - b^2)^4$

Remark 2.1. The first few values of the Bell polynomials of the second kind in (18) for $1 \le n, k \le 4$ can be computed by the explicit formula (19) and listed in Table 1.

Remark 2.2. For new results and applications about the Bell polynomials of the second kind $B_{n,k}$, please refer to the papers [6,7,18,19,25,28,36,39–42] and closely related references therein.

3 Explicit formulas for degenerate Narumi and Cauchy numbers

In this section, we state and prove an explicit formula for degenerate Narumi numbers $\mathcal{N}_n^{(\alpha)}(\lambda)$, derive an explicit formula for degenerate Cauchy numbers $C_n(\lambda)$, and list the first seven explicit expressions of degenerate Narumi numbers $\mathcal{N}_n^{(\alpha)}(\lambda)$ for $0 \le n \le 6$.

Theorem 3.1. *For $n \in \mathbb{N}$, degenerate Narumi numbers $\mathcal{N}_n^{(\alpha)}(\lambda)$ can be computed by*

$$\mathcal{N}_n^{(\alpha)}(\lambda) = (n-1)! \sum_{k=1}^{n} \frac{(-1)^k}{\lambda^{k-1}} \left[\sum_{\ell=1}^{k} (-1)^\ell \ell \binom{k}{\ell} \binom{\lambda\ell - 1}{n-1} \right] \left[\sum_{\ell=1}^{k} \frac{(-1)^\ell \langle\alpha\rangle_\ell}{(k+\ell)!} \sum_{j=0}^{\ell} (-1)^j \binom{k+\ell}{\ell - j} S(k+j, j) \right]. \tag{27}$$

Proof. Let $F_{\lambda,\alpha}(t)$ denote the generating function of degenerate Narumi numbers $\mathcal{N}_n^{(\alpha)}(\lambda)$. If applying $f(u) = \left(\frac{e^u - 1}{u}\right)^\alpha$ and $u = g(t) = \frac{(1+t)^\lambda - 1}{\lambda}$ to the Faà di Bruno formula (12), then we can write

$$\begin{aligned} \frac{\mathrm{d}^n}{\mathrm{d}t^n} F_{\lambda,\alpha}(t) = \sum_{k=1}^{n} \frac{\mathrm{d}^k}{\mathrm{d}u^k} \left(\frac{e^u - 1}{u}\right)^\alpha B_{n,k}\Bigg(&\frac{\lambda(1+t)^{\lambda-1}}{\lambda}, \frac{\lambda(\lambda-1)(1+t)^{\lambda-2}}{\lambda}, \\ &\dots, \frac{\lambda(\lambda-1)\cdots[\lambda-(n-k)](1+t)^{\lambda-(n-k+1)}}{\lambda} \Bigg). \end{aligned} \tag{28}$$

Utilizing the formula (25) and the Faà di Bruno formula (12), we arrive at

$$\begin{aligned} \lim_{u \to 0} \frac{\mathrm{d}^k}{\mathrm{d}u^k} \left(\frac{e^u - 1}{u}\right)^\alpha &= \lim_{u \to 0} \sum_{i=1}^{k} \langle\alpha\rangle_i v^{\alpha - i} B_{k,i}\left(\left(\frac{e^u - 1}{u}\right)', \left(\frac{e^u - 1}{u}\right)'', \dots, \left(\frac{e^u - 1}{u}\right)^{(k-i+1)} \right) \\ &= \sum_{i=1}^{k} \langle\alpha\rangle_i B_{k,i}\left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k-i+2}\right) \\ &= \sum_{i=1}^{k} \langle\alpha\rangle_i \frac{k!}{(k+i)!} \sum_{j=0}^{i} (-1)^{i-j} \binom{k+i}{i-j} S(k+j, j), \end{aligned} \tag{29}$$

where $v = v(u) = \frac{e^u - 1}{u} \to 1$ and

$$v^{(\ell)}(u) = \left(\frac{e^u - 1}{u}\right)^{(\ell)} = \left(\sum_{j=1}^{\infty} \frac{u^{j-1}}{j!}\right)^{(\ell)} = \left[\sum_{j=0}^{\infty} \frac{u^j}{(j+1)!}\right]^{(\ell)} \to \frac{1}{\ell + 1}$$

as $u \to 0$ for $\ell \in \mathbb{N}$.

Employing (13) and (17), we acquire

$$\begin{aligned} &B_{n,k}\left(\frac{\lambda(1+t)^{\lambda-1}}{\lambda}, \frac{\lambda(\lambda-1)(1+t)^{\lambda-2}}{\lambda}, \dots, \frac{\lambda(\lambda-1)\cdots(\lambda-(n-k))(1+t)^{\lambda-(n-k+1)}}{\lambda}\right) \\ &= \frac{(1+t)^{k\lambda-n}}{\lambda^k} B_{n,k}(\langle\lambda\rangle_1, \langle\lambda\rangle_2, \dots, \langle\lambda\rangle_{n-k+1}) \\ &= \frac{(1+t)^{k\lambda-n}}{\lambda^k}(-1)^k\lambda\frac{(n-1)!}{k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1} \\ &\to \frac{(-1)^k}{\lambda^{k-1}}\frac{(n-1)!}{k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1} \end{aligned} \tag{30}$$

as $t \to 0$.

Taking $t \to 0$, which is equivalent to $u \to 0$, on both sides of (28) and making use of (29) and (30) give

$$\begin{aligned} &\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}t^n}F_{\lambda,\alpha}(t) \\ &= \sum_{k=1}^{n}\lim_{u\to0}\frac{\mathrm{d}^k}{\mathrm{d}u^k}\left(\frac{e^u-1}{u}\right)^\alpha \lim_{t\to0}B_{n,k}\left(\frac{\lambda(1+t)^{\lambda-1}}{\lambda}, \frac{\lambda(\lambda-1)(1+t)^{\lambda-2}}{\lambda}, \right. \\ &\qquad \left. \dots, \frac{\lambda(\lambda-1)\cdots[\lambda-(n-k)](1+t)^{\lambda-(n-k+1)}}{\lambda}\right) \\ &= \sum_{k=1}^{n}\left[\sum_{i=1}^{k}\langle\alpha\rangle_i\frac{k!}{(k+i)!}\sum_{j=0}^{i}(-1)^{i-j}\binom{k+i}{i-j}S(k+j,j)\right]\left[\frac{(-1)^k}{\lambda^{k-1}}\frac{(n-1)!}{k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right] \\ &= (n-1)!\sum_{k=1}^{n}\left[\frac{(-1)^k}{\lambda^{k-1}}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]\left[\sum_{i=1}^{k}\frac{\langle\alpha\rangle_i}{(k+i)!}\sum_{j=0}^{i}(-1)^{i-j}\binom{k+i}{i-j}S(k+j,j)\right]. \end{aligned}$$

Considering the equation (11) leads to the formula (27). The proof of Theorem 3.1 is complete. □

Corollary 3.1. *For $n \in \mathbb{N}$, degenerate Cauchy numbers $C_n(\lambda)$ can be computed by*

$$C_n(\lambda) = (n-1)!\sum_{k=1}^{n}\frac{(-1)^k}{(k+1)!}\frac{1}{\lambda^{k-1}}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}. \tag{31}$$

Proof. This follows from setting $\alpha = 1$ in the formula (27) and simplifying. □

Remark 3.1. From the explicit formula (27) in Theorem 3.1, we can obtain the first seven explicit expressions of degenerate Narumi numbers $\mathcal{N}_n^{(\alpha)}(\lambda)$ for $0 \le n \le 6$ as follows:

$$\begin{aligned} &\mathcal{N}_0^{(\alpha)}(\lambda) = 1, \quad \mathcal{N}_1^{(\alpha)}(\lambda) = \frac{\alpha}{2}, \quad \mathcal{N}_2^{(\alpha)}(\lambda) = \frac{1}{12}\alpha(3\alpha+6\lambda-5), \\ &\mathcal{N}_3^{(\alpha)}(\lambda) = \frac{1}{8}\alpha[\alpha^2+\alpha(6\lambda-5)+4\lambda^2-10\lambda+6], \\ &\mathcal{N}_4^{(\alpha)}(\lambda) = \frac{1}{240}\alpha[15\alpha^3+30\alpha^2(6\lambda-5)+5\alpha(84\lambda^2-180\lambda+97)+120\lambda^3-580\lambda^2+960\lambda-502], \\ &\mathcal{N}_5^{(\alpha)}(\lambda) = \frac{1}{96}\alpha[3\alpha^4+10\alpha^3(6\lambda-5)+5\alpha^2(60\lambda^2-120\lambda+61)+2\alpha(180\lambda^3-690\lambda^2+910\lambda-401) \\ &\qquad +8(6\lambda^4-45\lambda^3+140\lambda^2-196\lambda+95)], \\ &\mathcal{N}_6^{(\alpha)}(\lambda) = \frac{1}{4032}\alpha[63\alpha^5+315\alpha^4(6\lambda-5)+315\alpha^3(52\lambda^2-100\lambda+49)+7\alpha^2(6480\lambda^3-22320\lambda^2+26370\lambda \\ &\qquad -10543)+42\alpha(744\lambda^4-4320\lambda^3+9910\lambda^2-10410\lambda+4075) \\ &\qquad +8(252\lambda^5-2478\lambda^4+11970\lambda^3-31983\lambda^2+41328\lambda-19087)]. \end{aligned}$$

Remark 3.2. The explicit formula (31) in Corollary 3.1 is slightly different from the explicit formula (8).

4 Explicit formulas for degenerate Narumi polynomials and numbers

In this section, with the aid of the explicit formulas (19) and (20), we establish two explicit formulas for degenerate Narumi polynomials $N_n^{(\alpha)}(x,\lambda)$, deduce a different formula for degenerate Narumi numbers $\mathcal{N}_n^{(\alpha)}(\lambda)$ from the formula (27), and list the first five values of degenerate Narumi polynomials $N_n^{(\alpha)}(x,\lambda)$ for $0 \le n \le 4$.

Theorem 4.1. *For $n \in \mathbb{N}$, degenerate Narumi polynomials $N_n^{(\alpha)}(x,\lambda)$ can be computed by*

$$\begin{aligned}
N_n^{(\alpha)}(x,\lambda) = (n-1)!\sum_{k=1}^{n}&\left[\frac{(-1)^k}{\lambda^{k-1}}\sum_{\ell=1}^{k}(-1)^{\ell}\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]\sum_{\ell=1}^{k}(-1)^{\ell}\left(1-\frac{2x}{\alpha}\right)^{\alpha-\ell}\frac{\langle\alpha\rangle_\ell}{\ell!(k+\ell)!}\\
&\times\left\{\sum_{p=1}^{\ell-1}(-1)^p\binom{\ell}{p}\sum_{q=0}^{k-\ell}\binom{k+\ell}{2p+q}\left(1-\frac{x}{\alpha}\right)^{2p+q}\left(\frac{x}{\alpha}\right)^{k+\ell-(2p+q)}\right.\\
&\times\left[\sum_{j=0}^{p}(-1)^j\binom{p}{j}\sum_{m=0}^{j}\binom{j}{m}(p-j)^{2p+q-m}\langle 2p+q\rangle_m\right]\\
&\times\left[\sum_{j=0}^{\ell-p}(-1)^j\binom{\ell-p}{j}\sum_{m=0}^{j}\binom{j}{m}(\ell-p-j)^{k+\ell-(2p+q)-m}\langle k+\ell-(2p+q)\rangle_m\right]\\
&\left.+\left[\left(1-\frac{x}{\alpha}\right)^{k+\ell}+(-1)^{\ell}\left(\frac{x}{\alpha}\right)^{k+\ell}\right]\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\sum_{m=0}^{j}\binom{j}{m}(\ell-j)^{k+\ell-m}\langle k+\ell\rangle_m\right\}
\end{aligned}\tag{32}$$

and

$$\begin{aligned}
N_n^{(\alpha)}(x,\lambda) = (n-1)!\sum_{k=1}^{n}&\left[\frac{(-1)^k}{k!}\frac{1}{\lambda^{k-1}}\sum_{\ell=1}^{k}(-1)^{\ell}\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]\\
&\times\left\{\sum_{q=1}^{k}\frac{\langle\alpha\rangle_q}{q!}\left(1-\frac{2x}{\alpha}\right)^{\alpha-q}\sum_{r+s=q}\sum_{\ell+m=k}(-1)^s\binom{q}{r}\binom{k}{\ell}\frac{(1-x/\alpha)^{r+\ell}}{\binom{r+\ell}{r}}\frac{(x/\alpha)^{s+m}}{\binom{s+m}{s}}\right.\\
&\left.\times\left[\sum_{j=0}^{r}(-1)^{r-j}\binom{\ell+r}{r-j}S(\ell+j,j)\right]\left[\sum_{j=0}^{s}(-1)^{s-j}\binom{m+s}{s-j}S(m+j,j)\right]\right\}.
\end{aligned}\tag{33}$$

Proof. Let $F_{\lambda,\alpha,x}(t)$ denote the generating function of degenerate Narumi polynomials $N_n^{(\alpha)}(x,\lambda)$ in (10). Applying $f(u)=\left(\frac{e^u-1}{u}\right)^{\alpha}e^{xu}$ and $u=g(t)=\frac{(1+t)^{\lambda}-1}{\lambda}$ to the Faà di Bruno formula (12) yields

$$\begin{aligned}
\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}t^n}F_{\lambda,\alpha,x}(t) = \lim_{t\to0}\sum_{k=1}^{n}&\frac{\mathrm{d}^k}{\mathrm{d}u^k}\left[\left(\frac{e^u-1}{u}\right)^{\alpha}e^{xu}\right]B_{n,k}\\
&\times\left(\frac{\lambda(1+t)^{\lambda-1}}{\lambda},\frac{\lambda(\lambda-1)(1+t)^{\lambda-2}}{\lambda},\dots,\frac{\lambda(\lambda-1)\cdots[\lambda-(n-k)](1+t)^{\lambda-(n-k+1)}}{\lambda}\right).
\end{aligned}$$

Since

$$\left(\frac{e^u-1}{u}\right)^{\alpha}e^{xu} = \left[\frac{e^{u(1-x/\alpha)}-e^{ux/\alpha}}{u}\right]^{\alpha} = G^{\alpha}_{x/\alpha,1-x/\alpha}(u),$$

where $G_{a,b}(u)$ is defined by (21), we obtain

$$
\begin{aligned}
&\lim_{u\to0}\frac{\mathrm{d}^k}{\mathrm{d}u^k}\left[\left(\frac{e^u-1}{u}\right)^{\alpha}e^{xu}\right]\\
&\quad=\lim_{u\to0}\frac{\mathrm{d}^k}{\mathrm{d}u^k}G^{\alpha}_{x/\alpha,1-x/\alpha}(u)\\
&\quad=\lim_{u\to0}\sum_{\ell=1}^{k}\frac{\mathrm{d}^\ell w^\alpha}{\mathrm{d}w^\ell}B_{k,\ell}\big(G'_{x/\alpha,1-x/\alpha}(u),G'_{x/\alpha,1-x/\alpha}(u),\dots,G^{(k-\ell+1)}_{x/\alpha,1-x/\alpha}(u)\big)\\
&\quad=\sum_{\ell=1}^{k}\langle\alpha\rangle_\ell\left(1-\frac{2x}{\alpha}\right)^{\alpha-\ell}B_{k,\ell}\big(G'_{x/\alpha,1-x/\alpha}(0),G'_{x/\alpha,1-x/\alpha}(0),\dots,G^{(k-\ell+1)}_{x/\alpha,1-x/\alpha}(0)\big)\\
&\quad=\sum_{\ell=1}^{k}\langle\alpha\rangle_\ell\left(1-\frac{2x}{\alpha}\right)^{\alpha-\ell}B_{k,\ell}\left(\frac{(1-x/\alpha)^2-(x/\alpha)^2}{2},\dots,\frac{(1-x/\alpha)^{k-\ell+2}-(x/\alpha)^{k-\ell+2}}{k-\ell+2}\right),
\end{aligned}
\tag{34}
$$

where

$$
w=w(u)=G_{x/\alpha,1-x/\alpha}(u)\to1-\frac{2x}{\alpha},\quad u\to0
$$

and we used the formulas (22) and (23). Consequently, we acquire

$$
\begin{aligned}
\lim_{t\to0}\frac{\mathrm{d}^n}{\mathrm{d}t^n}F_{\lambda,\alpha,x}(t)=\sum_{k=1}^{n}&\left[\sum_{\ell=1}^{k}\langle\alpha\rangle_\ell\left(1-\frac{2x}{\alpha}\right)^{\alpha-\ell}B_{k,\ell}\left(\frac{(1-x/\alpha)^2-(x/\alpha)^2}{2},\frac{(1-x/\alpha)^2-(x/\alpha)^3}{3},\right.\right.\\
&\left.\left.\dots,\frac{(1-x/\alpha)^{k-\ell+2}-(x/\alpha)^{k-\ell+2}}{k-\ell+2}\right)\right]\left[\frac{(-1)^k}{\lambda^{k-1}}\frac{(n-1)!}{k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right],
\end{aligned}
\tag{35}
$$

where we used the limit in (30). Further making use of the explicit formula (19) in Lemma 2.1 leads to

$$
\begin{aligned}
N_n^{(\alpha)}(x,\lambda)=\sum_{k=1}^{n}&\left[\frac{(-1)^k}{\lambda^{k-1}}\frac{(n-1)!}{k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]\\
&\times\sum_{\ell=1}^{k}\langle\alpha\rangle_\ell\left(1-\frac{2x}{\alpha}\right)^{\alpha-\ell}\left\{\frac{(-1)^\ell}{\ell!}\frac{k!}{(k+\ell)!}\sum_{p=1}^{\ell-1}(-1)^p\binom{\ell}{p}\sum_{q=0}^{k-\ell}\binom{k+\ell}{2p+q}\right.\\
&\times\left[\left(1-\frac{x}{\alpha}\right)^{2p+q}\sum_{j=0}^{p}(-1)^j\binom{p}{j}\sum_{m=0}^{j}\binom{j}{m}(p-j)^{2p+q-m}\langle2p+q\rangle_m\right]\\
&\times\left[\left(\frac{x}{\alpha}\right)^{k+\ell-(2p+q)}\sum_{j=0}^{\ell-p}(-1)^j\binom{\ell-p}{j}\sum_{m=0}^{j}\binom{j}{m}(\ell-p-j)^{k+\ell-(2p+q)-m}\langle k+\ell-(2p+q)\rangle_m\right]\\
&\left.+\frac{(1-x/\alpha)^{k+\ell}+(-1)^\ell(x/\alpha)^{k+\ell}}{\ell!}\frac{k!}{(k+\ell)!}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\sum_{m=0}^{j}\binom{j}{m}(\ell-j)^{k+\ell-m}\langle k+\ell\rangle_m\right\},
\end{aligned}
$$

which can be simplified as (32).

Employing (20) in (35) arrives at

$$
\begin{aligned}
N_n^{(\alpha)}(x,\lambda)=\sum_{k=1}^{n}&\left[\frac{(-1)^k}{\lambda^{k-1}}\frac{(n-1)!}{k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]\\
&\times\left\{\sum_{q=1}^{k}\langle\alpha\rangle_q\left(1-\frac{2x}{\alpha}\right)^{\alpha-q}\frac{1}{q!}\sum_{r+s=q}\sum_{\ell+m=k}(-1)^s\binom{q}{r}\binom{k}{\ell}\frac{(1-x/\alpha)^{r+\ell}}{\binom{r+\ell}{r}}\frac{(x/\alpha)^{s+m}}{\binom{s+m}{s}}\right.\\
&\left.\times\left[\sum_{j=0}^{r}(-1)^{r-j}\binom{\ell+r}{r-j}S(\ell+j,j)\right]\left[\sum_{j=0}^{s}(-1)^{s-j}\binom{m+s}{s-j}S(m+j,j)\right]\right\},
\end{aligned}
$$

which can be rearranged as (33). The proof of Theorem 4.1 is complete. □

Remark 4.1. The first five values of degenerate Narumi polynomials $N_n^{(\alpha)}(x,\lambda)$ for $0 \le n \le 4$ can be computed and listed as follows:

$$N_0^{(\alpha)}(x,\lambda)=1,\quad N_1^{(\alpha)}(x,\lambda)=\frac{\alpha}{2}+x,\quad N_2^{(\alpha)}(x,\lambda)=\frac{\alpha^2}{4}+\alpha\left(\frac{\lambda}{2}+x-\frac{5}{12}\right)+x(\lambda+x-1),$$
$$\begin{aligned}N_3^{(\alpha)}(x,\lambda)&=\frac{1}{8}\{\alpha^3+\alpha^2(6\lambda+6x-5)+2\alpha[2\lambda^2+\lambda(12x-5)+6x^2-11x+3]\\&\quad+8x[\lambda^2+3\lambda(x-1)+x^2-3x+2]\},\\ N_4^{(\alpha)}(x,\lambda)&=\frac{\alpha^4}{16}+\frac{\alpha^3}{8}(6\lambda+4x-5)+\frac{\alpha^2}{48}[84\lambda^2+36\lambda(6x-5)+72x^2-192x+97]\\&\quad+\alpha\left[\frac{\lambda^3}{2}+\lambda^2\left(7x-\frac{29}{12}\right)+\lambda\left(9x^2-\frac{33x}{2}+4\right)+2x^3-\frac{17x^2}{2}+\frac{19x}{2}-\frac{251}{120}\right]\\&\quad+x[\lambda^3+\lambda^2(7x-6)+\lambda(6x^2-18x+11)+x^3-6x^2+11x-6].\end{aligned}$$

Corollary 4.1. *For $n\in\mathbb{N}$, degenerate Narumi number $\mathcal{N}_n^{(\alpha)}(\lambda)$ can be computed by*

$$\begin{aligned}\mathcal{N}_n^{(\alpha)}(\lambda)&=(n-1)!\sum_{k=1}^{n}\frac{(-1)^k}{\lambda^{k-1}}\left[\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]\\&\quad\times\left[\sum_{\ell=1}^{k}\frac{\langle\alpha\rangle_\ell}{\ell!(k+\ell)!}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\sum_{m=0}^{j}\binom{j}{m}(\ell-j)^{k+\ell-m}\langle k+\ell\rangle_m\right].\end{aligned}\tag{36}$$

Proof. This follows from taking $x\to 0$ in (32). ☐

Remark 4.2. The explicit formula (27) in Theorem 3.1 is different from the explicit formula (36) in Corollary 4.1.

Remark 4.3. If taking $x\to 0$ in (33), then we deduce (27) readily.

5 Explicit formulas for Narumi polynomials and numbers

In this section, we present two explicit formulas for the Narumi polynomials $N_n^{(\alpha)}(x)$ and derive two explicit formulas for the Narumi numbers $N_n^{(\alpha)}$.

Theorem 5.1. *For $n\in\mathbb{N}$, the Narumi polynomials $N_n^{(\alpha)}(x)$ have the explicit formulas*

$$\begin{aligned}N_n^{(\alpha)}(x)&=\sum_{r=0}^{n}s(n,r)r!\sum_{k=1}^{r}\frac{(-1)^k\langle\alpha\rangle_k}{k!(r+k)!}\left(1-\frac{2x}{\alpha}\right)^{\alpha-k}\\&\quad\times\left\{\sum_{\ell=1}^{k-1}(-1)^\ell\binom{k}{\ell}\sum_{q=0}^{r-k}\binom{r+k}{2\ell+q}\left(1-\frac{x}{\alpha}\right)^{2\ell+q}\left(\frac{x}{\alpha}\right)^{r+k-(2\ell+q)}\right.\\&\quad\times\left[\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\sum_{m=0}^{j}\binom{j}{m}(\ell-j)^{2\ell+q-m}\langle 2\ell+q\rangle_m\right]\\&\quad\times\left[\sum_{j=0}^{k-\ell}(-1)^j\binom{k-\ell}{j}\sum_{m=0}^{j}\binom{j}{m}(k-\ell-j)^{r+k-(2\ell+q)-m}\langle r+k-(2\ell+q)\rangle_m\right]\\&\quad\left.+\left[(-1)^k\left(1-\frac{x}{\alpha}\right)^{r+k}+\left(\frac{x}{\alpha}\right)^{r+k}\right]\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{m=0}^{j}\binom{j}{m}(k-j)^{r+k-m}\langle r+k\rangle_m\right\}\end{aligned}\tag{37}$$

and

$$N_n^{(\alpha)}(x)=\sum_{q=0}^{n}s(n,q)\sum_{k=1}^{q}\frac{\langle\alpha\rangle_k}{k!}\biggl(1-\frac{2x}{\alpha}\biggr)^{\alpha-k}\Biggl\{\sum_{r+s=k}\sum_{\ell+m=q}(-1)^s\binom{k}{r}\binom{q}{\ell}\frac{(1-x/\alpha)^{r+\ell}}{\binom{r+\ell}{r}}\frac{(x/\alpha)^{s+m}}{\binom{s+m}{s}}\times\Biggl[\sum_{j=0}^{r}(-1)^{r-j}\binom{\ell+r}{r-j}S(\ell+j,j)\Biggr]\Biggl[\sum_{j=0}^{s}(-1)^{s-j}\binom{m+s}{s-j}S(m+j,j)\Biggr]\Biggr\}. \tag{38}$$

Proof. Taking $\ln(1+t)=u$ in (3) yields

$$\biggl(\frac{e^u-1}{u}\biggr)^{\alpha}e^{xu}=\sum_{n=0}^{\infty}N_n^{(\alpha)}(x)\frac{(e^u-1)^n}{n!},$$

where, by (1),

$$\sum_{n=0}^{\infty}N_n^{(\alpha)}(x)\frac{(e^u-1)^n}{n!}=\sum_{n=0}^{\infty}N_n^{(\alpha)}(x)\sum_{i=n}^{\infty}S(i,n)\frac{u^i}{i!}=\sum_{i=0}^{\infty}\Biggl[\sum_{n=0}^{i}S(i,n)N_n^{(\alpha)}(x)\Biggr]\frac{u^i}{i!}.$$

This implies that

$$\lim_{u\to0}\frac{\mathrm{d}^i}{\mathrm{d}u^i}\Biggl[\biggl(\frac{e^u-1}{u}\biggr)^{\alpha}e^{xu}\Biggr]=\sum_{n=0}^{i}S(i,n)N_n^{(\alpha)}(x),\quad i\ge0.$$

Making use of the result in (34) gives

$$\sum_{n=0}^{i}S(i,n)N_n^{(\alpha)}(x)=\sum_{\ell=1}^{i}\langle\alpha\rangle_\ell\biggl(1-\frac{2x}{\alpha}\biggr)^{\alpha-\ell}B_{i,\ell}\biggl(\frac{(1-x/\alpha)^2-(x/\alpha)^2}{2},\dots,\frac{(1-x/\alpha)^{i-\ell+2}-(x/\alpha)^{i-\ell+2}}{i-\ell+2}\biggr)$$

for $i\in\mathbb{N}$. In [43, p. 171, Theorem 12.1], it is stated that if b_α and a_k are a collection of constants independent of n, then

$$a_n=\sum_{\alpha=0}^{n}S(n,\alpha)b_\alpha\quad\text{if and only if}\quad b_n=\sum_{k=0}^{n}s(n,k)a_k.$$

Accordingly, we derive

$$\begin{aligned}N_i^{(\alpha)}(x)&=\sum_{n=0}^{i}s(i,n)\sum_{k=1}^{n}\langle\alpha\rangle_k\biggl(1-\frac{2x}{\alpha}\biggr)^{\alpha-k}B_{n,k}\biggl(\frac{(1-x/\alpha)^2-(x/\alpha)^2}{2},\dots,\frac{(1-x/\alpha)^{n-k+2}-(x/\alpha)^{n-k+2}}{n-k+2}\biggr)\\&=\sum_{n=0}^{i}s(i,n)\sum_{k=1}^{n}\langle\alpha\rangle_k\biggl(1-\frac{2x}{\alpha}\biggr)^{\alpha-k}\Biggl\{\frac{(-1)^k}{k!}\frac{n!}{(n+k)!}\sum_{\ell=1}^{k-1}(-1)^\ell\binom{k}{\ell}\sum_{q=0}^{n-k}\binom{n+k}{2\ell+q}\\&\quad\times\Biggl[\biggl(1-\frac{x}{\alpha}\biggr)^{2\ell+q}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\sum_{m=0}^{j}\binom{j}{m}(\ell-j)^{2\ell+q-m}\langle2\ell+q\rangle_m\Biggr]\\&\quad\times\Biggl[\biggl(\frac{x}{\alpha}\biggr)^{n+k-(2\ell+q)}\sum_{j=0}^{k-\ell}(-1)^j\binom{k-\ell}{j}\sum_{m=0}^{j}\binom{j}{m}(k-\ell-j)^{n+k-(2\ell+q)-m}\langle n+k-(2\ell+q)\rangle_m\Biggr]\\&\quad+\frac{n!}{k!(n+k)!}\Biggl[\biggl(1-\frac{x}{\alpha}\biggr)^{n+k}+(-1)^k\biggl(\frac{x}{\alpha}\biggr)^{n+k}\Biggr]\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{m=0}^{j}\binom{j}{m}(k-j)^{n+k-m}\langle n+k\rangle_m\Biggr\},\end{aligned}$$

where we used the formula (19) in Lemma 2.1. The explicit formula (37) is thus proved.

If we use the formula (20), then

$$\begin{aligned}N_i^{(\alpha)}(x) &= \sum_{n=0}^{i} s(i,n)\sum_{k=1}^{n}\langle\alpha\rangle_k\left(1-\frac{2x}{\alpha}\right)^{\alpha-k} B_{n,k}\left(\frac{(1-x/\alpha)^2-(x/\alpha)^2}{2},\dots,\frac{(1-x/\alpha)^{n-k+2}-(x/\alpha)^{n-k+2}}{n-k+2}\right)\\ &= \sum_{n=0}^{i} s(i,n)\sum_{k=1}^{n}\langle\alpha\rangle_k\left(1-\frac{2x}{\alpha}\right)^{\alpha-k}\frac{1}{k!}\sum_{r+s=k}\sum_{\ell+m=n}(-1)^s\binom{k}{r}\binom{n}{\ell}\frac{(1-x/\alpha)^{r+\ell}}{\binom{r+\ell}{r}}\frac{(x/\alpha)^{s+m}}{\binom{s+m}{s}}\\ &\quad\times\left[\sum_{j=0}^{r}(-1)^{r-j}\binom{\ell+r}{r-j}S(\ell+j,j)\right]\left[\sum_{j=0}^{s}(-1)^{s-j}\binom{m+s}{s-j}S(m+j,j)\right],\end{aligned}$$

which can be rewritten as (38). The proof of Theorem 5.1 is complete. □

Corollary 5.1. *For $n \in \mathbb{N}$, the Narumi numbers $N_n^{(\alpha)}$ have the explicit formulas*

$$N_n^{(\alpha)} = \sum_{r=0}^{n} s(n,r)r!\sum_{k=1}^{r}\frac{\langle\alpha\rangle_k}{k!(r+k)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{m=0}^{j}\binom{j}{m}(k-j)^{r+k-m}\langle r+k\rangle_m \tag{39}$$

and

$$N_n^{(\alpha)} = \sum_{q=0}^{n} s(n,q)q!\sum_{k=1}^{q}\frac{\langle\alpha\rangle_k}{(k+q)!}\sum_{j=0}^{k}(-1)^{k-j}\binom{q+k}{k-j}S(q+j,j). \tag{40}$$

Proof. These follow from taking $x \to 0$ in (37) and (38), respectively. □

Remark 5.1. When taking $\alpha = 1$ in (39) or (40), we derive the explicit formula (9) for the Cauchy numbers $C_n = N_n^{(1)}$ for $n \in \mathbb{N}$.

Remark 5.2. The first six values of the Narumi polynomials $N_n^{(\alpha)}(x)$ for $0 \le n \le 5$ can be computed by

$$N_0^{(\alpha)}(x) = 1,\quad N_1^{(\alpha)}(x) = \frac{\alpha}{2} + x,\quad N_2^{(\alpha)}(x) = \frac{\alpha^2}{4} + \alpha\left(x - \frac{5}{12}\right) + (x-1)x,$$

$$N_3^{(\alpha)}(x) = \frac{1}{8}[\alpha^3 + \alpha^2(6x-5) + 2\alpha(6x^2-11x+3) + 8x(x^2-3x+2)],$$

$$N_4^{(\alpha)}(x) = \frac{\alpha^4}{16} + \frac{1}{8}\alpha^3(4x-5) + \alpha^2\left(\frac{3x^2}{2} - 4x + \frac{97}{48}\right) + \alpha\left(2x^3 - \frac{17x^2}{2} + \frac{19x}{2} - \frac{251}{120}\right) + x(x^3-6x^2+11x-6),$$

$$\begin{aligned}N_5^{(\alpha)}(x) = \frac{1}{96}[&3\alpha^5 + 10\alpha^4(3x-5) + 5\alpha^3(24x^2-84x+61) + 2\alpha^2(120x^3-660x^2+1025x-401)\\ &+4\alpha(60x^4-460x^3+1140x^2-991x+190) + 96x(x^4-10x^3+35x^2-50x+24)].\end{aligned}$$

The first nine values of the Narumi numbers $N_n^{(\alpha)}$ for $0 \le n \le 8$ are

$$N_0^{(\alpha)} = 1,\quad N_1^{(\alpha)} = \frac{\alpha}{2},\quad N_2^{(\alpha)} = \frac{1}{12}\alpha(3\alpha-5),\quad N_3^{(\alpha)} = \frac{1}{8}\alpha(\alpha^2-5\alpha+6),$$

$$N_4^{(\alpha)} = \frac{1}{240}\alpha(15\alpha^3 - 150\alpha^2 + 485\alpha - 502),$$

$$N_5^{(\alpha)} = \frac{1}{96}\alpha(3\alpha^4 - 50\alpha^3 + 305\alpha^2 - 802\alpha + 760),$$

$$N_6^{(\alpha)} = \frac{\alpha}{4032}(63\alpha^5 - 1575\alpha^4 + 15435\alpha^3 - 73801\alpha^2 + 171150\alpha - 152696),$$

$$N_7^{(\alpha)} = \frac{\alpha}{1152}(9\alpha^6 - 315\alpha^5 + 4515\alpha^4 - 33817\alpha^3 + 139020\alpha^2 - 295748\alpha + 252336),$$

$$N_8^{(\alpha)} = \frac{\alpha}{34560}(135\alpha^7 - 6300\alpha^6 + 124110\alpha^5 - 1334760\alpha^4 + 8437975\alpha^3 - 31231500\alpha^2 + 62333204\alpha - 51360816).$$

Acknowledgements: The authors thank anonymous referees for their careful corrections to, valuable comments on, and helpful suggestions to the original version of this paper.

Funding information: The work of D. Lim was partially supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) NRF-2021R1C1C1010902.

Author contributions: All authors contributed equally to the manuscript and read and approved the final version.

Conflict of interest: The authors declare that they have no competing interests.

Data availability statement: Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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