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SH Surface Waves in a Homogeneous Gradient-Elastic Half-Space with Surface Energy

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Abstract. The existence of SH surface waves in a half-space of homogeneous material (i.e. anti-plane shear wave motions which decay exponentially with the distance from the free surface) is shown to be possible within the framework of the generalized linear continuum theory of gradient elasticity with surface energy. As is well-known such waves cannot be predicted by the classical theory of linear elasticity for a homogeneous half-space, although there is experimental evidence supporting their existence. Indeed, this is a drawback of the classical theory which is only circumvented by modelling the half-space as a layered structure (Love waves) or as having non-homogeneous material properties. On the contrary, the present study reveals that SH surface waves may exist in a homogeneous half-space if the problem is analyzed by a continuum theory with appropriate microstructure. This theory, which was recently introduced by Vardoulakis and co-workers, assumes a strain-energy density expression containing, besides the classical terms, volume strain-gradient and surface-energy gradient terms.

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1. Introduction

In the theory of wave motion in solids, the criterion for surface waves is that the displacement decays exponentially with distance from the free surface. As is well-known, the standard linear theory of elasticity predicts surface waves in a *homogeneous* half-space for the plane-stress/strain case (Rayleigh waves) but not for the anti-plane shear case (see e.g. Knowles [1], Achenbach [2], Eringen and Suhubi [3]). However, anti-plane shear surface waves (i.e. horizontally polarized or SH surface waves) have been detected in the context of both non-destructive testing (see e.g. Kraut [4]) and seismology (see e.g. [3], and Bullen and Bolt [5]). In order to explain the occurrence of these waves, *non-homogeneous* models for the half-space were proposed in the form of either a layered structure (Love waves [6], [2, 3]) or a material with mechanical properties increasing with depth (see [7–9] and References therein). To the best of our knowledge, no linear elastic theory has successfully been proposed to predict SH surface waves in a homogeneous half-space. The nonlocal integral-type elasticity theory of Eringen [10, 11] is also incapable of explaining the occurrence of these waves [11].

In mathematical language, the situation concerning inexistence of SH surface waves in a linearly elastic homogeneous (isotropic or anisotropic) half-space is tantamount to the violation of the pertinent *complementing* (or *consistency*) condition in a half-space $(-\infty < x < \infty, y \geqslant 0)$ for the system consisting of a Helmholtz partial differential equation (governing time-harmonic SH motions), a zero Neumann boundary condition at $y = 0$ (corresponding to zero traction at the surface) and a finiteness condition at $(x^2 + y^2)^{\frac{1}{2}} \to \infty$. In general, the complementing or consistency condition on boundary data (boundary conditions) in a boundary-value-problem is a suitability condition of them to the governing differential equation (or to the system of governing differential equations) (see e.g. Agmon *et al*. [12, 13]). Among others also, Stakgold briefly discusses this issue ([14], p. 64; 79–80; 85; 265) and gives a simple example concerning steady-state heat conduction ([15], p. 171). In particular, within the classical elasticity theory, Thompson [16] pointed out that the complementing condition implies that *all* surface waves propagate with non-zero velocity. Obviously, the complementing condition is satisfied for the plane-stress/strain case (and, thus, surface waves of the Rayleigh type are predicted by the classical theory) but not for the anti-plane shear case. This can be considered a rather major defect of the classical elasticity theory from the mathematics point of view, since a simple zero Neumann condition does not conform to the governing Helmholtz equation in a half-plane domain. Finally, we mention that a rigorous proof of the latter statement is provided by Vekua ([17], p. 316).

In the present study, the linear theory of gradient-elastic materials with surface energy is employed to investigate SH surface waves in a homogeneous half-space. Indeed, it is shown that this generalized continuum theory, which was recently introduced by Vardoulakis and co-workers [18–20], is capable of predicting SH surface motions. The theory is based on Mindlin's generalized elasticity theory with microstructure [21] and on Casal's 1-D constitutive model with surface energy [22–24] assuming a strain-energy density expression that contains, together with the classical terms, *volume-energy* and *surface-energy* strain-gradient terms. As far as the volume gradient term is concerned this theory is similar to the simple Aifantis' model of gradient elasticity [25, 26].

Here, free time-harmonic motions are considered and their analysis is based on the use of two-sided Laplace (or Fourier) transforms and on a parametric study of the resulting *dispersion equation*. It is also shown that *cut-off* frequencies exist and these are related to the characteristic material lengths introduced in the theory. In this way, we provide a regularization of the corresponding ill-posed problem of classical elasticity described above. This regularization, however, does not hold for *any* frequency but holds for frequencies higher than the cut-off frequency.

The present analysis could be useful in wave-propagation studies for homogeneous materials with microstructure such as materials with crystal lattices, polycrystals, granular materials and polymers. Also, recent studies [19, 20, 27, 28] suggest that the theory of gradient elasticity with surface energy yields adequate results for other interesting problems too, and therefore, that this theory is quite promising in examining situations where classical elasticity gives physically unsatisfactory results. Of course, we should mention that a similar course of action, through the use of different generalized continuum theories (couple-stress, micropolar, nonlocal or gradient-type theories), was taken up in the past by, e.g., Muki and Sternberg [29], Weitsman [30], Eringen and co-workers [10, 31, 32], Maugin [33], Nowinski [34], and Aifantis [26]. However, in defence of the classical theory, it should be noted that not in any case examined in the aforementioned studies were the new results found to be fully acceptable and worth the extra mathematical complexity.

2. Basic Preliminaries

The linear theory of gradient elasticity newly introduced by Vardoulakis and co-workers [18–20] will be utilized here to analyze SH surface waves. Generally, the central concept in gradient-type theories is the following relation between two spatially dependent properties, say A and B, of a material, where A determines B (see e.g. Maugin [33])

$$B(\mathbf{r}) = \mathcal{B}(A(\mathbf{r}), \nabla A(\mathbf{r}), \nabla\nabla A(\mathbf{r}), \ldots), \tag{1}$$

where $\mathbf{r}$ is the position vector and ∇ the gradient operator. As Eringen [32] and Maugin [33] note, the functional $\mathcal{B}$ in (1) may *alternatively* be approximated by a series of multiple volume integrals. They also provide references to earlier attempts for formulating nonlocal theories by the mechanicians of the 19th century (e.g. Voigt, Boltzmann, Cosserats, Duhem and Rayleigh). In the words of Eringen, 'However, a full construct of theories and applications did not materialize until recently. Such formalisms possess intrinsic dangers, requiring utmost care to avoid divergences, indeterminacies, inexistence and illusory or inconsistent results'.

For *linear elastic* materials, particularly, Mindlin's [21] theory provides a general framework in developing *strain-gradient* theories. In this section therefore, first we will briefly present the basic equations of this theory and then give their modified version due to Vardoulakis and co-workers [18–20].

Mindlin's theory introduced the idea of the *unit cell* (micro-medium), which may be interpreted as the periodic structure of a crystal lattice, a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material. Appropriate kinematical quantities are then defined to describe geometrical changes in both the macro- and micro-medium. Next, with respect to a Cartesian coordinate system $Ox_1x_2x_3$, the following Ansatz for the potential energy-density (potential energy per unit macro-volume) is taken:

$$W = W(\varepsilon_{qr}, \gamma_{qr}, \kappa_{qrs}), \tag{2}$$

where $\varepsilon_{qr} \equiv \frac{1}{2}(\partial_r u_q + \partial_q u_r)$ is the usual strain tensor defined in terms of the displacement vector $u_q, \partial_s \equiv \partial/\partial x_s$, the indices (q, r, s) span the range $(1, 2, 3), \gamma_{qr} \equiv \partial_q u_r - \psi_{qr}$ is the relative deformation with ψ_{qr} denoting the micro-deformation (i.e. the displacement-gradient in the micro-medium), and $\kappa_{qrs} \equiv \partial_q \psi_{rs}$ is the micro-deformation gradient. Then, appropriate definitions for the stresses follow from the variation of W:

$$\tau_{qr} \equiv \frac{\partial W}{\partial \varepsilon_{qr}}, \quad \alpha_{qr} \equiv \frac{\partial W}{\partial \gamma_{qr}}, \quad m_{qrs} \equiv \frac{\partial W}{\partial \kappa_{qrs}}, \tag{3a,b,c}$$

where $(\tau_{qr}, \alpha_{qr}, m_{qrs})$ are the Cauchy stress (symmetric), relative stress (asymmetric), and double stress tensors.

Further, from the variational equation of motion (by taking independent variations δu_q and $\delta\psi_{qr})$ and by assuming that the micro-medium is a cube with edges of length $2h$, one may obtain the following twelve stress *equations of motion* [21]:

$$\partial_q \sigma_{qr} + f_r = \rho'(\partial_{tt} u_r), \tag{4a}$$

$$\partial_q m_{qrs} + \alpha_{rs} + \Phi_{rs} = \tfrac{1}{3}\rho h^2(\partial_{tt}\psi_{rs}), \tag{4b}$$

and the twelve traction *boundary conditions*

$$t_r = n_q \sigma_{qr}, \quad T_{rs} = n_q m_{qrs}, \tag{5a,b}$$

where $\rho' \equiv \rho_M + \rho, \rho_M$ is the mass of macro-material per unit macro-volume, ρ is the mass of micro-material per unit macro-volume, $\sigma_{qr} \equiv \tau_{qr} + \alpha_{qr}$ is the total stress tensor, n_q are the components of the unit vector outnormal to the boundary, f_r is the body force per unit volume and t_r is the surface force per unit area, Φ_{rs} is the double force per unit volume (see e.g. Love [35] for an interpretation of this force system) and T_{rs} is the double force per unit area, and ∂_t denotes time differentiation. Regarding the double forces, we notice from Mindlin's [21] paper that the diagonal terms of Φ_{rs} and T_{rs} are double forces without moment and the off-diagonal terms are double forces with moment, while the antisymmetric part of Φ_{rs} is the body couple and that of T_{rs} is the Cosserat couple-stress vector. Also, in Φ_{rs} and T_{rs} the first subscript denotes the orientation of the lever arm between the forces and the second the orientation of the forces. The twenty-seven components of m_{qrs} are interpreted as double forces per unit area, with their first subscript designating the normal to the surface across which the component acts and the second and third subscripts having the same meaning as the two subscripts of T_{rs}.

It can also be shown [21] that Equations (3)–(5) contain the linear equations of a Cosserat continuum (see e.g. Mindlin and Tiersten [36], and Muki and Sternberg [29]) as a special case. However, we should mention that the above formulation involves, in its general form, a very large number of elastic constants and, therefore, applying it to practical situations may be extremely difficult. The particular theory proposed by Vardoulakis and co-workers [18–20] can be considered one of the simplest versions of Mindlin's elasticity theory with microstructure.

More specifically, Vardoulakis and co-workers suggested the following postulate for the strain-energy density function [18]:

$$W = \tfrac{1}{2}\lambda\varepsilon_{qq}\varepsilon_{rr} + \mu\varepsilon_{qr}\varepsilon_{rq} + \mu c(\partial_s\varepsilon_{qr})(\partial_s\varepsilon_{rq}) + \mu b_s\partial_s(\varepsilon_{qr}\varepsilon_{rq}), \tag{6}$$

where λ and μ are the standard Lame's constants, c is the gradient coefficient (having dimensions of [length]^2), $b_s \equiv b\nu_s$ with $\nu_s\nu_s = 1$ and b being a material length related to surface energy. Indeed, the last term in the r.h.s. of (6) is associated with surface energy since, in view of the divergence theorem, it can be written in the form [18]

$$\int_{(V)} \partial_s(b_s\varepsilon_{qr}\varepsilon_{rq})\,\mathrm{d}V = b\int_{(S)} (\varepsilon_{qr}\varepsilon_{rq})(\nu_s n_s)\,\mathrm{d}S, \tag{7}$$

where $\int_{(V)}$ denotes integration over the volume V of the body, $\int_{(S)}$ integration over the surface S enclosing V, and n_s are the components of the unit vector outnormal to the surface. Furthermore, the particular case $\nu_s \equiv -n_s$ was considered which corresponds physically to a weakening of the body along the direction normal to the surface.

Compatible with the above W-expression is taking: (i) $\gamma_{qr} = 0$ and, therefore, $\psi_{qr} \equiv \partial_q u_r, \kappa_{qrs} \equiv \partial_q\partial_r u_s = \kappa_{rqs}$ and $m_{qrs} \equiv \partial W/\partial\kappa_{qrs} = m_{rqs}$ (this defines the so-called restricted Mindlin continuum), (ii) $\rho_M = 0$ and $\rho' = \rho$, so as to let the micro-medium to merge with the macro-medium, and (iii) the relative stress α_{qr} to be workless. Accordingly, the respective variational equation of motion is obtained by taking as the only independent variation in the potential energy-density the quantity δu_q since the ψ_{qr} are no longer independent of u_q ([21], [18]). The stress equations of motion in the absence of body forces and the traction boundary conditions along a smooth boundary are then written as

$$\partial_q(\tau_{qr} + \alpha_{qr}) = \rho(\partial_{tt}u_r), \tag{8a}$$

$$\partial_q m_{qrs} + \alpha_{rs} = \tfrac{1}{3}\rho h^2(\partial_{tt}\psi_{rs}), \tag{8b}$$

$$\begin{aligned} &n_r\tau_{rs} - n_q n_r n_s\partial_s m_{qrs} - 2n_r(\delta_{q\ell} - n_q n_\ell)\partial_\ell m_{qrs} \\ &\quad + (n_q n_r n_\ell(\delta_{\ell j} - n_\ell n_j)\partial_j - n_q(\partial_{r\ell} - n_r n_\ell)\partial_\ell)m_{qrs} \\ &\quad + \tfrac{1}{3}\rho h^2 n_r(\partial_{tt}\psi_{rs}) = P_s, \end{aligned} \tag{9a}$$

$$n_q n_r m_{qrs} = R_s, \tag{9b}$$

where $\delta_{q\ell}$ is the Kronecker delta, P_s is the surface force per unit area, and R_s is the surface double force (without moment) per unit area.

Finally, the constitutive equations follow by combining (3a) and (3c) with (6) [18]:

$$\tau_{qr} = \lambda\delta_{qr}\varepsilon_{ss} + 2\mu\varepsilon_{qr} + 2\mu b_s(\partial_s\varepsilon_{qr}), \tag{10}$$

$$m_{sqr} = 2\mu[b_s\varepsilon_{qr} + c(\partial_s\varepsilon_{qr})]. \tag{11}$$

Notice that the relative stress α_{qr} can explicitly be obtained only by (8b) and (11). Obtaining α_{qr} permits, in turn, the determination of the total stress σ_{qr} through (10) and the definition equality $\sigma_{qr} \equiv \tau_{qr} + \alpha_{qr}$.

In closing this exposition of basic notions and relations, we notice that positive definiteness of the potential-energy density suggests the following restrictions of the material constants (Vardoulakis and Sulem [18]; see also Refs. [21], [36] and [37]):

$$(3\lambda + 2\mu) > 0, \quad \mu > 0, \quad c > 0, \quad -1 < (b/c^{\frac{1}{2}}) < 1, \tag{12a,b,c,d}$$

whereas attempts to determine the gradient coefficient c have been made within a simpler gradient elasticity theory (Altan and Aifantis [25]) providing the estimate

$$c = (0.25h)^2. \tag{13}$$

3. Governing Equations for a Time-Harmonic Anti-Plane Shear State in a Half-Space

Interest now is focussed on anti-plane shear (i.e. horizontally polarized or SH) motions in a gradient-elastic half-space with surface energy. With respect to an $Oxyz$ Cartesian coordinate system, the half-space occupies the region $(-\infty < x < \infty, y \geqslant 0)$ and is thick enough in the z-direction to allow an anti-plane shear state when the loadings act in the same direction (see Figure 1). In this case, any problem is essentially two-dimensional depending on (x, y).

Then, in view of (6)–(11) and by also taking $b_x = b_z = 0, b_y \equiv b \neq 0$ for the present case of SH motions in the half-space $y \geqslant 0$, we have

$$u_x = u_y = 0, u_z \equiv w(x, y, t) \neq 0, \tag{14a,b}$$

$$\tau_{xz} = \mu\frac{\partial w}{\partial x} + \mu b\frac{\partial^2 w}{\partial x\partial y}, \tag{15a}$$

$$\tau_{yz} = \mu\frac{\partial w}{\partial y} + \mu b\frac{\partial^2 w}{\partial y^2}, \tag{15b}$$

$$m_{xxz} = \mu c\frac{\partial^2 w}{\partial x^2}, \tag{16a}$$

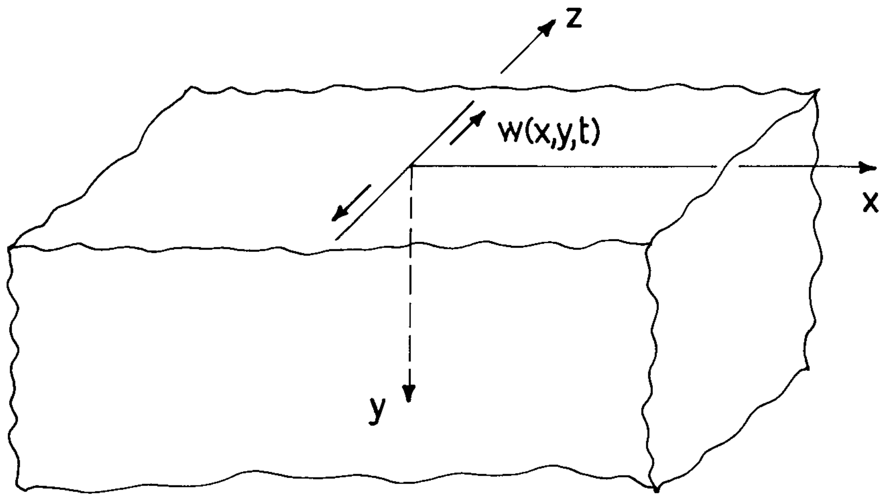

Figure 1. A gradient-elastic half-space with surface energy in an anti-plane shear state.

$$m_{xyz} = \mu c \frac{\partial^2 w}{\partial x \partial y}, \tag{16b}$$

$$m_{yxz} = \mu b \frac{\partial w}{\partial x} + \mu c \frac{\partial^2 w}{\partial x \partial y}, \tag{16c}$$

$$m_{yyz} = \mu b \frac{\partial w}{\partial y} + \mu c \frac{\partial^2 w}{\partial y^2}, \tag{16d}$$

$$\sigma_{xz} = \tau_{xz} + \alpha_{xz}, \tag{17a}$$

$$\sigma_{yz} = \tau_{yz} + \alpha_{yz}, \tag{17b}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = \rho \frac{\partial^2 w}{\partial t^2}, \tag{18}$$

$$\frac{\partial m_{xxz}}{\partial x} + \frac{\partial m_{yxz}}{\partial y} + \alpha_{xz} = I \frac{\partial^2 \psi_{xz}}{\partial t^2}, \tag{19a}$$

$$\frac{\partial m_{xyz}}{\partial x} + \frac{\partial m_{yyz}}{\partial y} + \alpha_{yz} = I \frac{\partial^2 \psi_{yz}}{\partial t^2}, \tag{19b}$$

where (u_x, u_y, u_z) are the displacements, (τ_{xz}, τ_{yz}), $(m_{xxz}, \ldots, m_{yyz})$, $(\sigma_{xz}, \sigma_{yz})$ and $(\alpha_{xz}, \alpha_{yz})$ are, respectively, the Cauchy stresses, double stresses, total stresses and relative (workless) stresses, $\psi_{xz} \equiv \partial w/\partial x$ and $\psi_{yz} \equiv \partial w/\partial y$ are the micro-deformations, and $I \equiv \frac{1}{3}\rho h^2$ is the micro-inertia coefficient.

Furthermore, if a *steady-state* response of the half-space is assumed, i.e.

$$w(x,y,t) = w(x,y) \cdot \mathrm{e}^{-i\omega t}, \tag{20}$$

with $i \equiv (-1)^{\frac{1}{2}}$ and ω being the frequency, the total stresses are explicitly obtained through (15), (17) and (19) as

$$\sigma_{xz} = \mu g \frac{\partial w}{\partial x} - \mu c \nabla^2 \left(\frac{\partial w}{\partial x} \right), \tag{21a}$$

$$\sigma_{yz} = \mu g \frac{\partial w}{\partial y} - \mu c \nabla^2 \left(\frac{\partial w}{\partial y} \right), \tag{21b}$$

and (18) and (21) provide the field equation of the response

$$c\nabla^4 w - g\nabla^2 w - k^2 w = 0, \tag{22}$$

where $\nabla^2 \equiv (\partial^2/\partial x^2) + (\partial^2/\partial y^2), \nabla^4 \equiv \nabla^2\nabla^2$ and

$$g = 1 - \frac{\omega^2 I}{\mu}, \tag{23}$$

$$k = \frac{\omega}{V}, \quad V = \left(\frac{\mu}{\rho} \right)^{\frac{1}{2}}. \tag{24a,b}$$

We notice in the above relations that V is the shear-wave velocity in the absence of gradient effects (viz. within the classical theory of elasticity), and the dimensionless quantity g may take *negative, zero*, or *positive* values depending upon the particular values of ω and I. The latter observation means, of course, that the character of the partial differential equation (22) may change with g. By the way, in a material with a shear modulus $\mu = O(10^9\, Nm^{-2})$ and mass density $\rho = O(10^3\, kgm^{-3})$, the case $g < 0$ occurs for a rather high excitation frequency $\omega > 173\, GHz$ if the internal length is of the order $h = O(10^{-8}\, m)$ – which is typical for several *crystal lattices* – but the same case occurs also for a much lower frequency $\omega > 17.3\, MHz$ if $h = O(10^{-4}\, m)$ – which is typical for several *granular materials*. Also, we should observe that (22) exhibits a dispersive character, a fact which is rather typical in gradient continuum theories (see e.g. [21], [33] and [36]). Finally, for low frequencies ω a good approximation could be $g = 1$, but here we rather chose to employ in the sequel the exact expression for g (i.e. Equation (23)).

4. General Form of SH Surface Waves

The criterion for surface waves is that the displacement decays exponentially with distance from the free surface. Then, if we consider plane wave solutions of the form $\exp[i(qx - \omega t)]$ with a dispersion relation $\omega = \omega(q)$, a distinct harmonic component of propagation of an SH wave satisfying the governing equations (14)–(19) in the half-space $y \geqslant 0$ will be expressed as

$$\begin{aligned}\overline{w}(x, y, t) &= [B(q) \cdot \mathrm{e}^{-\beta(q) \cdot y} + C(q) \cdot \mathrm{e}^{-\gamma(q) \cdot y}] \cdot \mathrm{e}^{i[qx - \omega(q) \cdot t]} \\ &\equiv w^*(q, y) \cdot \exp[iq(x - C_{ph} \cdot t)], \end{aligned} \tag{25}$$

where q is the wavenumber, ω is the frequency, C_{ph} is the phase velocity defined by $C_{ph} = \omega/q$, $B(q)$ and $C(q)$ are arbitrary amplitude functions, and

$$\beta(q) \equiv \beta = (q^2 - \sigma^2)^{\frac{1}{2}}, \tag{26a}$$

$$\gamma(q) \equiv \gamma = (q^2 + \tau^2)^{\frac{1}{2}}, \tag{26b}$$

with

$$\sigma = \frac{[(g^2 + 4ck^2)^{\frac{1}{2}} - g]^{\frac{1}{2}}}{(2c)^{\frac{1}{2}}}, \tag{27a}$$

$$\tau = \frac{[(g^2 + 4ck^2)^{\frac{1}{2}} + g]^{\frac{1}{2}}}{(2c)^{\frac{1}{2}}} \tag{27b}$$

being real and positive quantities.

We should notice that (25) results by applying the two-sided Laplace transform

$$f^*(p, y) = \int_{-\infty}^{\infty} f(x, y) \cdot \mathrm{e}^{-px}\, \mathrm{d}x, \quad \text{with} \quad p \equiv iq, \tag{28a,b}$$

to the field equation (22) and thus getting

$$w^*(p, y) = B(p) \cdot \mathrm{e}^{-\beta(p) \cdot y} + C(p) \cdot \mathrm{e}^{-\gamma(p) \cdot y} \quad \text{for} \quad y \geqslant 0, \tag{29}$$

with $\beta(p) \equiv \beta = i(p^2 + \sigma^2)^{\frac{1}{2}}$ and $\gamma(p) \equiv \gamma = (\tau^2 - p^2)^{\frac{1}{2}}$, as the general solution (bounded as $y \to \infty$) of the resulting ordinary differential equation

$$c\frac{d^4 w^*}{dy^4} + (2cp^2 - g)\frac{d^2 w^*}{dy^2} + (cp^4 - gp^2 - k^2)w^* = 0. \tag{30}$$

We also notice that the transform (28a) is equivalent to the Fourier transform

$$f^*(q, y) = \int_{-\infty}^{\infty} f(x, y) \cdot \mathrm{e}^{-iqx}\, \mathrm{d}x, \tag{31}$$

whereas the requirement of solution *finiteness* at infinity leads to a choice of branch cuts for β and γ so that the radicals have nonnegative real parts (see Figure 2).

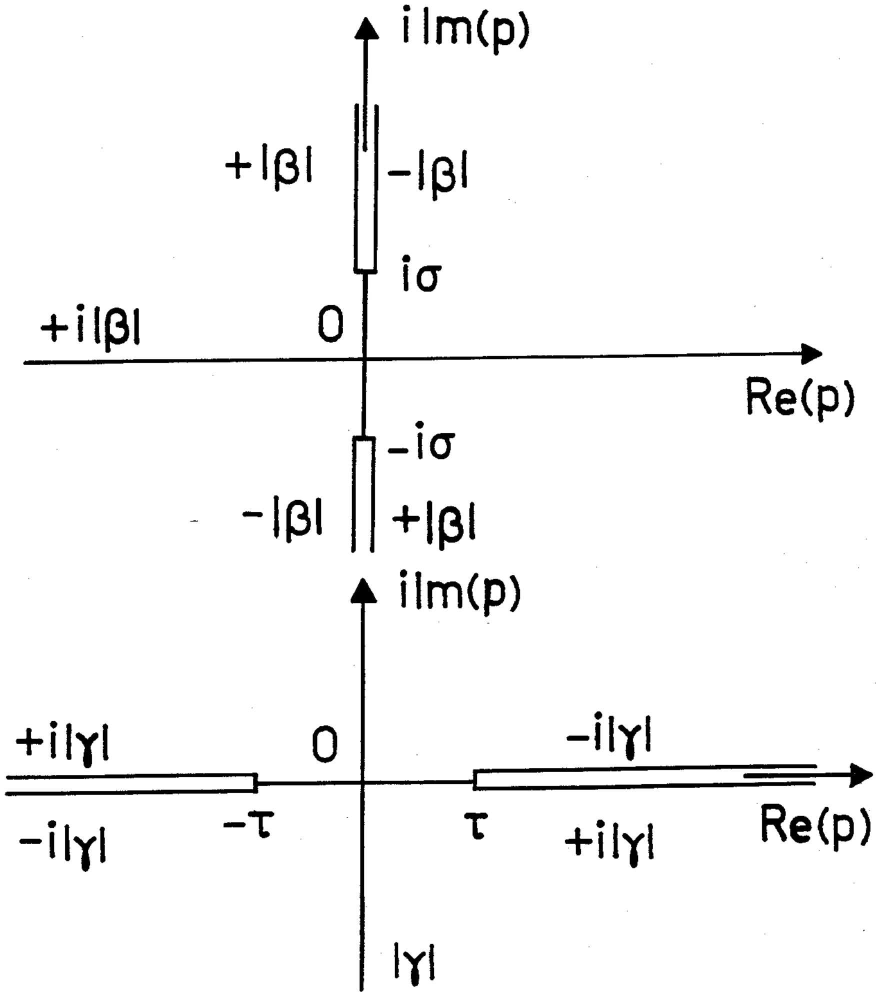

Figure 2. The cut complex p-plane for the functions $\beta(p)$ and $\gamma(p)$.

Finally, if a number of distinct harmonic components of propagation were possible, there would be a *dispersion relation* $\omega = \omega(q)$ for each component. Therefore, a general solution (synthesis) can be expressed by a Fourier integral (inversion integral)

$$w(x, y, t) = \frac{1}{2\pi} \int_{\Gamma_1} w^*(q, y) \cdot \mathrm{e}^{i[qx - \omega(q) \cdot t]} \, \mathrm{d}q, \tag{32}$$

or, equivalently, by a two-sided Laplace transform inversion

$$w(x, y, t) = \frac{1}{2\pi i} \int_{\Gamma_2} w^*(p, y) \cdot \mathrm{e}^{px - i\omega(p) \cdot t} \, \mathrm{d}p, \tag{33}$$

where Γ_1 and Γ_2 are straight paths in the pertinent complex planes and are parallel, respectively, to the axes $Re(q)$ and $Im(p)$. Then, the amplitude functions $B(q)$ and $C(q)$ in (25) (or $B(p)$ and $C(p)$ in (29)) denote the relative dominance of a particular harmonic component.

Here, we assume *free* motions and therefore the frequency should be taken *real-valued*. As is also evident from Equation (25), displacements associated with positive [negative] imaginary wavenumbers decay [grow] exponentially with x, thus representing no progressive waves but rather localized *standing* wave motions (these modes are known as leaky or evanescent modes – see e.g. Bullen and Bolt [5]). As is well-known, for a particular mode the frequency at which the wavenumber changes from real to imaginary (or complex) values is called the *cut-off* frequency. After these preliminaries, we will see indeed which conditions should be satisfied for the existence of SH surface waves in a gradient-elastic half-space with surface energy, and how cut-off frequencies for these waves can be determined.

In view now of the criterion for the existence of surface waves stated at the beginning of this section, we explore the possibility of solutions of the form (25) with *real* and *positive* functions $\beta(q)$ and $\gamma(q)$ (or $\beta(p)$ and $\gamma(p)$). The latter restriction is satisfied if and only if q is real (p is imaginary) such that $-\infty < q \equiv Im(p) < -\sigma$ or $\sigma < q \equiv Im(p) < \infty$. Then, surface SH-waves are represented by

$$\begin{aligned}\overline{w}_s(x,y,t) \;&=\; [B(q)\cdot \mathrm{e}^{-|\beta|\cdot y} + C(q)\cdot \mathrm{e}^{-|\gamma|\cdot y}]\cdot \mathrm{e}^{i[qx-\omega(q)\cdot t]} \\ &\equiv\; w_s^*(q,y)\cdot \mathrm{e}^{i[qx-\omega(q)\cdot t]} \equiv w_s^*(p,y)\cdot \mathrm{e}^{px-i\omega(p)\cdot t}, \end{aligned} \tag{34}$$

where

$$|\beta| = (q^2-\sigma^2)^{\frac{1}{2}}, \tag{35a}$$

$$|\gamma| = (q^2+\tau^2)^{\frac{1}{2}} \tag{35b}$$

are real and positive functions since q itself is real and $\sigma < |q|$.

Next, the appropriate *dispersion (*or *frequency) equation* can be obtained by enforcing the pertinent boundary conditions at the half-space surface. These are zero traction conditions (since no force acts on the surface – free motions) which from (9) read as

$$\sigma_{yz}(x,y=0) = 0 \quad \text{for} \quad -\infty < x < \infty, \tag{36a}$$

$$m_{yyz}(x,y=0) = 0 \quad \text{for} \quad -\infty < x < \infty. \tag{36b}$$

The above conditions are Laplace transformed according to (28a) into

$$\sigma_{yz}^*(p=iq,y=0) = 0, \tag{37a}$$

$$m^*_{yyz}(p = iq, y = 0) = 0. \tag{37b}$$

In addition, general forms of the transformed stresses $\sigma^*_{yz}(p = iq, y \geqslant 0)$ and $m^*_{yyz}(p = iq, y \geqslant 0)$ can be obtained from (21b), (16d), (28a) and (29) as

$$\sigma^*_{yz}(p = iq, y \geqslant 0) = -\mu c\tau^2 B\beta \cdot \mathrm{e}^{-\beta y} + \mu c\sigma^2 C\gamma \cdot \mathrm{e}^{-\gamma y}, \tag{38}$$

$$m^*_{yyz}(p = iq, y \geqslant 0) = \mu[(c\beta - b)B\beta \cdot \mathrm{e}^{-\beta y} + (c\gamma - b)C\gamma \cdot \mathrm{e}^{-\gamma y}]. \tag{39}$$

Of course, when only the *surface* wave solution (34) is considered and not the *general* one in (25), β and γ in Equations (38) and (39) are understood as, respectively, $|\beta|$ and $|\gamma|$ which were defined in (35).

Combining now (37) with (38) and (39) provides the linear homogeneous system for the unknown functions B and C in the case of SH surface waves:

$$-\mu c\tau^2|\beta| \cdot B + \mu c\sigma^2|\gamma| \cdot C = 0, \tag{40a}$$

$$(c \cdot |\beta| - b) \cdot |\beta| \cdot B + (c \cdot |\gamma| - b) \cdot |\gamma| \cdot C = 0, \tag{40b}$$

which has a nontrivial solution if and only if

$$-c[\sigma^2(q^2 - \sigma^2)^{\frac{1}{2}} + \tau^2(q^2 + \tau^2)^{\frac{1}{2}}] + ba^2 = 0, \tag{41}$$

with

$$a^2 \equiv \sigma^2 + \tau^2 = \frac{(g^2 + 4ck^2)^{\frac{1}{2}}}{c} > 0, \tag{42}$$

and q being a real number such that $\sigma < |q| < \infty$.

Therefore, Equation (41) constitutes the dispersion equation for surface waves. An immediate observation on this is that SH surface waves do exist only when $c \neq 0$ and $b > 0$; the cases $(c = 0)$ or $(c \neq 0$ and $b = 0)$ or $(c \neq 0$ and $b < 0)$ imply the non-existence of such motions. This finding means that the inclusion of the surface- energy strain gradient term (i.e. gradient anisotropy) is necessary for predicting surface SH-waves. It is also noted that, unlike the case of an inhomogeneous elastic half-space where transcendental equations appear giving rise to an *infinity of modes* [8, 9], the governing dispersion equation here for the considered gradient elastic medium with surface energy is an irrational algebraic equation. Accordingly, a *single* mode of SH surface waves may exist that is directly related to the parameter $(b/c^{\frac{1}{2}})$. This, in fact, is shown from our numerical results presented below.

5. Dispersion Curves and Numerical Results

It is seen from the results of the last section that the phase velocity C_{ph} and the wavenumber q are interrelated through the dispersion equation (41). The objective of this section is to produce dispersion curves from Equation (41), i.e. curves of (C_{ph}/V vs. qh), (C_{ph}/V vs. λ/h) and (q_d vs. ω_d), where the following relations define, respectively, the wavelength, a dimensionless wavenumber and a dimensionless frequency:

$$\lambda = \frac{2\pi}{q}, \tag{43}$$

$$q_d = c^{\frac{1}{2}} q, \tag{44}$$

$$\omega_d = \frac{\omega}{\omega_m}, \quad \omega_m \equiv \frac{3^{\frac{1}{2}} V}{h}. \tag{45a,b}$$

In particular, the definitions in (45) along with Equation (24b) allow us to write (23) in the form

$$g = 1 - \omega_d^2, \tag{46}$$

and present subsequent results in a convenient manner.

Results were obtained here for *specific* values of the gradient coefficient c and the surface-energy length b w.r.t. the internal length h. In other words, we express b and c in terms of h. Particularly, in what follows we accept the validity of the estimate (13), i.e. that $c = (0.25\,h)^2$, and also we consider three distinct cases where the dimensionless parameter $b_d \equiv b/c^{\frac{1}{2}}$ takes the values 0.1, 0.5, and 0.75. The latter choice of the values of b_d is, of course, in accordance with the restriction (12d). Moreover, (45) leads to the relation $k \equiv \omega/V = 3^{\frac{1}{2}} \omega_d/h$ and further, along with (13), yields $4ck^2 = (\frac{3}{4})\omega_d^2$ and

$$\frac{C_{ph}}{V} = \frac{3^{\frac{1}{2}}}{4} \frac{\omega_d}{q_d}, \quad qh = 4q_d. \tag{47a,b}$$

For convenience also, Equation (41) is normalized to yield the form

$$-[\sigma_d^2 (q_d^2 - \sigma_d^2)^{\frac{1}{2}} + \tau_d^2 (q_d^2 + \tau_d^2)^{\frac{1}{2}}] + b_d a_d^2 = 0, \tag{48}$$

through the substitutions

$$\sigma_d = c^{\frac{1}{2}} \sigma, \quad \tau_d = c^{\frac{1}{2}} \tau, \quad a_d = c^{\frac{1}{2}} a. \tag{49a,b,c}$$

Of course, the following restriction should also accompany (48):

$$\sigma_d < |q_d| < \infty. \tag{50}$$

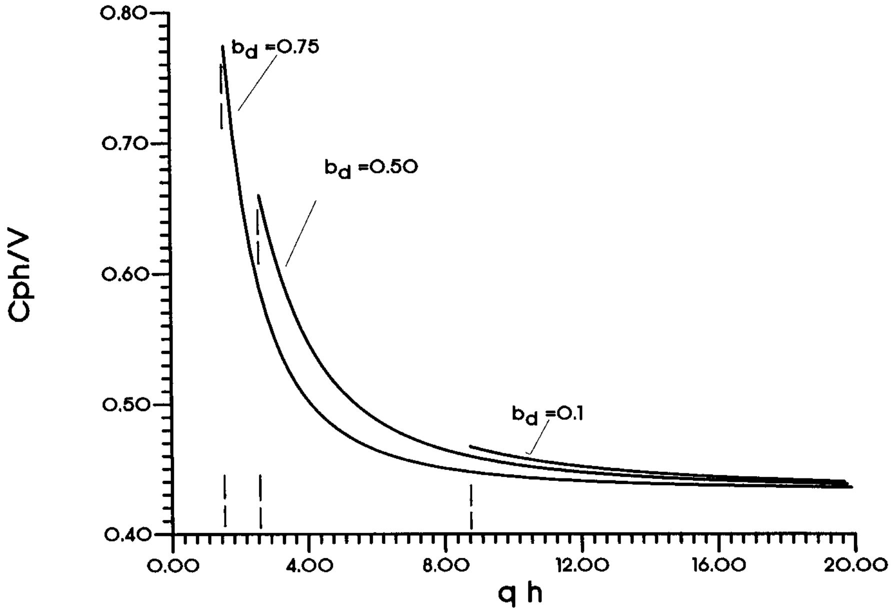

Figure 3. Dispersion curves for the propagation of SH surface waves showing the variation of the normalized phase velocity (C_{ph}/V) with the normalized wavenumber qh.

The solution of the irrational dispersion equation (48) was obtained here by a combination of exact analysis, use of the symbolic-manipulations program MATHEMATICA and FORTRAN programming. It should also be mentioned that: (i) due to the process of rationalization of Equation (48) *extraneous* roots will appear, so a check must always be performed to find out which roots do satisfy the original equation, and (ii) the occurrence of extraneous roots or complex roots marks the cut-off frequency.

The final form of (48) is

$$A_1(b_d, \omega_d) \cdot q_d^4 + A_2(b_d, \omega_d) \cdot q_d^2 + A_3(b_d, \omega_d) = 0, \tag{51}$$

that is a quartic equation for q_d which is *parametric* in b_d and ω_d. Of course, by assigning the specific values $b_d = 0.1, 0.5$, or 0.75 mentioned earlier, (51) becomes parametric only in ω_d. The coefficients, A_1, A_2 and A_3 are complicated functions, the determination of which was accomplished by using MATHEMATICA.

Our numerical results reveal that *propagation* of SH surface waves takes place when the normalized frequency $\omega_d \equiv \omega h/3^{\frac{1}{2}}V$ lies in the ranges

$$\omega_d \geqslant 2.35703 \quad \text{for} \quad b_d = 0.1, \tag{52a}$$

$$\omega_d \geqslant 0.990505 \quad \text{for} \quad b_d = 0.5, \tag{52b}$$

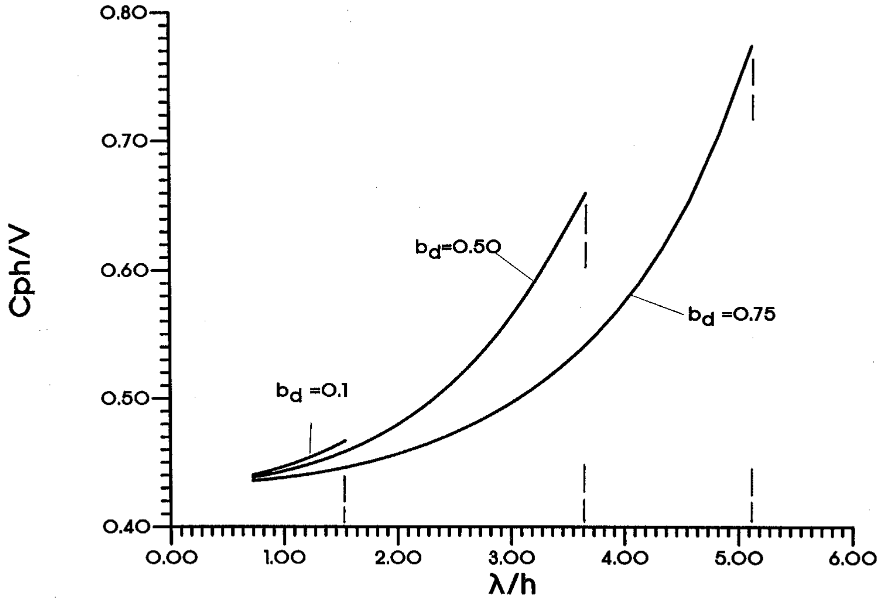

Figure 4. Dispersion curves for the propagation of SH surface waves showing the variation of the normalized phase velocity (C_{ph}/V) with the normalized wavelength λ/h.

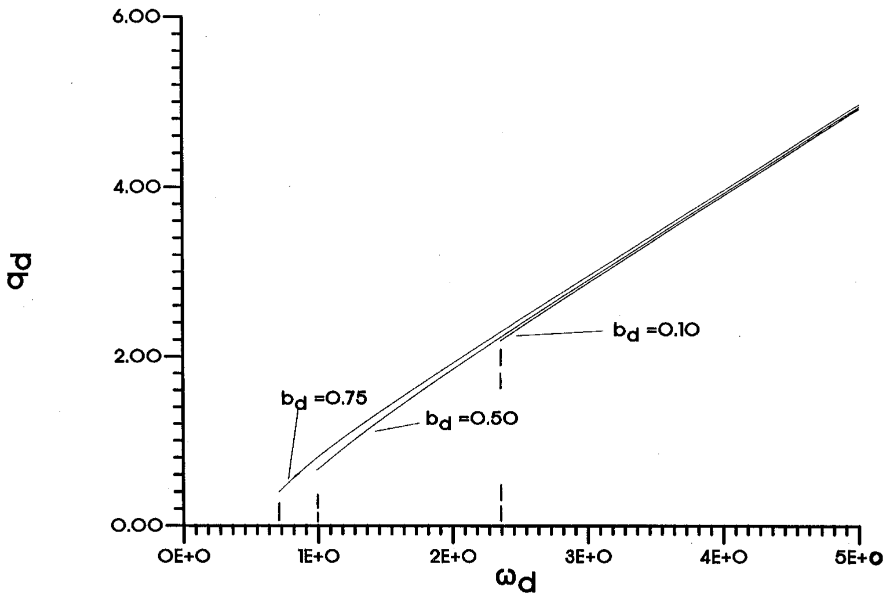

Figure 5. Dispersion curves for the propagation of SH surface waves showing the variation of the normalized wavenumber $q_d \equiv c^{\frac{1}{2}} q$ with the normalized frequency $\omega_d \equiv h\omega/3^{1/2}V$.

$$\omega_d \geqslant 0.70736 \quad \text{for} \quad b_d = 0.75. \tag{52c}$$

In these cases, q_d is *real* and satisfies both (48) and (50).

Also, when ω_d lies in the following ranges, *real* but *extraneous* roots for q_d occur (these, of course, are to be rejected):

$$0.99550 < \omega_d < 2.35703 \quad \text{for} \quad b_d = 0.1, \tag{53a}$$

$$0.86650 < \omega_d < 0.990505 \quad \text{for} \quad b_d = 0.5, \tag{53b}$$

$$0.66400 < \omega_d < 0.70736 \quad \text{for} \quad b_d = 0.75. \tag{53c}$$

Finally, in the following cases the equalities mark the frequencies $\omega = \omega_d \cdot \omega_m$ for which *complex* roots for q_d first occur:

$$\omega_d \leqslant 0.9955 \quad \text{for} \quad b_d = 0.1, \tag{54a}$$

$$\omega_d \leqslant 0.8665 \quad \text{for} \quad b_d = 0.5, \tag{54b}$$

$$\omega_d \leqslant 0.6640 \quad \text{for} \quad b_d = 0.75. \tag{54c}$$

Some results in the form of graphs are given now. Figure 3 presents the variation of the normalized phase velocity C_{ph}/V with the normalized wavenumber qh. This was obtained by solving the dispersion equation (48) in the range (50) and taking into account the relations (44), (45), and (47). Propagation of SH surface waves is implied here when qh is greater than the value at which each graph starts in the left-hand side of the figure. This range of qh-values extends of course beyond $qh = 20.0$. One could observe that for $qh > 16.00$ (i.e. for high-frequency waves) the values of $b_d \equiv b/c^{\frac{1}{2}}$ play no significant role. Figure 4 presents the graph C_{ph}/V vs. the normalized wavelength λ/h. In this case, cut-off wavelengths are the ones at which each graph stops in the right-hand side of the Figure. Finally, Figure 5 depicts the variation of the normalized wavenumber $q_d \equiv c^{\frac{1}{2}}q$ with the normalized frequency $\omega_d \equiv \omega h/3^{\frac{1}{2}}V$ for, again, three different values of b_d. Here, cut-off frequencies are those at which each graph starts in the left-hand side of the figure.

In closing, we present Table 1 where some numerical results are summarized for two different materials. These materials have the following constants. *Material I:* shear modulus $\mu = 2.1 \times 10^9\, Nm^{-2}$, mass density $\rho = 1190\, kgr/m^3, V \equiv (\mu/\rho)^{\frac{1}{2}} = 1321\, m\, sec^{-1}$, internal characteristic length $h = 10^{-8}\, m, \omega_m \equiv 3^{\frac{1}{2}} V/h = 2.28881 \times 10^{11}\, Hz$. *Material II:* $\mu = 30.5 \times 10^9\, Nm^{-2}, \rho = 2717\, kgr/m^3, V = 3350\, m\, sec^{-1}, h = 4 \times 10^{-4}\, m, \omega_m = 1.45059 \times 10^7\, Hz$. In particular, the latter constants (i.e. μ, ρ and h) are for Dionysos Marble as measured in the study of Vardoulakis *et al*. [38]. One can immediately observe that the value of the internal length h plays a significant role in the occurrence of a cut-off

Table I. The range of values of the normalized frequency ω_d and frequencies ω (for Material I and II) for which SH surface waves exist or do not exist.

			Material I	**Material II**
b_d	ω_d	Roots of Equation (48)	ω (in Hz)	ω (in Hz)
		real		
	2.35703		5.394×10^{11}	3.419×10^{7}
0.10		real but extraneous		
	0.99550		2.278×10^{11}	1.444×10^{7}
		complex		
		real		
	0.990505		2.267×10^{11}	1.437×10^{7}
0.50		real but extraneous		
	0.866500		1.983×10^{11}	1.257×10^{7}
		complex		
		real		
	0.70736		1.619×10^{11}	1.026×10^{7}
0.75		real but extraneous		
	0.66400		1.519×10^{11}	0.963×10^{7}
		complex		

frequency. The greater the internal length is, the cut-off frequency occurs at a lower value.

6. Concluding Remarks

The present study explored the possibility of predicting SH surface waves in a *homogeneous* elastic half-space by a generalized continuum theory. Indeed, the linear theory of gradient elasticity with surface energy [18–21] proved successful in predicting the occurence of these waves. It is noticed that no other linear elasticity theory (either isotropic or anisotropic), of those attempting to resolve the issue of SH surface waves in a homogeneous half-space, has given satisfactory results. Among the latter theories, e.g., classical linear elasticity [2, 3, 5, 6], non-local integral-type elasticity [10, 11] and simple gradient-type (without surface-energy effects) elasticity [25, 26] are included.

It was also shown here, by employing a simple integral-transform analysis, that a dispersion equation is obtained for the propagation of a *single* mode of SH surface waves and that this equation interrelates basically the phase velocity with the wave number. Numerical results were presented in a normalized fashion. An application employing material constants of a granular macromorphic rock (Dionysos Marble) was also included. The results generally show the dependence of cut-off frequencies upon the size of the unit cell (micro-medium), which may be interpreted as the periodic structure of a crystal lattice, a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material.

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