Granul. Comput. (2017) 2:85–94
DOI 10.1007/s41066-016-0029-y

ORIGINAL ARTICLE

Shadowed sets of dynamic fuzzy sets

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Received: 9 October 2015 / Accepted: 19 September 2016 / Published online: 27 September 2016

Abstract In practical situations, fuzzy sets with time-varying membership degrees are frequently encountered. In this paper, we interpret dynamic fuzzy sets by means of shadowed sets. We provide an analytic solution to computing the pair of thresholds by searching for a balance of uncertainty in the framework of shadowed sets. Subsequently, we construct errors-based three-way approximations of shadowed sets and present an alternative decision-theoretic formulation for calculating the pair of thresholds. Finally, we employ several examples to illustrate how to calculate thresholds for making a decision by means of dynamic loss functions.

Keywords Fuzzy set · Shadowed set · Decision-theoretic rough sets · Three-way decision

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1 Introduction

Fuzzy set theory, proposed by Zadeh (1965) in 1965, is a powerful mathematical tool to describe uncertainty information. The concept of membership function, taking its values in the unit interval, is a fundamental notion of fuzzy set theory. However, according to (Pedrycz 1998), excessive precision of fuzzy sets has been questioned due to the conceptual shortcoming associated with precise numeric values, and we need to distinguish objects by a lot of levels of fuzziness, since human only process about seven plus or minus two units of information in practice. Hence, a new tread of fuzzy set theory is motivated by the approximations of fuzzy sets by less number of membership degrees.

Indeed, approximations of fuzzy sets by finite memship grades have been extensively investigated in the literature (Banerjee and Pal 1996; Chakrabarty et al. 1998; Chanas 2001; Dubois and Prade 1980; Grzegorzewski 2002, 2013; Klir and Bo 1995; Liang et al. 2013; Nasibov and Peker 2008; Zadeh 2008). Shadowed sets, amongst others, represent one of the related theories and has gained a growing interests in recent years (Cattaneo and Ciucci 2003a, b, 2008; Mitra and Kundu 2011; Pedrycz 1999, 2004, 2005, 2009; Pedrycz et al. 2009; Wang and Wang 2012; Zhou et al. 2011). In Pedrycz's model (Pedrycz 1998), an element with membership degree close to 1 will be evaluated to 1, we handle the element as well as 1; an element with membership grade close to 0 will be reduced to 0, we handle the element as well as 0; and the other elements would be put into a shadowed region. In practical situations, membership grade of fuzzy sets varies with time. Such a type of fuzzy sets is always called dynamic fuzzy sets (Wang et al. 1988; Solana-Cipres et al. 2009; Lopes et al. 2013). For example, a 20-year-old man was viewed as an old man in the primitive society, a 20-year-old man was viewed as a middle-aged man in feudal society, and a 20-year-old man was viewed as a young man now. Therefore, dynamic fuzzy sets are much more expressive than static fuzzy set, and it is thus of great significance to express shadowed sets by means of dynamic fuzzy sets.

Much research work has focused on three-way decision theory (Feng and Mi 2016; Liu et al. 2016; Ma et al. 2014; Lang and Yang 2015; Min and Xu 2016; Sang et al. 2016; Skowron et al. 2016; Zhang and Min 2016; Yao 2003, 2008, 2010, 2011a, b; Hu 2014; Hu et al. 2016; Li et al. 2016; Yu et al. 2016) and shadowed sets (Deng and Yao 2014) by loss functions. For example, Li and Zhou 2011 evaluated the cost and benefit of assigning an instance to a specific subcategory and defined a general loss function for supervised learning. Liang and Liu (Liang et al. 2013; Liang and Liu 2014) presented triangular fuzzy decision-theoretic rough sets and systematic studies on three-way decisions with interval-valued decision-theoretic rough sets. Liu et al. (2011, 2012, 2013) proposed stochastic decision-theoretic rough sets, interval-valued decision-theoretic rough sets, fuzzy decision-theoretic rough sets, and dynamic decision-theoretic rough sets. Deng and Yao (2013, 2014) presented decision-theoretic three-way approximations of fuzzy sets on the basis of decision-theoretic rough sets. More specifically, they computed a pair of thresholds for three-way approximations of fuzzy sets by loss functions, and classified a set of objects into three regions by the pair of thresholds. In other words, one of the following three decisions can be made for each object: elevate the membership grade to 1, reduce the membership grade to 0, and change the membership grade to a third intermediate value. In practical situations, dynamic loss functions are of interest, because such functions are frequently encountered; moreover, relevant studies on shadowed sets by dynamic loss functions have not been conducted so far. Therefore, it is urgent to further study dynamic loss functions for making a decision by three-way decision.

The purpose of this paper is to further investigate shadowed sets. Section 2 recalls the basic principles of shadowed sets. Section 3 introduces shadowed sets of dynamic fuzzy sets. Section 4 provides errors-based interpretation of shadowed sets. Section 5 shows how to compute the thresholds based on decision-theoretic rough sets. This paper is completed with some concluding remarks in Sect. 6.

2 Preliminaries

In this section, we review some concepts of shadowed sets.

Definition 2.1 Pedrycz (1998) Let A be a fuzzy set, the shadowed set S_{μ_A} of A is defined as

$$S_{\mu_A}(x) = \begin{cases} 1, & \mu_A(x) \geq \alpha; \\ 0, & \mu_A(x) \leq \beta; \\ [0,1], & \beta < \mu_A(x) < \alpha. \end{cases} \tag{1}$$

In Pedrycz's model, an optimal pair of thresholds is computed by the objective function as

$$\text{Elevated area}_{(\alpha,\beta)}(\mu_{\mathrm{A}}) + \text{Reduced area}_{(\alpha,\beta)}(\mu_{\mathrm{A}}) = \text{Shadowed area}_{(\alpha,\beta)}(\mu_{\mathrm{A}}). \tag{2}$$

The objective function is constructed on elevated area, reduced area, and shadowed area, which is shown in Fig. 1. In other words, the shadowed area is the sum of the elevated area and reduced area, and it is difficult to compute a pair of thresholds satisfying this condition in practical situations. Instead, Pedrycz (1998) proposed another approach to computing the pair of thresholds when the universe U is finite by minimizing the absolute difference as

$$\begin{aligned} V_{(\alpha,\beta)}(\mu_A) &= |\text{Elevated area}_{(\alpha,\beta)}(\mu_{\mathrm{A}}) + \text{Reduced area}_{(\alpha,\beta)}(\mu_{\mathrm{A}}) \\ &\quad - \text{Shadowed area}_{(\alpha,\beta)}(\mu_{\mathrm{A}})| \\ &= |\sum_{\mu_A(x)\geq\alpha}(1-\mu_A(x)) + \sum_{\mu_A(x)\leq\beta}(\mu_A(x)) \\ &\quad - Card(\{x \in U \mid \beta < \mu_A(x) < \alpha\})|, \end{aligned} \tag{3}$$

where card$(\cdot)$ denotes the cardinality of a set $\cdot$, and an optimal pair of thresholds α and β can be derived by minimizing the objective function $V_{(\alpha,\beta)}(\mu_A)$. Similarly, it is also difficult to compute the pair of thresholds α and β since minimizing $V_{(\alpha,\beta)}(\mu_A)$ involves two parameters α and β. For convenience, by $\alpha+\beta=1$, the objective function is simplified as

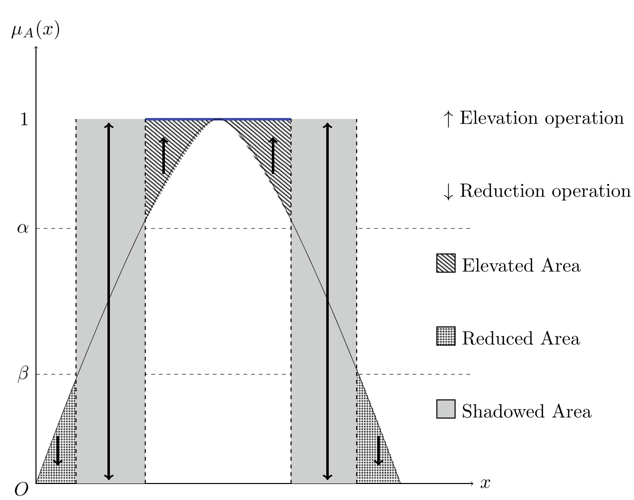

Fig. 1 Shadowed sets

$$
\begin{aligned}
&V_{(\alpha,1-\alpha)}(\mu_A) \\
&= |\text{Elevated area}_{(\alpha,1-\alpha)}(\mu_\text{A}) + \text{Reduced area}_{(\alpha,1-\alpha)}(\mu_\text{A}) \\
&\quad - \text{Shadowed area}_{(\alpha,1-\alpha)}(\mu_\text{A})| \\
&= |\sum_{\mu_A(x)\geq\alpha}(1-\mu_A(x)) + \sum_{\mu_A(x)\leq 1-\alpha}(\mu_A(x)) \\
&\quad - Card(\{x\in U \mid 1-\alpha(<\mu_A(x)<\alpha\})|. \qquad (4)
\end{aligned}
$$

Subsequently, Deng and Yao (2014) expressed the objective function to further investigate shadowed sets in terms of the errors. In their approach, for an object x with membership grade $\mu_A(x)$, the elevation operation changes the membership grade from $\mu_A(x)$ to 1, and the reduction operation changes the membership grade from $\mu_A(x)$ to 0. The errors induced by elevation and reduction operations are shown as

$$E_e(\mu_A(x)) = 1-\mu_A(x), E_r(\mu_A(x)) = \mu_A(x) - 0 = \mu_A(x). \qquad (5)$$

The errors $E_e(\mu_A)$ and $E_r(\mu_A)$ induced by the elevation and reduction operations for a fuzzy set A of the universe U, respectively, are shown as

$$E_e(\mu_A) = \sum_{\mu_A(x)\geq\alpha}(1-\mu_A(x)), E_r(\mu_A) = \sum_{\mu_A(x)\leq\beta}(\mu_A(x)). \qquad (6)$$

The error for the shadowed area is not clear because of the unit interval [0, 1] as the membership grade when $\beta<\mu(x)<\alpha$. By computing the difference between $\mu(x)$ and the maximum 1 and the minimum value 0 and summarizing them up, we express the error induced by shadowed area as

$$
\begin{aligned}
E_s(\mu_A) = &\sum_{\beta<\mu_A(x)<\alpha}(1-\mu_A(x)) \\
&+\sum_{\beta<\mu_A(x)<\alpha}(\mu_A(x)) = Card\{x|\beta<\mu_A(x)<\alpha\},
\end{aligned} \qquad (7)
$$

where $Card\{x|\beta<\mu_A(x)<\alpha\}$ denotes the cardinality of $\{x|\beta<\mu_A(x)<\alpha\}$.

The objective function by the error-based interpretation of the three areas is expressed as

$$
\begin{aligned}
V_{(\alpha,\beta)}(\mu_A) &= |E_e(\mu_A)+E_r(\mu_A)-E_s(\mu_A)| \\
&= |\sum_{\mu_A(x)\geq\alpha}(1-\mu_A(x)) + \sum_{\mu_A(x)\leq\beta}(\mu_A(x)) \\
&\quad - \sum_{\beta<\mu_A(x)<\alpha}(1-\mu_A(x)) - \sum_{\beta<\mu_A(x)<\alpha}(\mu_A(x))|.
\end{aligned} \qquad (8)
$$

The objective function is constructed on elevated area, reduced area, and shadowed area, as shown in Fig. 2. In other words, the objective function is a kind of tradeoff of errors

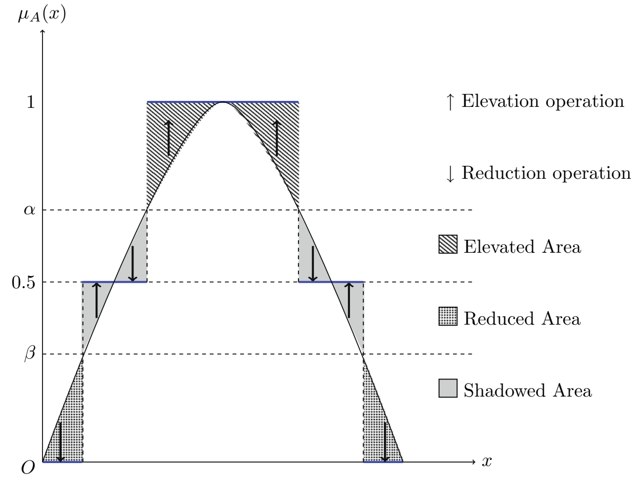

Fig. 2 Errors-based shadowed sets

produced by three regions. On one hand, $E_s(\mu_A)$ consists of the errors of elevation and reduction operations, and it is impossible to elevate $\mu_A(x)$ to 1 and reduce $\mu_A(x)$ to 0 simultaneously. On the other hand, we are not able to allocate any numeric membership grade to the elements in the shadowed area. That is, any numeric value of the unit interval [0, 1] could be permitted to reflect the uncertainty. Therefore, it is necessary to investigate that which numeric value is representative of the membership grade of elements in the shadowed area.

In fuzzy sets, 0.5 is a semantically meaningful value to represent the membership grades of objects in the boundary region, and a three-way approximation of a fuzzy set is shown by replacing the unit interval [0, 1] with 0.5 as

$$T_{\mu_A}(x) = \begin{cases} 1, & \mu(x)\geq\alpha; \\ 0, & \mu(x)\leq\beta; \\ 0.5, & \beta<\mu(x)<\alpha. \end{cases} \qquad (9)$$

By analyzing $T_{\mu_A}(x)$, we see that the correspondences between areas of elevation and reduction and errors of elevation and reduction operations remain to be the same. However, the errors of the shadowed region are revised as

$$E_{s_{0.5}}(\mu_A) = \sum_{0.5<\mu_A(x)<\alpha}(1-\mu_A(x)) + \sum_{\beta<\mu_A(x)<0.5}(\mu_A(x)). \qquad (10)$$

By $E_e(\mu_A), E_r(\mu_A)$, and $E_{s_{0.5}}(\mu_A)$, we have

$$
\begin{aligned}
E_{(\alpha,\beta)}(\mu_A) &= |E_e(\mu_A)+E_r(\mu_A)-E_{s_{0.5}}(\mu_A)| \\
&= |\sum_{\mu_A(x)\geq\alpha}(1-\mu_A(x)) + \sum_{\mu_A(x)\leq\beta}(\mu_A(x)) \\
&\quad - \sum_{0.5<\mu_A(x)<\alpha}(\mu_A(x)-0.5) \\
&\quad - \sum_{\beta<\mu_A(x)<0.5}(0.5-\mu_A(x))|. \qquad (11)
\end{aligned}
$$

The total errors of the three areas are minimized instead of searching for a tradeoff between different areas. Correspondingly, the total errors as the summation of errors of all objects are expressed as

$$E_{(\alpha,\beta)}(\mu_A) = \sum_{x \in U} E_{(\alpha,\beta)}(\mu_A(x)), \tag{12}$$

where

$$E_{(\alpha,\beta)}(\mu_A(x)) = \begin{cases} 1-\mu(x), & \mu(x) \geq \alpha; \\ 0.5-\mu(x), & \beta < \mu(x) \leq 0.5; \\ \mu(x)-0.5, & 0.5 < \mu(x) < \alpha; \\ \mu(x)-0, & \mu(x) \leq \beta. \end{cases} \tag{13}$$

The total error is minimized by minimizing the error of each individual object, and we search for a pair of thresholds α and β, such that $E_{(\alpha,\beta)}(\mu_A(x))$ is minimized for each object. We consider the following actions and associated errors for minimizing the error of each object:

$$\begin{aligned} \text{Elevate to } 1 &: 1-\mu_A(x); \\ \text{Reduce to } 0 &: \mu_A(x)-0; \\ \text{Reduce or elevate to } 0.5 &: |\mu_A(x)-0.5|. \end{aligned}$$

That is, the absolute differences between $\mu_A(x)$ and three values 1, 0.5, and 0, respectively, are the associated errors.

By considering various costs of the actions of elevation and reduction, Deng and Yao (2014) presented an analytic solution of computing the pair of thresholds α and β, in which one of the three actions for an object with a membership grade can be taken as follows: elevate the membership grade to 1, reduce the membership grade to 0, and change the membership grade to 0.5. More specially, there are two situations for the third case: reduce the membership grade to 0.5 if $\alpha > \mu_A(x) \geq 0.5$ and elevate the membership grade to 0.5 if $\beta < \mu_A(x) < 0.5$. Each action will incur error, and the costs of different actions are not necessarily the same.

In Table 1, the set of actions $\{a_e, a_r, a_{s_\downarrow}, a_{s_\uparrow}\}$ describes four possible actions on changing the membership grade. For simplicity, we use $\{e, r, s_\downarrow, s_\uparrow\}$ to denote the four actions, where the elevation action a_e elevate the membership grade of x from $\mu_A(x)$ to 1, the reduction action a_r reduce the membership grade of x from $\mu_A(x)$ to 0, the elevation $a_{s_\uparrow}$ elevate the membership grade of x from $\mu_A(x)$ to 0.5 if $\mu_A(x) < 0.5$, and the reduction $a_{s_\downarrow}$ reduce the membership grade of x from $\mu_A(x)$ to 0.5 if $\mu_A(x) > 0.5$. The fuzzy membership grade $\mu_A(x)$ represents the state of object in the second column, and the errors of different actions are given in the fourth column, and the losses of different actions are given in the fifth column.

Table 1 Loss function

Action	Fuzzy set membership grade	Three-way membership grade	Error	Loss
a_e	$\mu_A(x) \geq \alpha$	1	$1-\mu_A(x)$	λ_e
a_r	$\mu_A(x) \leq \beta$	0	$\mu_A(x)$	λ_r
$a_{s_\downarrow}$	$0.5 \leq \mu_A(x) < \alpha$	0.5	$\mu_A(x)-0.5$	$\lambda_{s_\downarrow}$
$a_{s_\uparrow}$	$\beta < \mu_A(x) < 0.5$	0.5	$0.5-\mu_A(x)$	$\lambda_{s_\uparrow}$

Suppose $\lambda_e > 0, \lambda_r > 0, \lambda_{s_\downarrow} > 0, \lambda_{s_\uparrow} > 0, \lambda_{s_\downarrow} \leq \lambda_r$, and $\lambda_{s_\uparrow} \leq \lambda_e$, $R_a(x) = \lambda_a E_a(\mu_A(x))$ denotes the loss for taking actions $\{e, r, s_\downarrow, s_\uparrow\}$, and the losses of four actions for an object can be computed as

$$\begin{aligned} R_e(x) &= \lambda_e E_e(\mu_A(x)) \\ R_r(x) &= \lambda_r E_r(\mu_A(x)) \\ R_{s_\downarrow}(x) &= \lambda_{s_\downarrow} E_{s_\downarrow}(\mu_A(x)) \\ R_{s_\uparrow}(x) &= \lambda_{s_\uparrow} E_{s_\uparrow}(\mu_A(x)) \end{aligned} \tag{14}$$

According to the minimum losses of actions, Deng and Yao (2014) immediately have three rules as

(E) *If* $\mu_A(x) \geq \alpha$, *then* $T_{\mu_A}(x) = 1$;
(R) *If* $\mu_A(x) \leq \beta$, *then* $T_{\mu_A}(x) = 0$;
(S) *If* $\beta < \mu_A(x) < \alpha$, *then* $T_{\mu_A}(x) = 0.5$,

where

$$\alpha = \frac{2\lambda_e + \lambda_{s_\downarrow}}{2(\lambda_e + \lambda_{s_\downarrow})} \text{ and } \beta = \frac{\lambda_{s_\uparrow}}{2(\lambda_r + \lambda_{s_\uparrow})}.$$

3 Shadowed sets of dynamic fuzzy sets

In Pedrycz (1998), Pedrycz interpreted and determined the required pair of thresholds by a framework of shadowed sets, which solves the issue with Zadeh's proposal. In practice, the membership function of a fuzzy set varies with time, referred as dynamic fuzzy set, and it is necessary to study shadowed sets of dynamic fuzzy sets.

Definition 3.1 Wang et al. (1988) Let U be a universe, $R^+ = [0, +\infty)$, $T \subseteq R^+$, and $A \in \mathcal{F}(U)$, where $\mathcal{F}(U)$ is the set of all fuzzy sets of U, and the membership function μ_A is defined as follows:

$$\mu_A : U \times T \longrightarrow [0,1] : (x,t) \longrightarrow \mu_A(x,t), \tag{15}$$

where $x \in U$ and $t \in T$. Then, $\{A(t) | t \in T\}$ is called a dynamic fuzzy set.

The classical membership function μ_A^* of a fuzzy set A is static, such as $\mu_A^* : U \longrightarrow [0,1] : x \longrightarrow \mu_A^*(x)$, which is only the function of the object. Furthermore, for a dynamic fuzzy set, $\mu_A(x,t)$ is the membership degree of x to A at the time $t \in T$, which is more complex than the membership function of fuzzy set. Intuitively, an object with a full membership grade 1 is a typical instance of the concept at the time t, an object with a membership grade 0 is viewed as a non-instance of the concept at the time t, and a higher membership grade implies that an object belongs more to the concept at the time t.

Example 3.2 Let $U = \{x_1, x_2, x_3, x_4, x_5\}$, and $A = \frac{x_1}{1-\frac{1}{t}} + \frac{x_2}{1-\frac{1}{t}} + \frac{x_3}{1-\frac{1}{2t}} + \frac{x_4}{1-\frac{1}{2t}} + \frac{x_5}{1-\frac{1}{3t}}$. By Definition 3.1, we have that $\mu_A(x_1,t) = 1 - \frac{1}{t}$, $\mu_A(x_2,t) = 1 - \frac{1}{t}$, $\mu_A(x_3,t) = 1 - \frac{1}{2t}$, $\mu_A(x_4,t) = 1 - \frac{1}{2t}$, and $\mu_A(x_5,t) = 1 - \frac{1}{3t}$.

Definition 3.3 Let A be a dynamic fuzzy set, the shadowed set S_{μ_A} of A is defined as

$$S_{\mu_A}(x,t) = \begin{cases} 1, & \mu(x,t) \geq \alpha(t); \\ 0, & \mu(x,t) \leq \beta(t); \\ [0,1], & \beta(t) < \mu(x,t) < \alpha(t). \end{cases} \quad (16)$$

In terms of three-way decisions, the shadowed set of a dynamic fuzzy set can be conveniently interpreted as three regions: the positive region defined by membership grade 1, the negative region defined by membership grade 0, and the boundary region defined by membership grade [0,1]. Moreover, dynamic shadowed sets give a new interpretation of approximations of dynamic fuzzy sets, and the membership function of a dynamic shadowed set is viewed as a modification of a membership function of a shadowed set. For example, for an object x at the time t, we elevate the membership grade from $\mu(x,t)$ to 1 if $\mu(x,t) \geq \alpha(t)$; we reduce the membership grade from $\mu(x,t)$ to 0 if $\mu(x,t) \leq \beta(t)$, and we change the membership grade from $\mu(x,t)$ to [0,1] if $\beta(t) < \mu(x,t) < \alpha(t)$.

The pair of thresholds $\alpha(t)$ and $\beta(t)$ are important for computing three-way approximations of dynamic fuzzy sets. In what follows, we introduce a systematic way to compute the pair of thresholds $\alpha(t)$ and $\beta(t)$ by minimizing an objective function that characterizes the uncertainty of a dynamic shadowed set. Consider a dynamic shadowed set, as shown in Fig. 3. By comparing membership functions of a dynamic shadowed set, we identify three dynamic regions, as shown in Fig. 1: the dynamic elevated area, the dynamic reduced area, and the dynamic shadowed area.

According to Pedrycz's model, we provide an optimal pair of thresholds based on the three areas as

$$\text{Elevated area}_{(\alpha(t),\beta(t))}(\mu_A) + \text{Reduced area}_{(\alpha(t),\beta(t))}(\mu_A) = \text{Shadowed area}_{(\alpha(t),\beta(t))}(\mu_A). \quad (17)$$

The objective function is constructed on dynamic elevated area, dynamic reduced area, and dynamic shadowed area, which are shown in Fig. 3. In other words, the dynamic shadowed area is the sum of the dynamic elevated area and dynamic reduced area. As Pedrycz's model, it is difficult to

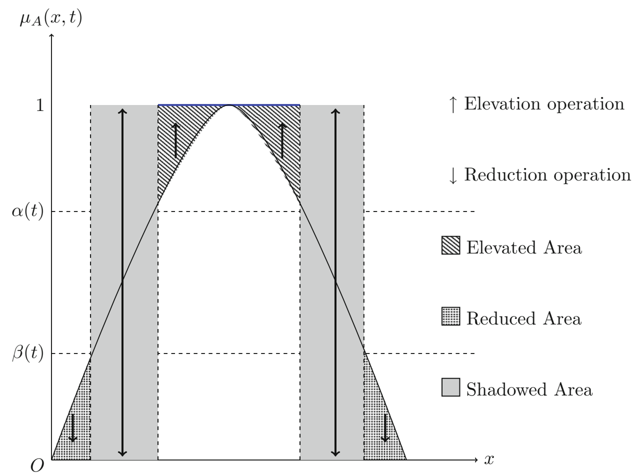

Fig. 3 Dynamic shadowed sets

compute a pair of thresholds which satisfying this condition in practical situations. Instead, we propose an approach to computing the thresholds when the universe U is finite by minimizing the following absolute difference as

$$\begin{aligned} &V_{(\alpha(t),\beta(t))}(\mu_A) \\ &= |\text{Elevated area}_{(\alpha(t),\beta(t))}(\mu_A) + \text{Reduced area}_{(\alpha(t),\beta(t))}(\mu_A) \\ &\quad - \text{Shadowed area}_{(\alpha(t),\beta(t))}(\mu_A)| \\ &= |\sum_{\mu_A(x,t) \geq \alpha(t)} (1 - \mu_A(x,t)) + \sum_{\mu_A(x,t) \leq \beta(t)} (\mu_A(x,t)) \\ &\quad - Card(\{x \in U | \beta(t) < \mu_A(x,t) < \alpha(t)\})|, \end{aligned} \quad (18)$$

where card(·) denotes the cardinality of a set ·, and an optimal pair of thresholds $\alpha(t)$ and $\beta(t)$ can be derived by minimizing the objective function $V_{(\alpha(t),\beta(t))}(\mu_A)$. Similarly, minimizing $V_{(\alpha(t),\beta(t))}(\mu_A)$ involves two parameters $\alpha(t)$ and $\beta(t)$. For convenience, by assuming that $\alpha(t) + \beta(t) = 1$, the objective function is simplified into

$$\begin{aligned} &V_{(\alpha(t),1-\alpha(t))}(\mu_A) \\ &= |\text{Elevated Area}_{(\alpha(t),1-\alpha(t))}(\mu_A) \\ &\quad + \text{Reduced area}_{(\alpha(t),1-\alpha(t))}(\mu_A) \\ &\quad - \text{Shadowed area}_{(\alpha(t),1-\alpha(t))}(\mu_A)| \\ &= |\sum_{\mu_A(x,t) \geq \alpha(t)} (1 - \mu_A(x,t)) + \sum_{\mu_A(x,t) \leq 1-\alpha(t)} (\mu_A(x,t)) \\ &\quad - Card(\{x \in U | 1 - \alpha(t) < \mu_A(x,t) < \alpha(t)\})|. \end{aligned} \quad (19)$$

4 Errors-based interpretation of dynamic shadowed sets

In this section, we present a detailed analysis of a objective function for shadowed sets in terms of errors of approximations. We also provide a new objective function by the total error of approximations for determining the thresholds $\alpha(t)$ and $\beta(t)$.

To further study shadowed sets, we express the objective function in terms of the errors introduced by a shadowed set approximation. For an object x with membership grade $\mu_A(x,t)$ at the time t, the elevation operation changes the membership grade from $\mu_A(x,t)$ to 1, the reduction operation changes the membership grade from $\mu_A(x,t)$ to 0, and the errors induced by elevation and reduction are shown as

$$\begin{aligned} E_e(\mu_A(x,t)) &= 1-\mu_A(x,t), E_r(\mu_A(x,t)) \\ &= \mu_A(x,t) - 0 = \mu_A(x,t). \end{aligned} \tag{20}$$

The errors $E_e(\mu_A)$ and $E_r(\mu_A)$ induced by the elevation and reduction operations for a dynamic shadowed set A of the universe U, respectively, are shown as

$$\begin{aligned} E_e(\mu_A) &= \sum_{\mu_A(x,t)\geq\alpha(t)} (1-\mu_A(x,t)), E_r(\mu_A) \\ &= \sum_{\mu_A(x,t)\leq\beta(t)} (\mu_A(x,t)). \end{aligned} \tag{21}$$

The error for the dynamic shadowed area is not clear because of the unit interval [0, 1] as the membership grade when $\beta(t)<\mu(x,t)<\alpha(t)$. By computing the difference between $\mu(x,t)$ and the maximum 1 and the minimum value 0 and summarizing them up, we have

$$\begin{aligned} E_s(\mu_A) = &\sum_{\beta(t)<\mu_A(x,t)<\alpha(t)} (1-\mu_A(x,t)) \\ &+ \sum_{\beta(t)<\mu_A(x,t)<\alpha(t)} (\mu_A(x,t)). \end{aligned} \tag{22}$$

Subsequently, we express the objective function in terms of errors by the error-based interpretation of the three areas as

$$\begin{aligned} V_{(\alpha(t),\beta(t))}(\mu_A) &= |E_e(\mu_A) + E_r(\mu_A) - E_s(\mu_A)| \\ &= |\sum_{\mu_A(x,t)\geq\alpha(t)} (1-\mu_A(x,t)) \\ &\quad + \sum_{\mu_A(x,t)\leq\beta(t)} (\mu_A(x,t)) \\ &\quad - \sum_{\beta(t)<\mu_A(x,t)<\alpha(t)} (1-\mu_A(x,t)) \\ &\quad - \sum_{\beta(t)<\mu_A(x,t)<\alpha(t)} (\mu_A(x,t))|. \end{aligned} \tag{23}$$

The objective function is constructed on dynamic elevated area, dynamic reduced area, and dynamic shadowed area, which is shown in Fig. 4. In other words, the objective function is a kind of tradeoff of errors produced by three regions. However, the rationale for such a tradeoff is not entirely clear. On one hand, $E_s(\mu_A)$ consists of the errors of elevation and reduction operations, and it is impossible to elevate $\mu_A(x,t)$ to 1 and reduce $\mu_A(x,t)$ to 0 if $\beta(t)<\mu_A(x,t)<\alpha(t)$ simultaneously. On the other hand, we are not able to allocate any numeric membership grade for the elements in the dynamic shadowed area. In other words, any numeric value of the unit interval [0,1] could be permitted to reflect the uncertainty. Therefore, it is necessary to investigate that which numeric value is meaningful to the membership grade of elements in the dynamic shadowed area.

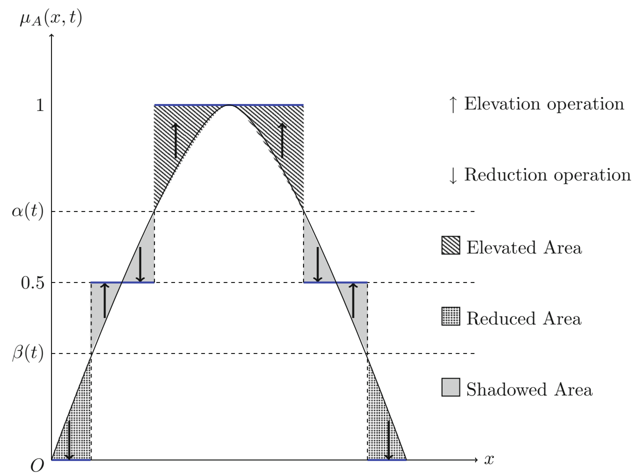

Fig. 4 Errors-based dynamic shadowed sets

In fuzzy sets, 0.5 is a semantically meaningful value to represent the membership grades of objects in the boundary region. Therefore, we define a three-way approximation of a dynamic shadowed set by replacing the unit interval [0,1] with 0.5, that is

$$T_{\mu_A}(x,t) = \begin{cases} 1, & \mu(x,t)\geq\alpha(t); \\ 0, & \mu(x,t)\leq\beta(t); \\ 0.5, & \beta(t)<\mu(x,t)<\alpha(t). \end{cases} \tag{24}$$

By analyzing $T_{\mu_A}(x,t)$, we see that the correspondences between areas of elevation and reduction and errors of elevation and reduction remain to be the same. However, we need to revise the errors of the dynamic shadowed region as

$$\begin{aligned} E_{s_{0.5}}(\mu_A) = &\sum_{0.5<\mu_A(x,t)<\alpha(t)} (1-\mu_A(x,t)) \\ &+ \sum_{\beta(t)<\mu_A(x,t)<0.5} (\mu_A(x,t)). \end{aligned} \tag{25}$$

By $E_e(\mu_A), E_r(\mu_A)$ and $E_{s_{0.5}}(\mu_A)$, we have

$$\begin{aligned} E_{(\alpha(t),\beta(t))}(\mu_A) &= E_e(\mu_A) + E_r(\mu_A) - E_{s_{0.5}}(\mu_A) \\ &= \sum_{\mu_A(x,t)\geq\alpha(t)} (1-\mu_A(x,t)) \\ &\quad + \sum_{\mu_A(x,t)\leq\beta(t)} (\mu_A(x,t)) \\ &\quad - \sum_{0.5<\mu_A(x,t)<\alpha(t)} (\mu_A(x,t)-0.5) \\ &\quad - \sum_{\beta(t)<\mu_A(x,t)<0.5} (0.5-\mu_A(x,t)). \end{aligned} \tag{26}$$

The total errors of the three areas are minimized instead of searching for a tradeoff between different areas. Correspondingly, we express the total error as the summation of errors of all objects as

$$E_{(\alpha(t),\beta(t))}(\mu_A) = \sum_{x \in U} E_{(\alpha(t),\beta(t))}(\mu_A(x,t)), \tag{27}$$

where

$$E_{(\alpha(t),\beta(t))}(\mu_A(x,t)) = \begin{cases} 1-\mu(x,t), & \mu(x,t) \geq \alpha(t); \\ 0.5-\mu(x,t), & \beta(t) < \mu(x,t) \leq 0.5; \\ \mu(x,t)-0.5, & 0.5 < \mu(x,t) < \alpha(t); \\ \mu(x,t)-0, & \mu(x,t) \leq \beta(t). \end{cases} \tag{28}$$

The total error will be minimized by minimizing the error of each individual object, and we can search for a pair of thresholds $\alpha(t)$ and $\beta(t)$, such that $E_{(\alpha(t),\beta(t))}(\mu_A(x,t))$ is minimized for each object. We consider the following actions and associated errors for minimizing the error of each object:

$$\begin{aligned} &\text{Elevate to } 1: 1-\mu_A(x,t); \text{Reduce to } 0: \mu_A(x,t) \\ &\quad -0; \text{Reduce or elevate to } 0.5: |\mu_A(x,t)-0.5|. \end{aligned} \tag{29}$$

That is, the absolute differences between $\mu_A(x,t)$ and three values 1, 0.5, and 0, respectively, are the associated errors. A minimized difference is obtained if $\mu_A(x,t)$ is changed into a value that is closest to $\mu_A(x,t)$.

Example 4.1 (Continued from Example 3.2) By taking $\alpha(t) = 1-\frac{1}{3t}$ and $\beta(t) = 1-\frac{1}{t}$, we have $\mu_A(x_1,t) \leq 1-\frac{1}{t}$, $1-\frac{1}{t} \leq \mu_A(x_2,t) \leq 1-\frac{1}{3t}$, $1-\frac{1}{t} \leq \mu_A(x_3,t) \leq 1-\frac{1}{3t}$, $\mu_A(x_4,t) \geq 1-\frac{1}{3t}$ and $\mu_A(x_5,t) \geq 1-\frac{1}{3t}$. Consequently, we have

$$\begin{aligned} &E_{(\alpha(t),\beta(t))}(\mu_A(x_1,t)) = \mu(x_1,t) = 1-\frac{1}{t}; \\ &E_{(\alpha(t),\beta(t))}(\mu_A(x_2,t)) = \mu(x_2,t)-0.5 = 1-\frac{1}{t}-0.5 = 0.5-\frac{1}{t}; \\ &E_{(\alpha(t),\beta(t))}(\mu_A(x_3,t)) = \mu(x_3,t)-0.5 = 1-\frac{1}{2t}-0.5 = 0.5-\frac{1}{2t}; \\ &E_{(\alpha(t),\beta(t))}(\mu_A(x_4,t)) = 1-\mu(x_4,t) = 1-(1-\frac{1}{2t}) = \frac{1}{2t}; \\ &E_{(\alpha(t),\beta(t))}(\mu_A(x_5,t)) = 1-\mu(x_5,t) = 1-(1-\frac{1}{3t}) = \frac{1}{3t}. \end{aligned} \tag{30}$$

5 Three-way approximations of dynamic shadowed sets

In this section, we introduce a framework for three-way approximations of dynamic shadowed sets.

Table 2 Dynamic loss function

Action	Dynamic membership grade	Three-way membership grade	Error	Loss
$a_e(t)$	$\mu_A(x,t) \geq \alpha(t)$	1	$1-\mu_A(x,t)$	$\lambda_e(t)$
$a_r(t)$	$\mu_A(x,t) \leq \beta(t)$	0	$\mu_A(x,t)$	$\lambda_r(t)$
$a_{s_\downarrow}(t)$	$0.5 \leq \mu_A(x,t) < \alpha(t)$	0.5	$\mu_A(x,t)-0.5$	$\lambda_{s_\downarrow}(t)$
$a_{s_\uparrow}(t)$	$\beta(t) < \mu_A(x,t) < 0.5$	0.5	$0.5-\mu_A(x,t)$	$\lambda_{s_\uparrow}(t)$

5.1 Three-way approximations of dynamic shadowed sets based on costs

In Sect. 4, we investigate dynamic shadowed sets by three membership grades of 0, 0.5, and 1. We take one of the following three actions for an object with a dynamic membership grade: elevate the membership grade to 1, reduce the membership grade to 0, and change the membership grade to 0.5. More specifically, there are two situations for the third case: reduce the membership grade to 0.5 if $\mu_A(x,t) \geq 0.5$ and elevate the membership grade to 0.5 if $\mu_A(x,t) < 0.5$. Each action will incur error, and the costs of different actions are not necessarily the same.

Table 2 summarizes information about three-way approximations of a dynamic shadowed set. The second column represents the membership grade, the fourth column represents the errors of different actions, and the fifth column represents the losses of different actions.

The set of actions $\{a_e(t), a_r(t), a_{s_\downarrow}(t), a_{s_\uparrow}(t)\}$ or $\{e, r, s_\downarrow, s_\uparrow\}$ denotes four actions on the variations of the membership grade. The symbol $a_e(t)$ means elevating the membership grade of x from $\mu_A(x,t)$ to 1, the symbol $a_r(t)$ means reducing the membership grade of x from $\mu_A(x,t)$ to 0, the symbol $a_{s_\uparrow}(t)$ means elevating the membership grade of x from $\mu_A(x,t)$ to 0.5 if $\mu_A(x,t) < 0.5$, and the symbol $a_{s_\downarrow}(t)$ means reducing the membership grade of x from $\mu_A(x,t)$ to 0.5 if $\mu_A(x,t) > 0.5$.

Each of the four losses $\lambda_e(t), \lambda_r(t), \lambda_{s_\downarrow}(t)$ and $\lambda_{s_\uparrow}(t)$ provides the unit cost, and the actual cost of each action is weighted by the magnitude of its error. Suppose $R_a(x,t) = \lambda_a(t)E_a(\mu_A(x,t))$ denote the loss for taking actions $\{e, r, s_\downarrow, s_\uparrow\}$, the losses of four actions for an object can be computed as

$$\begin{aligned} R_e(x,t) &= \lambda_e(t)E_e(\mu_A(x,t)) \\ R_r(x,t) &= \lambda_r(t)E_r(\mu_A(x,t)) \\ R_{s_\downarrow}(x,t) &= \lambda_{s_\downarrow}(t)E_{s_\downarrow}(\mu_A(x,t)) \\ R_{s_\uparrow}(x,t) &= \lambda_{s_\uparrow}(t)E_{s_\uparrow}(\mu_A(x,t)) \end{aligned} \tag{31}$$

Since only an action is taken for each object, the total loss of the approximation is computed by

$$R(t) = \sum_{x \in U} R_a(x,t) = \sum_{x \in U} \lambda_a(t) E_a(\mu_A(x,t)). \quad (32)$$

To minimize the total loss $R(t)$

$$\mathrm{argmin}_{a(t) \in \mathrm{action}} \mathrm{R}_{a(t)}(\mathrm{x}, \mathrm{t}), \quad (33)$$

where $a(t) \in \{e(t), r(t), s_\downarrow(t), s_\uparrow(t)\}$.

According to the value $\mu_A(x,t)$ of an object x, we have two groups of decision rules for obtaining three-way approximations of a dynamic shadowed set as follows:

1. When $\mu_A(x,t) \geq 0.5$, ($E1$) If $R(a_e|x) \leq R(a_r|x)$ and $R(a_e|x) \leq R(a_{s_\downarrow}|x)$, then take action a_e; ($R1$) If $R(a_r|x) \leq R(a_e|x)$ and $R(a_r|x) \leq R(a_{s_\downarrow}|x)$, then take action a_r; ($S1$) If $R(a_{s_\downarrow}|x) \leq R(a_e|x)$ and $R(a_{s_\downarrow}|x) \leq R(a_r|x)$, then take action $a_{s_\downarrow}$.
2. When $\mu_A(x,t) < 0.5$, ($E2$) If $R(a_e|x) \leq R(a_r|x)$ and $R(a_e|x) \leq R(a_{s_\uparrow}|x)$, then take action a_e; ($R2$) If $R(a_r|x) \leq R(a_e|x)$ and $R(a_r|x) \leq R(a_{s_\uparrow}|x)$, then take action a_r; ($S2$) If $R(a_{s_\uparrow}|x) \leq R(a_e|x)$ and $R(a_{s_\uparrow}|x) \leq R(a_r|x)$, then take action $a_{s_\uparrow}$.

5.2 Three-way approximations of dynamic shadowed sets based on dynamic loss functions

In this subsection, we present an analytic solution to computing three-way approximations of dynamic shadowed sets by considering dynamic loss functions satisfying certain properties.

Suppose $(c1): \lambda_e(t) > 0, \lambda_r(t) > 0, \lambda_{s_\downarrow}(t) > 0, \lambda_{s_\uparrow}(t) > 0$; $(c2): \lambda_{s_\downarrow}(t) \leq \lambda_r(t)$; $(c3): \lambda_{s_\uparrow}(t) \leq \lambda_e(t)$, Condition ($c1$) means that all costs are nonnegative; Condition ($c2$) means that the cost of reducing a membership grade $\mu_A(x,t) \geq 0.5$ to 0.5 is smaller than the cost of reducing it to 0; and Condition ($c3$) means that the cost of elevating a membership grade $\mu_A(x,t) < 0.5$ to 0.5 is smaller than the cost of elevating it to 1. We propose the decision rules as follows:

(1) When $\mu_A(x,t) \geq 0.5$, the rule ($E1$) is expressed as

$$\begin{aligned} R(a_e|x) \leq R(a_r|x) &\Leftrightarrow (1-\mu_A(x,t))\lambda_e(t) \leq (\mu_A(x,t)-0)\lambda_r(t) \\ &\Leftrightarrow \mu_A(x,t) \geq \frac{\lambda_e(t)}{\lambda_e(t)+\lambda_r(t)} = \gamma(t); \\ R(a_e|x) \leq R(a_{s_\downarrow}|x) &\Leftrightarrow (1-\mu_A(x,t))\lambda_e(t) \leq (\mu_A(x,t)-0.5)\lambda_{s_\downarrow}(t) \\ &\Leftrightarrow \mu_A(x,t) \geq \frac{2\lambda_e(t)+\lambda_{s_\downarrow}(t)}{2(\lambda_e(t)+\lambda_{s_\downarrow}(t))} = \alpha(t). \end{aligned} \quad (34)$$

The rule ($R1$) is shown as

$$\begin{aligned} R(a_r|x) \leq R(a_r|x) &\Leftrightarrow \mu_A(x,t) \leq \gamma(t); \\ R(a_r|x) \leq R(a_{s_\downarrow}|x) &\Leftrightarrow \mu_A(x,t)\lambda_r(t) \leq (\mu_A(x,t)-0.5)\lambda_{s_\downarrow}(t) \\ &\Leftrightarrow \mu_A(x,t) \leq \frac{-\lambda_{s_\downarrow}(t)}{2(\lambda_r(t)-\lambda_{s_\downarrow}(t))} = \gamma^-(t). \end{aligned} \quad (35)$$

The rule ($S1$) is depicted by

$$\begin{aligned} R(a_{s_\downarrow}|x) \leq R(a_e|x) &\Leftrightarrow \mu_A(x,t) \leq \alpha(t); \\ R(a_{s_\downarrow}|x) \leq R(a_{s_\downarrow}|x) &\Leftrightarrow \mu_A(x,t) \geq \gamma^-(t). \end{aligned} \quad (36)$$

Since $\gamma^-(t) \leq 0$ contradicts $\mu_A(x,t) \geq 0.5$, rule ($R1$) cannot be used. Thus, when $\mu_A(x,t) \geq 0.5$, the rules are concisely expressed as follows: (E1) If $\mu_A(x,t) \geq \alpha(t)$, then $T_{\mu_A}(x,t) = 1$; (S1) If $0.5 \leq \mu_A(x,t) < \alpha(t)$, then $T_{\mu_A}(x,t) = 0.5$.

(2) When $\mu_A(x,t) < 0.5$, the rule ($E2$) is expressed as

$$\begin{aligned} R(a_e|x) \leq R(a_r|x) &\Leftrightarrow (1-\mu_A(x,t))\lambda_e(t) \leq (\mu_A(x,t)-0)\lambda_r(t) \\ &\Leftrightarrow \mu_A(x,t) \geq \frac{\lambda_e(t)}{\lambda_e(t)+\lambda_r(t)} = \gamma(t); \\ R(a_e|x) \leq R(a_{s_\uparrow}|x) &\Leftrightarrow (1-\mu_A(x,t))\lambda_e(t) \leq (0.5-\mu_A(x,t))\lambda_{s_\uparrow}(t) \\ &\Leftrightarrow \mu_A(x,t) \geq \frac{\lambda_e(t)-0.5\lambda_{s_\uparrow}(t)}{\lambda_e(t)-\lambda_{s_\uparrow}(t)} = \gamma^+(t). \end{aligned} \quad (37)$$

The rule ($R2$) is shown as

$$\begin{aligned} R(a_r|x) \leq R(a_e|x) &\Leftrightarrow \mu_A(x,t) \leq \gamma(t); \\ R(a_r|x) \leq R(a_{s_\uparrow}|x) &\Leftrightarrow \mu_A(x,t)\lambda_r(t) \leq (0.5-\mu_A(x,t))\lambda_{s_\uparrow}(t) \\ &\Leftrightarrow \mu_A(x,t) \leq \frac{\lambda_{s_\uparrow}(t)}{2(\lambda_r(t)+\lambda_{s_\uparrow}(t))} = \beta(t). \end{aligned} \quad (38)$$

The rule ($S2$) is depicted by

$$\begin{aligned} R(a_{s_\uparrow}|x) \leq R(a_e|x) &\Leftrightarrow \mu_A(x,t) \leq \gamma^+(t); \\ R(a_{s_\uparrow}|x) \leq R(a_{s_\uparrow}|x) &\Leftrightarrow \mu_A(x,t) \geq \beta(t). \end{aligned} \quad (39)$$

Since $\gamma^+(t) \geq 1$ contradicts $\mu_A(x,t) < 0.5$, rule ($E2$) is impossible to apply. Thus, when $\mu_A(x,t) < 0.5$, the remaining rules are concisely expressed as follows: (R2) If $\mu_A(x,t) \leq \beta(t)$, then $T_{\mu_A}(x,t) = 0$; (S2) If $\beta(t) \leq \mu_A(x,t) < 0.5$, then $T_{\mu_A}(x,t) = 0.5$.

By combining the two sets of rules, we immediately have three rules as (E) If $\mu_A(x,t) \geq \alpha(t)$, then $T_{\mu_A}(x,t) = 1$; (R) If $\mu_A(x,t) \leq \beta(t)$, then $T_{\mu_A}(x,t) = 0$; (S) If $\beta(t) < \mu_A(x,t) < \alpha(t)$, then $T_{\mu_A}(x,t) = 0.5$, where

$$\alpha(t) = \frac{2\lambda_e(t)+\lambda_{s_\downarrow}(t)}{2(\lambda_e(t)+\lambda_{s_\downarrow}(t))} \quad \text{and} \quad \beta(\mathrm{t}) = \frac{\lambda_{s_\uparrow}(\mathrm{t})}{2(\lambda_\mathrm{r}(\mathrm{t})+\lambda_{s_\uparrow}(\mathrm{t}))}. \quad (40)$$

We derive two models by considering two special dynamic loss functions.

(1) Consider a dynamic loss function satisfying the condition: $\lambda_e(t) = \lambda_r(t) = \lambda_{s_\uparrow}(t) = \lambda_{s_\downarrow}(t) = \lambda(t)$, we obtain a special model of three-way approximations of dynamic fuzzy sets, and the pair of thresholds $\alpha(t)$ and $\beta(t)$ are given by

$$\alpha(t) = \frac{2\lambda_e(t) + \lambda_{s_\downarrow}(t)}{2(\lambda_e(t) + \lambda_{s_\downarrow}(t))} = \frac{3}{4} \quad \text{and}$$
$$\beta(t) = \frac{\lambda_{s_\uparrow}(t)}{2(\lambda_r(t) + \lambda_{s_\uparrow}(t))} = \frac{1}{4}. \tag{41}$$

(2) Consider a dynamic loss function satisfying the condition: $\frac{\lambda_{s_\downarrow}(t)}{\lambda_e(t)} = \frac{\lambda_{s_\uparrow}(t)}{\lambda_r(t)}$, by applying the condition $\alpha(t) + \beta(t) = 1$, we have that

$$\alpha(t) + \beta(t) = 1 \Leftrightarrow \frac{2\lambda_e(t) + \lambda_{s_\downarrow}(t)}{2(\lambda_e(t) + \lambda_{s_\downarrow}(t))} + \frac{\lambda_{s_\uparrow}(t)}{2(\lambda_r(t) + \lambda_{s_\uparrow}(t))} = 1 \Leftrightarrow \frac{\lambda_{s_\downarrow}(t)}{\lambda_e(t)} = \frac{\lambda_{s_\uparrow}(t)}{\lambda_r(t)}. \tag{42}$$

In practice, if the pair of thresholds is interpreted in terms of a dynamic loss function, then the user can provide a better estimation of the thresholds in time. Therefore, the dynamic decision-theoretic model gives an interpretation of the pair of thresholds, which is important to discuss approximations of dynamic fuzzy sets.

5.3 An example of three-way approximations of dynamic shadowed sets

In this subsection, we employ an example to illustrate the main ideas of three-way approximations of dynamic shadowed sets.

Suppose that a dynamic loss function is shown as

$$\lambda_e(t) = 3t + 4, \lambda_r(t) = 4t + 3, \lambda_{s_\downarrow}(t) = t + 1, \ \lambda_{s_\uparrow}(t) = 2t + 2. \tag{43}$$

The dynamic loss function satisfies conditions (c1)–(c3). According to Sect. 4, we compute an optimal pair of thresholds as

$$\alpha(t) = \frac{2\lambda_e(t) + \lambda_{s_\downarrow}(t)}{2(\lambda_e(t) + \lambda_{s_\downarrow}(t))} = \frac{2(3t+4) + t + 1}{2(3t+4+t+1)} = \frac{7t+9}{8t+10};$$
$$\beta(t) = \frac{\lambda_{s_\uparrow}(t)}{2(\lambda_r(t) + \lambda_{s_\uparrow}(t))} = \frac{2t+2}{2(4t+3+t+1)} = \frac{t+1}{5t+4}. \tag{44}$$

In contrast to the original shadowed sets, one advantage of three-way approximations of dynamic shadowed sets is that the optimal pair of thresholds $\alpha(t)$ and $\beta(t)$ is independent on particular membership functions. To illustrate how to make a three-way decision for an object, we employ a dynamic shadowed set and the dynamic membership function is a Gaussian membership function as

$$\mu_A(x,t) = \frac{1}{\sigma(t)\sqrt{2\pi}} e^{-\frac{(x-\delta(t))^2}{2\sigma^2(t)}}, \tag{45}$$

where $\delta(t)$ and variance $\sigma^2(t)$ are mathematical expectations and variance, respectively. If $\mu_A(x,t) = \frac{t+1}{2t+5} < 0.5$, then the losses of taking actions $a_e, a_{s_\uparrow}$ and a_r of x are

$$R(a_e|x) = \lambda_e(t)E_e(\mu_A(x,t)) = (3t+4) * (1 - \frac{t+1}{2t+5}) = \frac{3t^2 + 16t + 16}{2t+5};$$
$$R(a_{s_\uparrow}|x) = \lambda_{s_\uparrow}(t)(0.5 - E_{s_\uparrow}(\mu_A(x,t))) = (2t+2) * (1 - \frac{t+1}{2t+5}) = \frac{2t^2 + 10t + 8}{2t+5};$$
$$R(a_r|x) = \lambda_e(t)E_e(\mu_A(x,t)) = (4t+3) * (\frac{t+4}{2t+5}) = \frac{4t^2 + 19t + 12}{2t+5}. \tag{46}$$

Therefore, the loss of reduction action $a_{s_\uparrow}$ has the minimum cost by analyzing $R(a_e|x), R(a_{s_\uparrow}|x)$, and $R(a_r|x)$.

6 Conclusions

In this paper, we first have presented shadowed sets of dynamic fuzzy sets. Second, we have constructed three-way approximations of dynamic shadowed sets. Thirdly, we have computed the pair of thresholds for three-way approximations of dynamic shadowed sets. Finally, we have employed several examples to illustrate how to calculate thresholds for making a decision by dynamic loss functions.

There are still many interesting topics deserving further investigations on shadowed sets. For example, there are many types of fuzzy sets and dynamic loss functions, and it is of interest to investigate dynamic loss functions-based three-way approximations of dynamic shadowed sets. In the future, we will further investigate dynamic shadowed sets and discuss its application in knowledge discovery.

Acknowledgments We would like to thank the anonymous reviewers very much for their professional comments and valuable suggestions. This work was supported by the National Natural Science Foundation of China (Nos. 11371130, 61603063, 11526039, and 11571100), China Postdoctoral Science Foundation (Nos. 2013M542558 and 2015M580353), Planned Science and Technology Project of Hunan Province (No. 2015JC3055), and China Postdoctoral Science Special Foundation (No. 2016T90383).

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