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# Shafer-type inequalities for inverse trigonometric functions and Gauss lemniscate functions

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## Abstract

In this paper, we present Shafer-type inequalities for inverse trigonometric functions and Gauss lemniscate functions.

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## 1 Introduction

Shafer [1] indicated several elementary quadratic approximations of selected functions without proof. Subsequently, Shafer [2] established these results as analytic inequalities. For example, Shafer [2] proved that for  $x > 0$ ,

$$\frac{8x}{3 + \sqrt{25 + \frac{80}{3}x^2}} < \arctan x. \quad (1.1)$$

The inequality (1.1) can also be found in [3]. Also in [2], Shafer proved that for  $0 < x < 1$ ,

$$\frac{8x}{3\sqrt{1-x^2} + \sqrt{25 + \frac{5}{3}x^2}} < \arcsin x. \quad (1.2)$$

Zhu [4] proved that the function

$$F(x) = \frac{\left(\frac{8x}{\arctan x} - 3\right)^2 - 25}{x^2}$$

is strictly decreasing for  $x > 0$ , and

$$\lim_{x \rightarrow 0^+} F(x) = \frac{80}{3} \quad \text{and} \quad \lim_{x \rightarrow \infty} F(x) = \frac{256}{\pi^2}.$$

From this one derives the following double inequality:

$$\frac{8x}{3 + \sqrt{25 + \frac{80}{3}x^2}} < \arctan x < \frac{8x}{3 + \sqrt{25 + \frac{256}{\pi^2}x^2}}, \quad x > 0. \quad (1.3)$$

The constants  $80/3$  and  $256/\pi^2$  are the best possible. In [4], (1.3) is called Shafer-type inequality.

Using the Maple software, we find that

$$\arctan x \left( 3 + \sqrt{25 + \frac{80}{3}x^2} \right) = 8x + \frac{32}{4,725}x^7 - \frac{64}{4,725}x^9 + \frac{25,376}{1,299,375}x^{11} - \dots$$

This fact motivated us to present a new upper bound for  $\arctan x$ , which is the first aim of the present paper.

**Theorem 1.1** For  $x > 0$ ,

$$\arctan x < \frac{8x + \frac{32}{4,725}x^7}{3 + \sqrt{25 + \frac{80}{3}x^2}}. \tag{1.4}$$

The second aim of the present paper is to develop (1.2) to produce a symmetric double inequality.

**Theorem 1.2** For  $0 < x < 1$ , we have

$$\frac{8x}{3\sqrt{1-x^2} + \sqrt{25+ax^2}} < \arcsin x < \frac{8x}{3\sqrt{1-x^2} + \sqrt{25+bx^2}} \tag{1.5}$$

with the best possible constants

$$a = \frac{5}{3} = 1.666666\dots \quad \text{and} \quad b = \frac{256 - 25\pi^2}{\pi^2} = 0.938223\dots \tag{1.6}$$

Recently, some famous inequalities for trigonometric and inverse trigonometric functions have been improved (see, for example, [5–8]).

The lemniscate, also called the lemniscate of Bernoulli, is the locus of points  $(x, y)$  in the plane satisfying the equation  $(x^2 + y^2)^2 = x^2 + y^2$ . In polar coordinates  $(r, \theta)$ , the equation becomes  $r^2 = \cos(2\theta)$  and its arc length is given by the function

$$\operatorname{arcsl} x = \int_0^x \frac{1}{\sqrt{1-t^4}} dt, \quad |x| \leq 1, \tag{1.7}$$

where  $\operatorname{arcsl} x$  is called the arc lemniscate sine function studied by Gauss in 1797-1798. Another lemniscate function investigated by Gauss is the hyperbolic arc lemniscate sine function, defined as

$$\operatorname{arcslh} x = \int_0^x \frac{1}{\sqrt{1+t^4}} dt, \quad x \in \mathbb{R}. \tag{1.8}$$

Functions (1.7) and (1.8) can be found (see [9], Chapter 1, [10], p.259 and [11–19]).

Another pair of lemniscate functions, the arc lemniscate tangent  $\operatorname{arctl}$  and the hyperbolic arc lemniscate tangent  $\operatorname{arctlh}$ , have been introduced in [12], (3.1)-(3.2). Therein it has been proven that

$$\operatorname{arctl} x = \operatorname{arcsl} \left( \frac{x}{\sqrt[4]{1+x^4}} \right), \quad x \in \mathbb{R} \tag{1.9}$$

and

$$\operatorname{arctlh} x = \operatorname{arcslh} \left( \frac{x}{\sqrt[4]{1-x^4}} \right), \quad |x| < 1 \tag{1.10}$$

(see [12], Proposition 3.1).

In analogy with (1.1), we here establish Shafer-type inequalities for the lemniscate functions, which is the last aim of the present paper.

**Theorem 1.3** For  $0 < x < 1$ ,

$$\frac{10x}{5 + \sqrt{25 - 10x^4}} < \operatorname{arcsl} x \tag{1.11}$$

and

$$\frac{10x}{5 + \sqrt{25 - 15x^4}} < \operatorname{arctlh} x. \tag{1.12}$$

**Theorem 1.4** For  $x > 0$ ,

$$\frac{95x}{80 + \sqrt{225 + 285x^4}} < \operatorname{arcslh} x. \tag{1.13}$$

We present the following conjecture.

**Conjecture 1.1** For  $x > 0$ ,

$$\operatorname{arcslh} x < \frac{95x + \frac{931}{2,925}x^{13}}{80 + \sqrt{225 + 285x^4}} \tag{1.14}$$

and

$$\frac{1,210x}{940 + 9\sqrt{900 + 1,210x^4}} < \operatorname{arctl} x < \frac{1,210x + \frac{2,078,417}{280,800}x^{13}}{940 + 9\sqrt{900 + 1,210x^4}}. \tag{1.15}$$

## 2 Lemmas

The following lemmas have been proved in [17].

**Lemma 2.1** For  $|x| < 1$ ,

$$\operatorname{arcsl} x = \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi}(4n + 1) \cdot n!} x^{4n+1} = x + \frac{1}{10}x^5 + \frac{1}{24}x^9 + \dots \tag{2.1}$$

**Lemma 2.2** For  $0 < x < 1$ ,

$$\operatorname{arctlh} x = \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{3}{4})}{\Gamma(\frac{3}{4}) \cdot (4n + 1) \cdot n!} x^{4n+1} = x + \frac{3}{20}x^5 + \frac{7}{96}x^9 + \dots \tag{2.2}$$

### 3 Proofs of Theorems 1.1 to 1.4

*Proof of Theorem 1.1* The inequality (1.11) is obtained by considering the function  $f(x)$  defined by

$$f(x) = \arctan x - \frac{8x + \frac{32}{4,725}x^7}{3 + \sqrt{25 + \frac{80}{3}x^2}}, \quad x > 0.$$

Differentiation yields

$$f'(x) = -\frac{2g(x)}{(1+x^2)(9 + \sqrt{225 + 240x^2})^2 \sqrt{225 + 240x^2}},$$

where

$$g(x) = (-7,875 - 2,100x^2 + 112x^6 + 112x^8)\sqrt{225 + 240x^2} + 118,125 + 94,500x^2 + 2,800x^6 + 5,360x^8 + 2,560x^{10}.$$

We now show that

$$g(x) > 0, \quad x > 0. \tag{3.1}$$

By an elementary change of variable

$$t = \sqrt{225 + 240x^2}, \quad t > 15,$$

the inequality (3.1) is equivalent to

$$\begin{aligned} & \left[ -7,875 - 2,100\left(\frac{t^2 - 225}{240}\right) + 112\left(\frac{t^2 - 225}{240}\right)^3 + 112\left(\frac{t^2 - 225}{240}\right)^4 \right] t \\ & + 118,125 + 94,500\left(\frac{t^2 - 225}{240}\right) + 2,800\left(\frac{t^2 - 225}{240}\right)^3 + 5,360\left(\frac{t^2 - 225}{240}\right)^4 \\ & + 2,560\left(\frac{t^2 - 225}{240}\right)^5 \\ & = \frac{(2t^6 + 141t^5 + 4,515t^4 + 93,690t^3 + 1,562,400t^2 + 24,053,625t + 362,626,875)(t - 15)^4}{622,080,000} \\ & > 0 \quad \text{for } t > 15, \end{aligned}$$

which is true. Hence, we have

$$g(x) > 0 \quad \text{and} \quad f'(x) < 0 \quad \text{for } x > 0.$$

So,  $f(x)$  is strictly decreasing for  $x > 0$ , and we have

$$f(x) < f(0) = 0, \quad x > 0.$$

The proof is complete. □

**Remark 3.1** Let  $x_0 = 1.4243\dots$ . Then we have

$$\frac{8x + \frac{32}{4,725}x^7}{3 + \sqrt{25 + \frac{80}{3}x^2}} < \frac{8x}{3 + \sqrt{25 + \frac{256}{\pi^2}x^2}}, \quad 0 < x < x_0.$$

This shows that for  $0 < x < x_0$ , the upper bound in (1.11) is better than the upper bound in (1.3). In fact, for  $x \rightarrow 0$ , we have

$$\arctan x - \frac{8x}{3 + \sqrt{25 + \frac{256}{\pi^2}x^2}} = O(x^3),$$

$$\arctan x - \frac{8x}{3 + \sqrt{25 + \frac{80}{3}x^2}} = O(x^7),$$

and

$$\arctan x - \frac{8x + \frac{32}{4,725}x^7}{3 + \sqrt{25 + \frac{80}{3}x^2}} = O(x^9).$$

*Proof of Theorem 1.2* The double inequality (1.11) can be written for  $0 < x < 1$  as

$$b < F(x) < a,$$

where

$$F(x) = \frac{\left(\frac{8x}{\arcsin x} - 3\sqrt{1-x^2}\right)^2 - 25}{x^2}, \quad 0 < x < 1.$$

By an elementary change of variable,

$$x = \sin t, \quad 0 < t < \frac{\pi}{2},$$

we have

$$G(t) = F(\sin t) = \frac{\left(\frac{8 \sin t}{t} - 3 \cos t\right)^2 - 25}{\sin^2 t}, \quad 0 < t < \frac{\pi}{2}.$$

We now prove that  $F(x)$  is strictly decreasing for  $0 < x < 1$ . It suffices to show that  $G(t)$  is strictly decreasing for  $0 < t < \pi/2$ . Differentiation yields

$$\begin{aligned} -\frac{t^3 \sin^3 t}{16} G'(t) &= 8 \sin^3 t - 3t^2 \sin t - (2t^3 + 3t) \cos t + 3t \cos^3 t \\ &= 8 \left( \frac{3 \sin t - \sin(3t)}{4} \right) - 3t^2 \sin t - (2t^3 + 3t) \cos t \\ &\quad + 3t \left( \frac{\cos(3t) + 3 \cos t}{4} \right) \\ &= (6 - 3t^2) \sin t - 2 \sin(3t) + \frac{3}{4}t \cos(3t) - \left( \frac{3}{4}t + 2t^3 \right) \cos t \\ &= \frac{16}{945}t^9 - \frac{16}{4,725}t^{11} + \sum_{n=6}^{\infty} (-1)^n u_n(t), \end{aligned} \tag{3.2}$$

where

$$u_n(t) = \frac{21 + 2n + 48n^2 + 64n^3 + (6n - 21) \cdot 9^n}{4 \cdot (2n + 1)!} t^{2n+1}.$$

Elementary calculations reveal that for  $0 < t < \pi/2$  and  $n \geq 6$ ,

$$\begin{aligned} \frac{u_{n+1}(t)}{u_n(t)} &= \frac{t^2(135 + 290n + 240n^2 + 64n^3 + (54n - 135) \cdot 9^n)}{2(n + 1)(2n + 3)(21 + 2n + 48n^2 + 64n^3 + (6n - 21) \cdot 9^n)} \\ &< \frac{(\pi/2)^2}{n + 1} \frac{135 + 290n + 240n^2 + 64n^3 + (54n - 135) \cdot 9^n}{2(2n + 3)(21 + 2n + 48n^2 + 64n^3 + (6n - 21) \cdot 9^n)} \\ &< \frac{135 + 290n + 240n^2 + 64n^3 + (54n - 135) \cdot 9^n}{2(2n + 3)(21 + 2n + 48n^2 + 64n^3 + (6n - 21) \cdot 9^n)} \end{aligned}$$

and

$$\begin{aligned} &2(2n + 3)(21 + 2n + 48n^2 + 64n^3 + (6n - 21) \cdot 9^n) \\ &\quad - (135 + 290n + 240n^2 + 64n^3 + (54n - 135) \cdot 9^n) \\ &= (24n^2 - 102n + 9)9^n + 256n^4 + 512n^3 + 56n^2 - 194n - 9 > 0. \end{aligned}$$

We then obtain, for  $0 < t < \pi/2$  and  $n \geq 6$ ,

$$\frac{u_{n+1}(t)}{u_n(t)} < 1.$$

Hence, for every  $t \in (0, \pi/2)$ , the sequence  $n \mapsto u_n(t)$  is strictly decreasing for  $n \geq 6$ . We then obtain from (3.2)

$$-\frac{t^3 \sin^3 t}{16} G'(t) > t^9 \left( \frac{16}{945} - \frac{16}{4,725} t^2 \right) > 0, \quad 0 < t < \frac{\pi}{2},$$

which implies  $G'(t) < 0$  for  $0 < t < \pi/2$ . Hence,  $G(t)$  is strictly decreasing for  $0 < t < \pi/2$ , and  $F(x)$  is strictly decreasing for  $0 < x < 1$ . So, we have

$$\frac{256 - 25\pi^2}{\pi^2} = \lim_{t \rightarrow 1} F(t) < F(x) = \frac{(\frac{8x}{\arcsin x} - 3\sqrt{1 - x^2})^2 - 25}{x^2} < \lim_{t \rightarrow 0} F(t) = \frac{5}{3}$$

for all  $x \in (0, 1)$ , with the constants  $5/3$  and  $(256 - 25\pi^2)/\pi^2$  being best possible. The proof is complete. □

*Proof of Theorem 1.3* By (2.1), we find that for  $0 < x < 1$ ,

$$\begin{aligned} (25 - 10x^4) - \left( \frac{10x}{\operatorname{arcsl} x} - 5 \right)^2 &> (25 - 10x^4) - \left( \frac{10x}{x + \frac{1}{10}x^5 + \frac{1}{24}x^9} - 5 \right)^2 \\ &= \frac{10x^8(3,120 - 1,344x^4 - 120x^8 - 25x^{12})}{(120 + 12x^4 + 5x^8)^2}. \end{aligned}$$

Noting that

$$3,120 - 1,344t - 120t^2 - 25t^3 > 0 \quad \text{for } 0 < t < 1,$$

we obtain, for  $0 < x < 1$ ,

$$(25 - 10x^4) - \left(\frac{10x}{\operatorname{arcsl} x} - 5\right)^2 > 0,$$

which implies (1.11).

By (2.2), we find that for  $0 < x < 1$ ,

$$\begin{aligned} (25 - 15x^4) - \left(\frac{10x}{\operatorname{arct} h x} - 5\right)^2 &> (25 - 15x^4) - \left(\frac{10x}{x + \frac{1}{10}x^5 + \frac{1}{24}x^9} - 5\right)^2 \\ &= \frac{15x^8 A(x)}{(24,960 + 3,744x^4 + 1,820x^8 + 1,155x^{12})^2}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} A(x) &= (115,947,520 - 71,285,760x^8 - 4,204,200x^{16}) \\ &\quad + x^4(87,320,064 - 11,961,040x^8 - 1,334,025x^{16}). \end{aligned}$$

Noting that for  $0 < t < 1$ ,

$$115,947,520 - 71,285,760t - 4,204,200t^2 > 0$$

and

$$87,320,064 - 11,961,040t - 1,334,025t^2 > 0,$$

we obtain  $A(x) > 0$  for  $0 < x < 1$ . From (3.3), we obtain (1.12). The proof is complete.  $\square$

*Proof of Theorem 1.4* The inequality (1.13) is obtained by considering the function  $h(x)$  defined by

$$h(x) = \operatorname{arcsl} h x - \frac{95x}{80 + \sqrt{225 + 285x^4}}, \quad x > 0.$$

Differentiation yields

$$h'(x) = \frac{1}{\sqrt{1+x^4}} - \frac{475(16\sqrt{225+285x^4} + 45 - 57x^4)}{(80 + \sqrt{225+285x^4})^2 \sqrt{225+285x^4}}.$$

By an elementary change of variable

$$t = \sqrt{225 + 285x^4}, \quad x > 0 \quad \left(\text{or } x = \sqrt[4]{\frac{t^2 - 225}{285}}, t > 15\right), \tag{3.4}$$

we have

$$\begin{aligned} & \frac{1}{\sqrt{1+x^4}} - \frac{475(16\sqrt{225+285x^4} + 45 - 57x^4)}{(80 + \sqrt{225+285x^4})^2 \sqrt{225+285x^4}} \\ &= \frac{285}{\sqrt{17,100+285t^2}} + \frac{95(t^2 - 80t - 450)}{t(80+t)^2} = \frac{95I(t)}{t(80+t)^2}, \end{aligned}$$

where

$$I(t) = \frac{19,200t + 480t^2 + 3t^3}{\sqrt{17,100+285t^2}} + t^2 - 80t - 450, \quad t > 15.$$

We now prove that

$$h'(x) > 0, \quad x > 0.$$

It suffices to show that

$$I(t) > 0, \quad t > 15.$$

Differentiation yields

$$\begin{aligned} I'(t) &= \frac{6(192,000 + 9,600t + 90t^2 + 80t^3 + t^4)}{(60+t^2)\sqrt{17,100+285t^2}} + 2t - 80, \\ I''(t) &= \frac{6(576,000 - 565,200t - 4,800t^2 + 150t^3 + t^5)}{(60+t^2)^2\sqrt{17,100+285t^2}} + 2, \end{aligned}$$

and

$$I'''(t) = \frac{10,800(-18,840 - 1,920t + 1,271t^2 + 8t^3)}{(60+t^2)^3\sqrt{17,100+285t^2}} > 0 \quad \text{for } t > 15.$$

Thus, we have, for  $t > 15$ ,

$$I''(t) > I''(15) = 0 \implies I'(t) > I'(15) = 0 \implies I(t) > I(15) = 0.$$

Hence,  $h'(x) > 0$  holds for  $x > 0$ , and we have

$$h(x) > h(0) = 0, \quad x > 0.$$

The proof is complete. □

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors read and approved the final manuscript.

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