CHAPTER 7

SHALLOW WATER WAVES A COMPARISON OF THEORIES AND EXPERIMENTS

by

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ABSTRACT

A series of experiments were performed to determine the velocity field and other characteristics of large amplitude shallow water waves. The experimental results were compared with the predictions of a variety of wave theories including those commonly used in engineering practice. While no theory was found exceptionally accurate, the cnoidal wave theory of Keulegan and Patterson appears most adequate for the range of wavelengths and water depths studied.

1. INTRODUCTION

This paper presents the results of an experimental program performed to establish the major characteristics of waves in shallow water and to determine by detailed comparisons which of the many wave theories commonly used in engineering practice best describes them. Knowledge of the features of these waves is essential to the engineer interested, for example, in the forces on submerged structures, such as pilings. For this purpose, he needs reliable expressions for the velocity field within the wave Upon choosing a suitable design wave (that is, mean water depth, wave height, and wave period) he then proceeds to select a theory to describe that wave. This choice is not easily made. He must evaluate the applicability of at least a dozen theories. It may be hoped, for example, that a theory developed to a fifth order of approximation is more accurate than its lower order counterparts which may or may not actually be the case. We have, therefore, conducted an extensive series of experiments to measure the characteristics of large shallow water waves and have, further, compared these results with a large number of theories in order to suggest one as most applicable for practical use. In this comparison and selection, we have settled upon the horizontal particle velocity under the crest as the single most important feature, since, in applications, velocity is generally most critical and the velocity under the crest is the greatest attained at any

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depth. Other factors are of lesser importance; for example, the surface profile in all theories must be roughly similar since the wave height is imposed. Furthermore, a theory which prescribes the velocity field well is constrained to be good for other features, such as accelerations and pressures, a priori

The range of waves considered included relative depths, D/T^2, from 0.05 to 0 8 ft sec^{-2} with heights near the breaking limit. The maximum horizontal particle velocity in these waves has been compared with the predictions of twelve theories not all of which are expected to apply. Nevertheless, it is possible that a theory may, fortuitously, describe waves outside its range of analytical validity better than one developed for those waves. Therefore, our selection of theories has been rather broad, including all those commonly used in engineering practice.

Prior to presenting our experimental results and a comparison of these results with various theories, a general discussion on the arbitrariness which prevails in the development of these theories is presented. Even though it is realized that it is a difficult task to be fully aware of all of the mathematical intricacies leading sometimes to small differences, it is assumed that the reader is familiar with the subject. It is not a critical analysis of the shallow water wave theories, but rather it is attempted to call attention to some known facts which are pertinent prior to presenting our experimental results.

2. ARBITRARINESS OF WAVE THEORIES

In an Eulerian system of coordinates a surface wave problem generally involves three unknowns the free surface elevation η (or total water depth), the pressure p (generally known at the free surface), and the particle velocity $\vec{V}$, expressed as a function of space and time.

Since a general method of solution does not exist, a number of simplifying assumptions are generally made which apply to a succession of particular cases with varying accuracy.

First of all, it is assumed that a periodic progressive wave travelling over a horizontal bottom is characterized by a steady state profile, i.e.,

$$\eta, \vec{V}, p = f(x - Ct)$$

where C is a constant equal to the wave velocity or phase velocity (In fact, it has been observed that under certain conditions the wave profile in shallow water becomes unstable, asymmetric, and even degenerates into a succession of smaller undulations)

The problem now consists essentially of solving a system satisfying continuity, momentum and boundary conditions. However, these are not sufficient for solving the nonlinear problem and two more considerations are necessary.

This leads us to a discussion of the problem of rotationality and mass transport (since they are related).

It is first pointed out that a steady flow of arbitrary velocity distribution can always be superimposed on a given wave motion, so leading to any arbitrary mass transport velocity distribution (Dubreuil-Jacotin 1934) The arbitrariness in the calculation of wave motion is inherent in the arbitrariness which prevails in the assumptions which are used in the calculation of the mass transport.

The wave motion can be determined by assuming that there is no mass transport at all. These are the closed orbit theories, such as the exact solution of Gerstner (1809) in deep water, and the power series solution of Boussinesq and Kravtchenko and Daubert (1957) for shallow water. As a result of this assumption, the motion is found to be rotational and the vorticity is in the opposite direction to the particle rotation, i.e., in the opposite direction to what should be expected physically under the influence of a shearing stress due to wind blowing in the wave direction.

The wave motion can also be assumed to be irrotational, in which case a mass transport distribution is found as a result of nonlinearity (Wehausen and Laitone, 1960). These are the Stokesian wave theories which include Stokes (1847), Levi-Civita (1925), Struik (1926), and Nekrassov (1951).

Even though there is a given mass transport distribution which is a function of the vertical coordinate, the integrating constant is often determined by assuming that the average mass transport is nil for the sake of continuity, i.e., a steady flow is superimposed such that the average mass transport velocity is zero (Miche, 1944).

Let

$$U(z) = (\mu + 1)\, U_{\zeta=0}(z)$$

where $U(z)$ is the mass transport velocity as a function of vertical distance z and $U_{\zeta=0}$ is the mass transport in an irrotational wave ($\zeta=0$) of the same family and approximation.

The case where $\mu = -1$ corresponds to the Boussinesq (closed orbit) solution. $\mu = 0$ corresponds to the irrotational theory of Stokes If $\mu > 0$, the average vorticity is in the same direction as the orbit direction, such as due to a strong wind blowing in the wave direction. The case where $\mu < -1$ gives a negative vorticity and a negative mass transport at the free surface which can be due to a wind blowing locally in the opposite direction to the wave propagation (a frequent nearshore occurrence) In the case where $-1 < \mu < 0$, the vorticity is in the opposite direction and the mass transport is smaller than in the irrotational case.

It is recalled that Longuet-Higgins (1953) has demonstrated the importance of the viscous force at the bottom to explain the well-observed fact that mass transport at the bottom is always in the wave direction.

In addition to the first assumption regarding rotationality or mass transport, another condition is required. For example, for progressive monochromatic waves, it is attempted to establish a steady state solution such that the potential function $\varphi = f(x - Ct)$ where C is a constant equal to the wave velocity in which case the solution is unique. Although the steady state solutions are of the same form, C is undetermined, and for the determination of C, another condition is required. For example, the average horizontal velocity over a wave period at a given location may be taken to be zero, but the mass transport is then imposed to be minimum. Another condition is generally preferred. This condition consists of assuming that the average momentum over a wave length is zero by addition of a uniform motion, in this case, another expression for C is found which results in a different mass transport. Thus, it is realized that the calculation of wave theories is subject to some arbitrariness because different assumptions can lead to different values of C. (Le Méhauté, 1968)

Consider further the case of irrotational waves. The values of the wave characteristics depend upon the number of terms chosen for the power series expansion, either in terms of wave steepness H/L (Stokesian solution) or in terms of relative height H/D (cnoidal type solution). The deficiency of the calculation of mass transport is inherent in the deficiency of the power series solution. It has been pointed out (Koh and Le Méhauté, 1965) that the Stokesian power series solution is not uniformly convergent, and the validity of the solution is lost when the relative depth D/L tends to a small value (say, $D/L < 0.1$ for a fifth order solution) since the coefficient functions of D/L tend to infinity. The same occurs in the case of the cnoidal wave solution. There is no unique cnoidal theory, rather, the literature contains several theories which may not be identical. As in Stokesian theories, since all cnoidal representations are truncated series, the order of approximation is important because certain factors are zero in low order theory

There are two types of cnoidal theories. The oldest is intuitive in nature and the newer theories are straightforward and more rigorous. All are irrotational. The primary intuitive theory is that of Korteweg and de Vries (1895). The first and second terms of the series are deduced but no scheme is presented for extension to higher order terms. The terms which are found are unique. The rigorous theories are those of Keller (1948), Laitone (1960, 1962) and Chappelear (1962) which are all based on a perturbation expansion developed by Friedrichs (1948). The work of Keller confirms the results of Korteweg and de Vries, whereas that of Laitone and Chappelear gives a higher order term.

Unfortunately, even though rigor prevails, the newer theories diverge after the third term. According to a personal communication with L. Webb, the fourth terms exceed the third by factors of 10 to 25. The cnoidal wave theory of Keulegan and Patterson is not consistent mathematically as some second order terms are neglected while third order terms are included, but it may be the most appealing physically.

There are questions which can be raised as to the legitimacy of truncating these series, either Stokesian or cnoidal, just because the coefficients blow up. For true rigor it must be shown that the series are asymptotic in the pure mathematical sense. This has not been done and indeed it would be a difficult task.

As it would have been a tremendous task to consider all wave theories, a selection has been made for convenience. The following wave theories were chosen for comparison with our experiments (listed with the sources of equations and tables used in this study)

1. Linear Airy theory in Eulerian coordinates (Wiegel, 1964)
2. Linear Airy theory in Lagrangian coordinates (Biesel, 1952)
3. Linear long wave theory (Wiegel, 1964)
4. Stokes' waves, 2nd order of approximation (Wiegel, 1964)
5. Stokes' waves, 3rd order of approximation (Skjelbreia, 1959)
6. Stokes' waves, 5th order of approximation (Skjelbreia and Hendrickson, 1962)
7. Cnoidal theory of Keulegan and Patterson (Masch and Wiegel, 1961)
8. Cnoidal theory of Laitone, 1st order of approximation (Laitone, 1961)
9. Cnoidal theory of Laitone, 2nd order of approximation (Laitone, 1961)
10. Solitary wave theory of Boussinesq (Munk, 1949)
11. Solitary wave theory of McCowan (Munk, 1949)
12. Empirical modification of linear Airy theory by Goda (Goda, 1964)

While our experiments involved only periodic waves, solitary wave theory has been included in the comparisons, this is so since for the waves studied, the phenomenon of short high crests and long flat troughs is observed. It is possible, then, that solitary wave theory might describe these waves in the vicinity of their crests.

While not an analytical theory, the twelfth referenced empirical modification of Airy theory was included to determine the validity of its extension to waves of the type studied here.

It should also be noted that the source utilized for 3rd order Stokes' theory contains some small errors in its mathematical development. Nevertheless, it is a standard reference and was utilized without modification.

The theories chosen involving nonlinearity are irrotational and have mass transport in the wave direction. Since mass transport velocity is always small compared with particle velocity, this effect has not been measured as it is not judged to be a significant factor in the validity of these theories at this stage. For this reason, the comparison of experiment with 2nd order Stokes' theory is also valid for the theory of Miche (1944).

3. EXPERIMENTAL PROCEDURES AND RESULTS

The current experimental program was performed in the Tetra Tech wave tank which is illustrated in Figure 1. The tank is 105 feet long with an approximately four-foot square cross-section. The wave generator is of the versatile plunger type. The period of the waves can be varied by the motor speed control through a range of approximately one to twelve seconds.

In order to produce large amplitude shallow water waves in the test section, a convergence was combined with a sloping bottom prior to the constant depth and width region. This arrangement concentrates the wave energy in the test section and easily permits the generation of high amplitude shallow water waves. The wave is thus generated as a deep water "linear" wave and is transformed into a "nonlinear cnoidal type" shallow water wave through the transitional section.

The wave channel was entirely sealed nullifying mass transport. Provision for a return flow under the test station would result in a difficult to specify mass transport through the channel due to unknown head-loss Hence, a sealed channel was preferred, giving zero net mass transport, in the interest of well-defined conditions.

Water particle motions were determined photographically through observation of neutrally buoyant particles suspended in the water. Nitrile rubber, especially compounded to a specific gravity of unity, was used for this purpose, cut into one-eighth inch cubes, it was found to follow the water motion quite well and reflect sufficient light to photograph easily.

Lighting was provided by a stroboscope of accurately controllable flash rate allowing, with open camera shutter, the recording of several successive particle positions on a single negative

To provide a spatial scale, a rectangular grid of wires with one inch spacing was placed at the observation window and is visible, with the channel bottom, on each negative. In order to minimize parallax problems, a 200 mm telephoto lens was used in conjunction with a 35 mm camera. Data processing is then easily accomplished, particle image spacing and flash rate giving local velocity vectors, and position within the wave being known by reference to the bottom

The data scatter was found to be gratifyingly small and systematic errors (predominately due to wave reflection) are judged to be at most ± 5 %

Two sets of non-breaking waves were studied, one set with heights just below breaking and another with heights considerably lower Figures 2 through 5 show the measured horizontal particle velocities under the crest for the lower waves while Figures 6 through 9 give the same results for the limit waves. The experimental conditions (wave height and period and water depth) are indicated on the figures

It is noted from these figures that no theory is uniformly valid However, some general observations can be made. Firstly, Airy theory (the same in both Eulerian and Lagrangian coordinates) is surprisingly good for the shorter waves but departs significantly from the data as the period increases. Even for these longer waves, however, Airy theory provides a reasonable approximation to the velocity at the bottom The higher order Stokes' theories are never as good as simple Airy theory and, considering their much greater complexity, are not to be preferred

The Boussinesq, lst order cnoidal (Laitone) and linear long wave theories all show a constant velocity profile. The Boussinesq and cnoidal values are most suitable near the surface while (like Airy theory) the linear long wave result is in best agreement with the data near the bottom.

While no theory appears best in all cases, the cnoidal theory of Keulegan and Patterson (K & P) is seen from the figures to be the best compromise. This may be somewhat surprising considering, for example, the greater rigor of the Laitone cnoidal formulations which are uniformly worse in this confrontation with data. The question of analytical niceties aside, the Keulegan and Patterson cnoidal theory is seen to find practical justification for the long, high waves considered here.

Figure 10 shows a typical comparison between measured (photographically) and computed surface profiles. As stated earlier, the variation between theories regarding profile is not as great as that for particle velocity since the most significant feature, the wave height, is imposed by experiment.

The solitary wave theories of Boussinesq and McCowan are seen to match the experimental shape near the crest quite well, but are, of course, displaced upward. Better is the Keulegan and Patterson cnoidal profile which not only matches the shape adequately but also is accurately placed with respect to the still water level.

The appearance of humps within the troughs of the Stokes' waves at all orders is an indication that they are being applied here beyond their range of validity. Still, the 3rd and 5th orders appear preferable in gross characteristics to the simpler Airy theory. The formulation in Lagrangian coordinates is seen to improve Airy theory, even at the first order, since the profile is a trochoid with the crest somewhat narrower than the trough, however, the amount of correction is insufficient to produce adequate agreement with the data.

Again, one is led to select the cnoidal formulation of Keulegan and Patterson although the solitary wave theories may also be useful if only the shape of the crest is of interest

Determination of the maximum horizontal velocity, presented previously, is relatively simple since it always corresponds to the top of the particle orbit, which is readily located on the photographs, and occurs at $x = 0$ (at the crest) in all theories so that computations, too, are straightforward. Figure 11 is an example of data obtained when the particle motion is purely vertical (that is, at the sides of the particle orbit). In contrast to the horizontal case just described, comparison with theory is not straightforward. In particular, the phase position at which purely vertical motion occurs is not (except for Airy theory) immediately known, but must be found from the equation for horizontal velocity with $u = 0$. For this reason, and since vertical velocity is generally of less practical significance than horizontal velocity, we have not made a complete comparison with all the theories. Instead, we show in Figure 11 Airy theory, because of its simplicity, Keulegan and Patterson cnoidal theory, because of its success in the previous comparisons, and McCowan theory, in order to assess the applicability of the solitary wave approximation (Since the water particle motion is never purely vertical under a solitary wave, the theoretical curve is that of maximum vertical velocity.) It is seen that the McCowan theory is best with Keulegan and Patterson next and Airy theory least satisfactory

It was found in the course of the experiments that an increase in period beyond about four seconds, in water depths on the order of one-half foot, was accompanied by the development of a wave instability. That is, the high, narrow wave crest split into several smaller undulations upon entering the test section, as many as five being distinctly visible under some circumstances. The appearance of this instability precluded extension of the experiments to longer waves. The limit of stability, while not investigated in detail, was approximately $D/T^2 \approx 0.04$ ft sec^{-2}.

4. CONCLUSION

It has been found that for the range of relative depths, D/T^2, and relative heights, H/T^2, studied here, none of the commonly used wave theories are in exceptional agreement with data. However, for these large amplitude shallow water waves, the cnoidal theory of Keulegan and Patterson is perhaps the most generally acceptable description.

This conclusion may be compared with many previous classifications of wave theories (for example, Druet, 1965). Dean (1965) has compared the closeness of fit of several theories to the free surface boundary conditions as a criterion of relative validity. He found that this method is particularly valid for deep water waves, but may not be sufficient for shallow water waves.

For applications requiring information only near the surface or involving only the crest shape, the solitary wave theory of McCowan is an adequate approximation, while simple Airy theory may be applied at the bottom

These results should ease the problem of choice for the engineer concerned with the effects of large amplitude shallow water waves; perhaps, too, some assistance will be found by analysts in development of improved theories for those waves.

It should be recalled that the value of the vorticity has to be determined from the viscous effect which is particularly strong at the bottom and at the free surface under the influence of wind. A rotational theory at a second order of approximation, with an arbitrary vorticity distribution (and, consequently, mass transport) remains to be established. This theory should take into account not only the viscous effect due to the bottom boundary layer, but also the effect of wind shearing stresses at the free surface. This can be achieved as a solution of the vorticity transport equation with an approach similar to that proposed by Longuet-Higgins. One can only hope at this stage that the mathematical model for the distribution of vorticity can be empirically correlated with wind characteristics by making quantitative observations on mass transport.

Such an approach, combined with the application of stochastic processes, will provide the engineer with a much more realistic model of water waves than any high order Stokesian or cnoidal wave theory Further mathematical refinements of such theories would appear to be of little practical interest.

ACKNOWLEDGEMENTS

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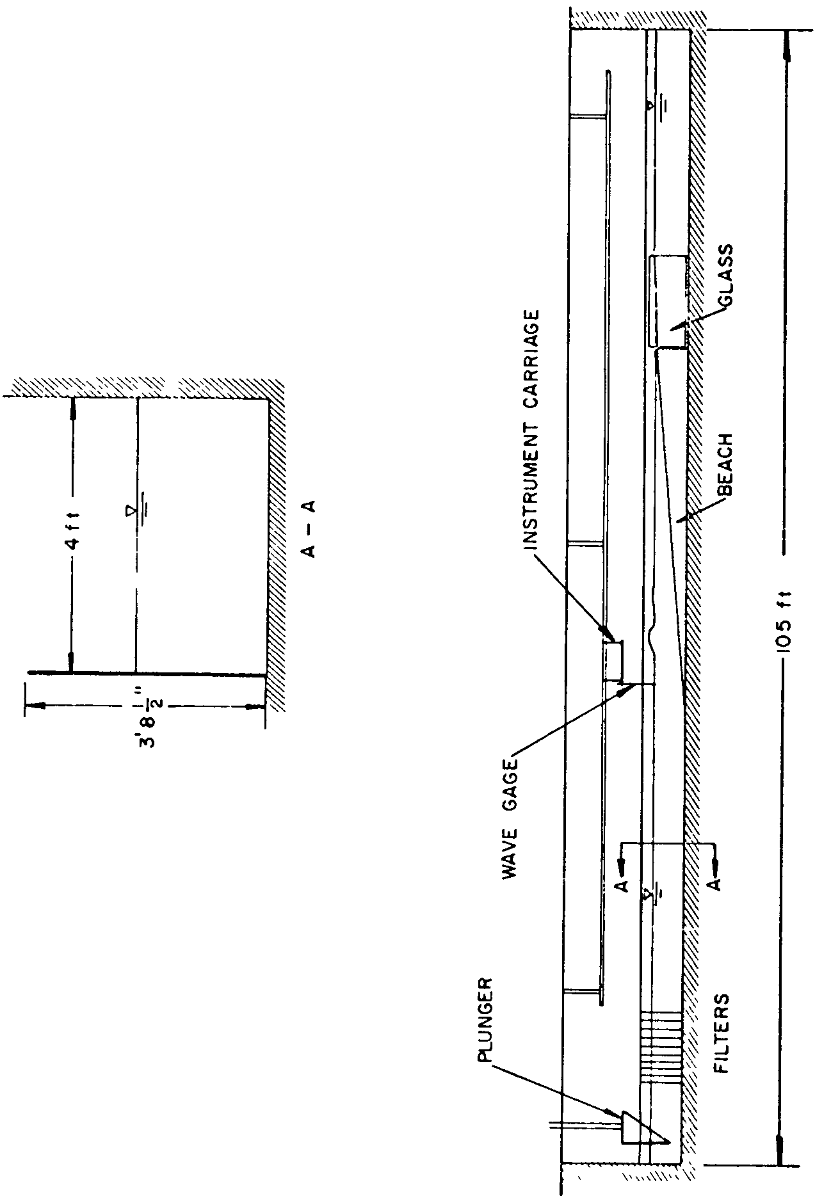

Figure 1. Diagram of the Tetra Tech wave tank.

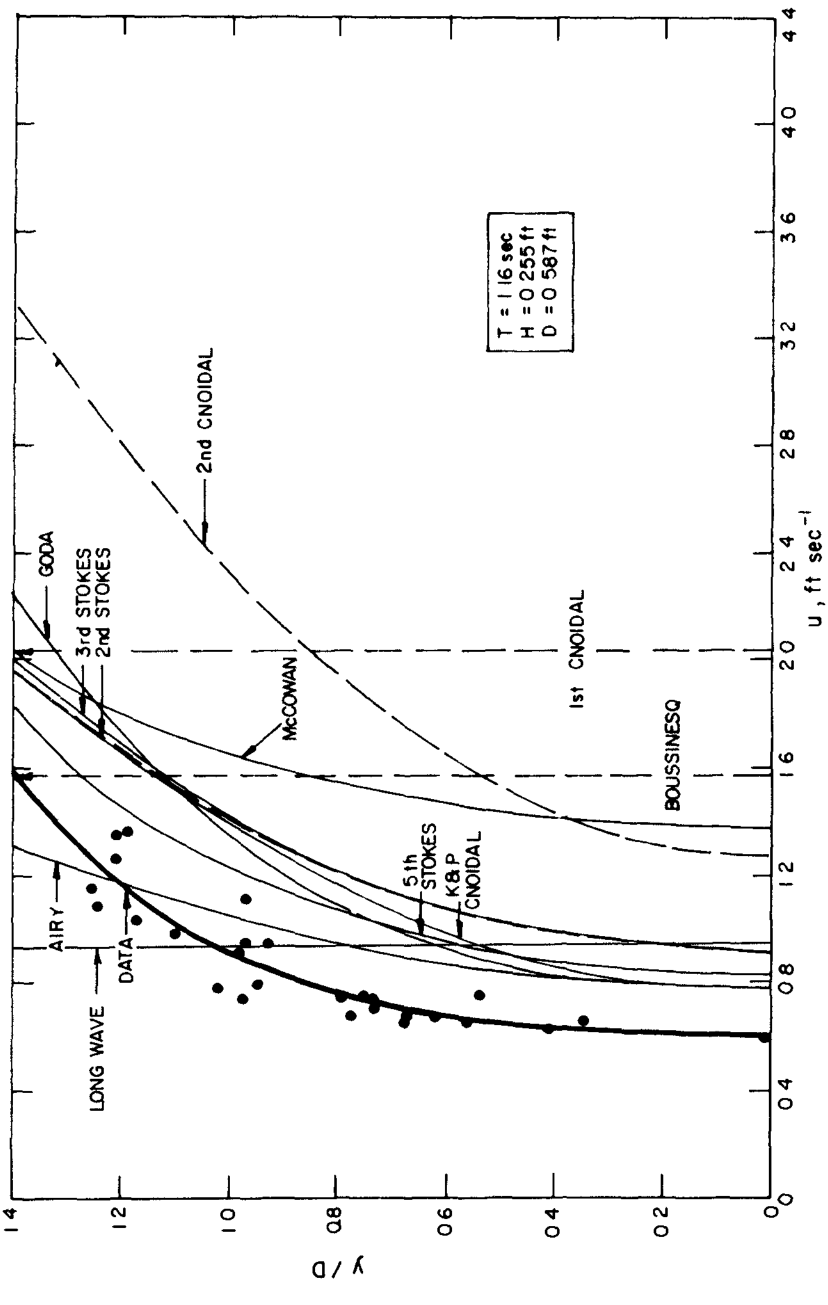

Figure 2. Horizontal Particle Velocity under the Crest - NON-BREAKING WAVE.

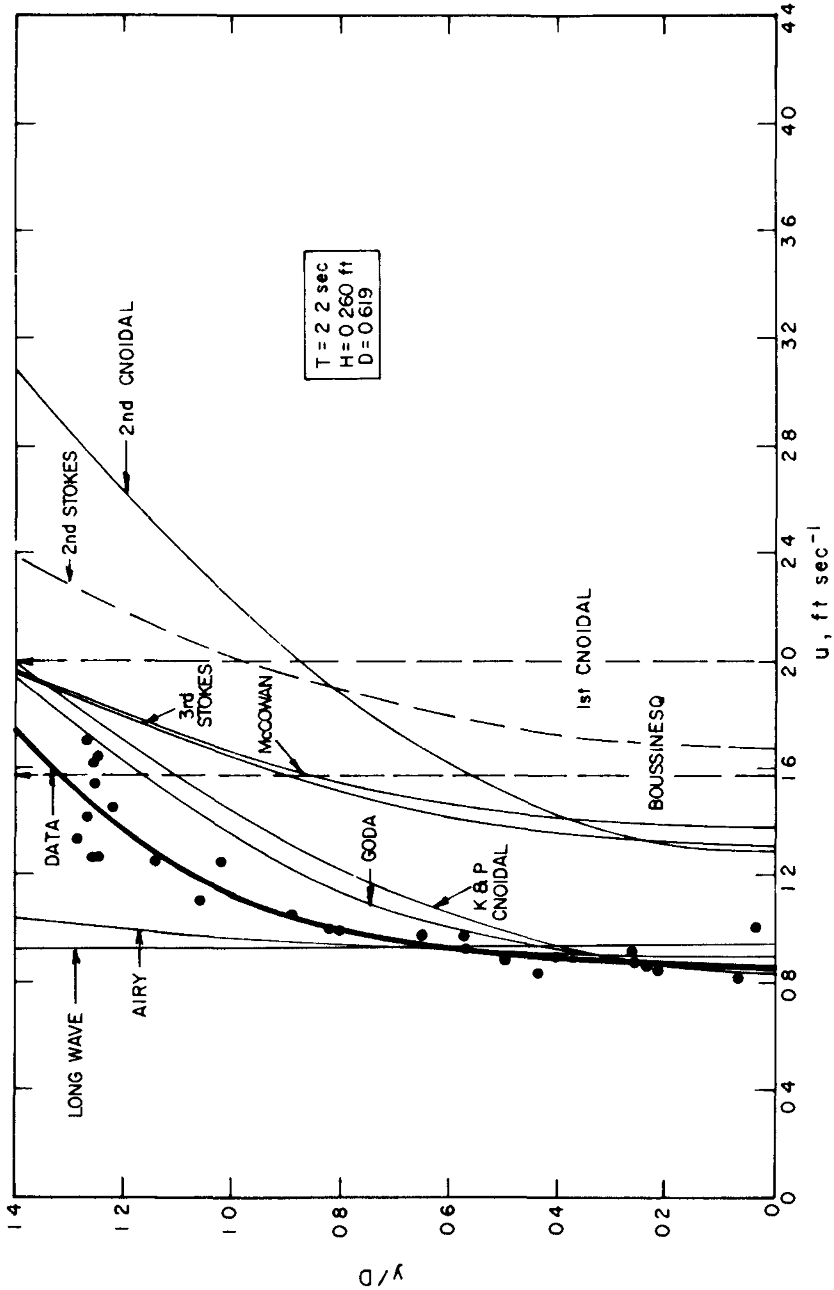

Figure 3. Horizontal Particle Velocity under the Crest - NON-BREAKING WAVE.

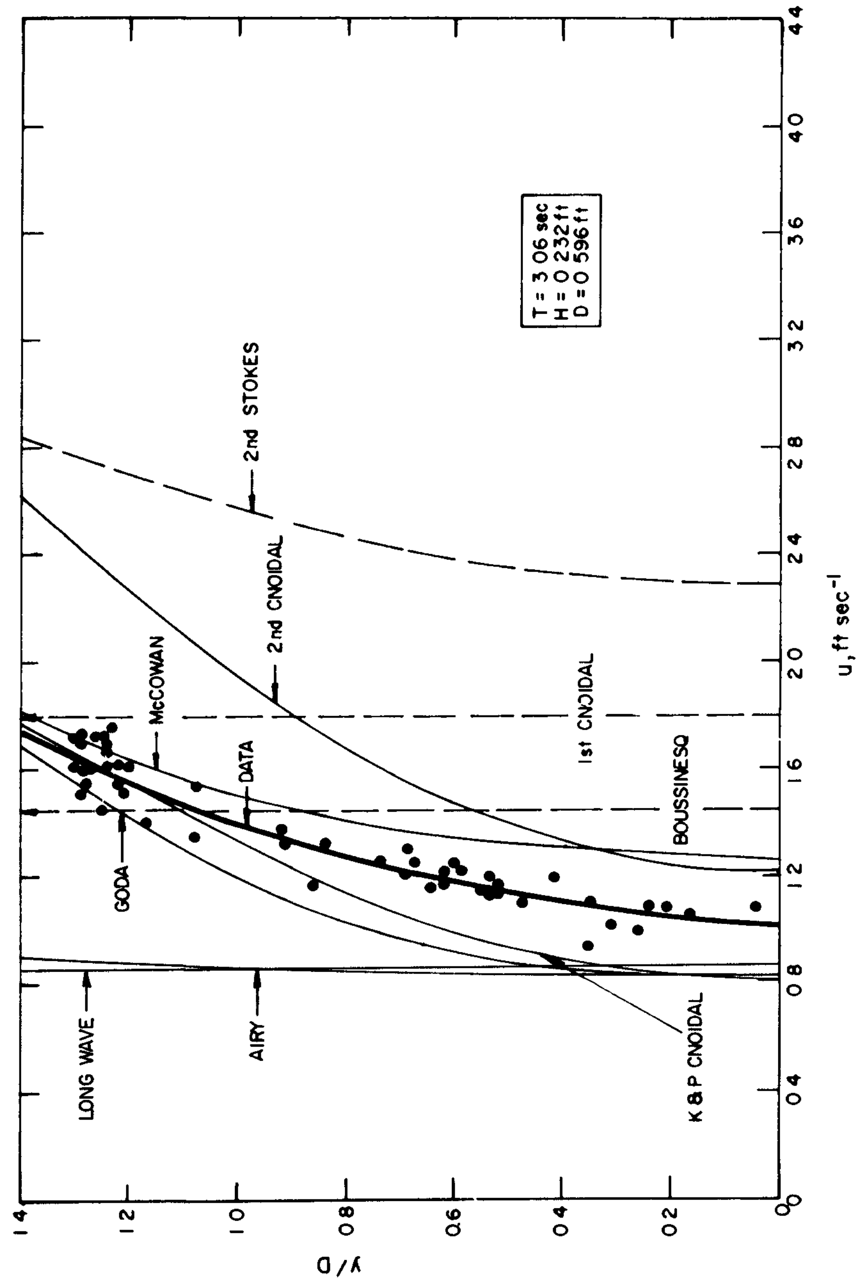

Figure 4. Horizontal Particle Velocity under the Crest - NON-BREAKING WAVE.

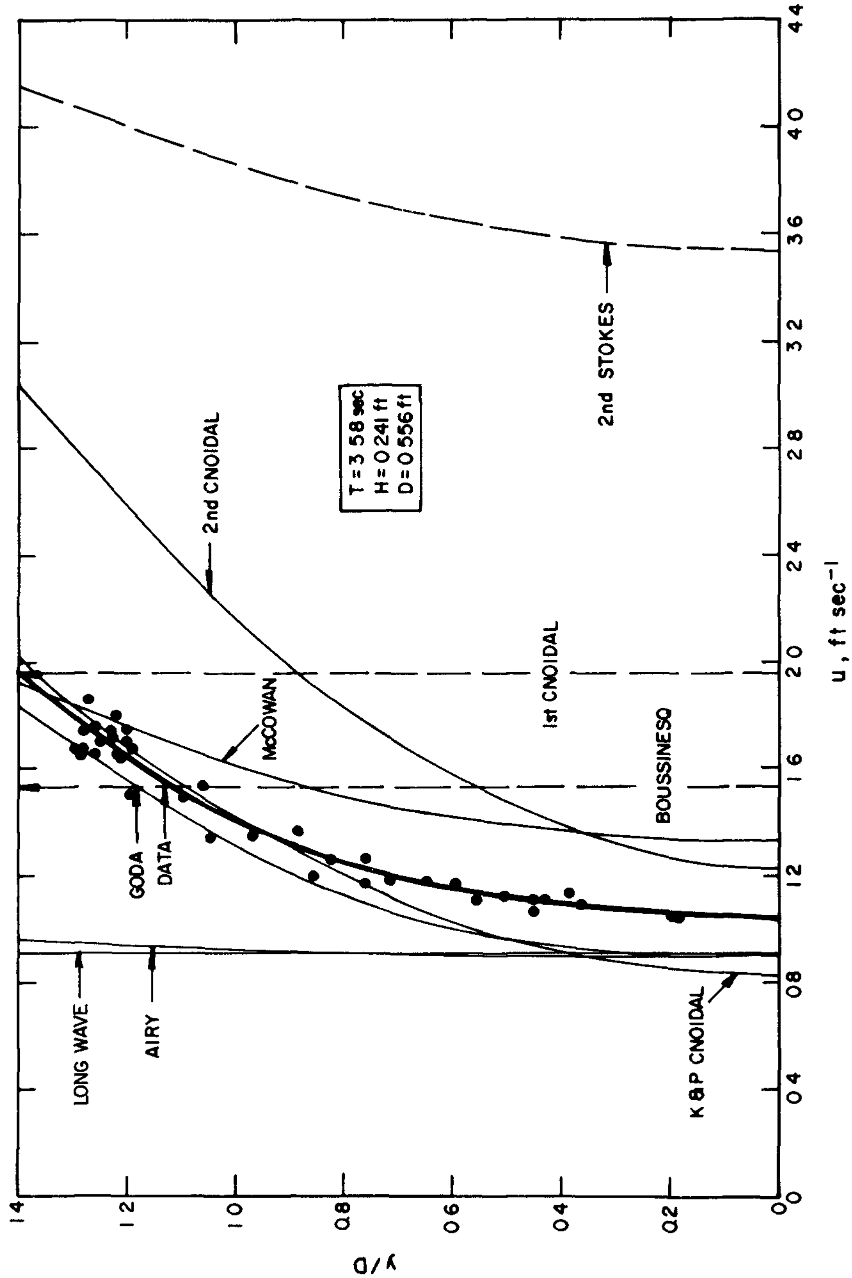

Figure 5. Horizontal Particle Velocity under the Crest - NON-BREAKING WAVE.

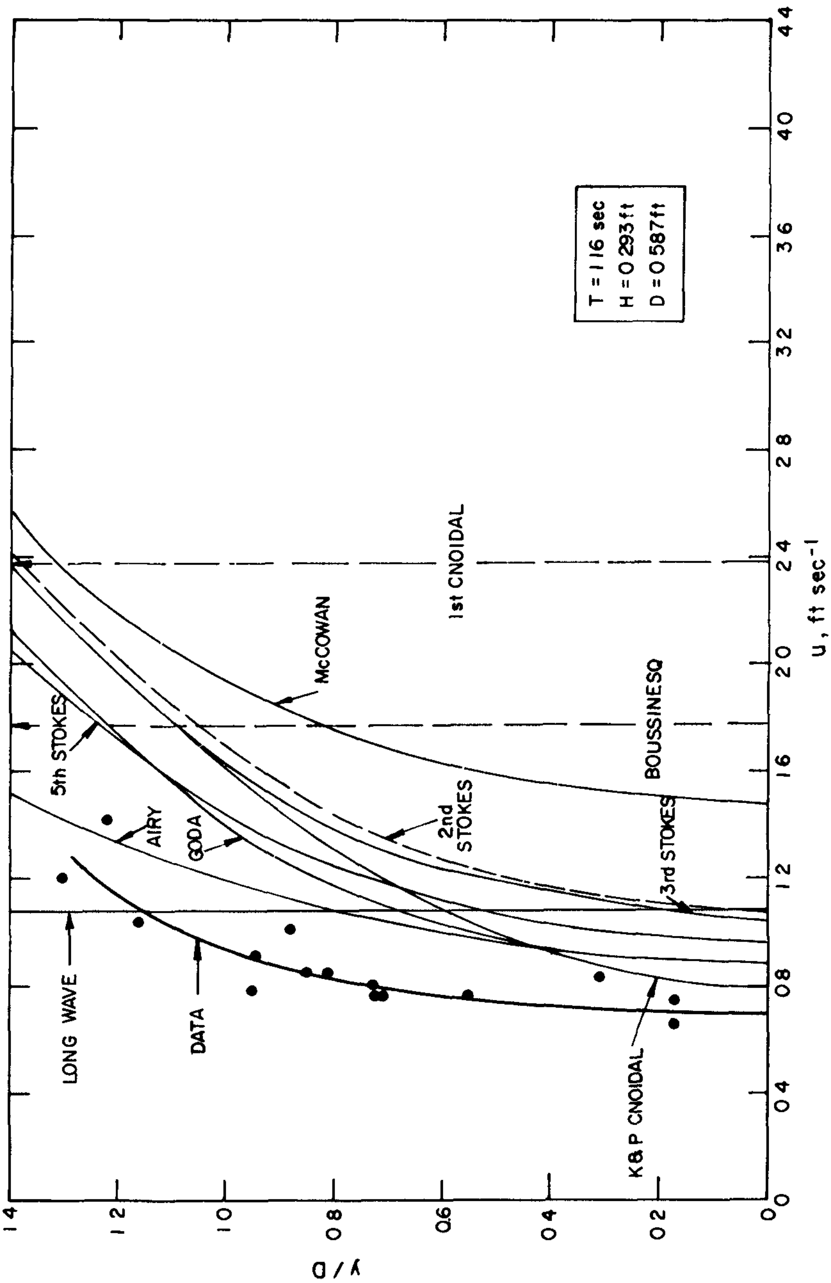

Figure 6. Horizontal Particle Velocity under the Crest - NEAR-BREAKING WAVE.

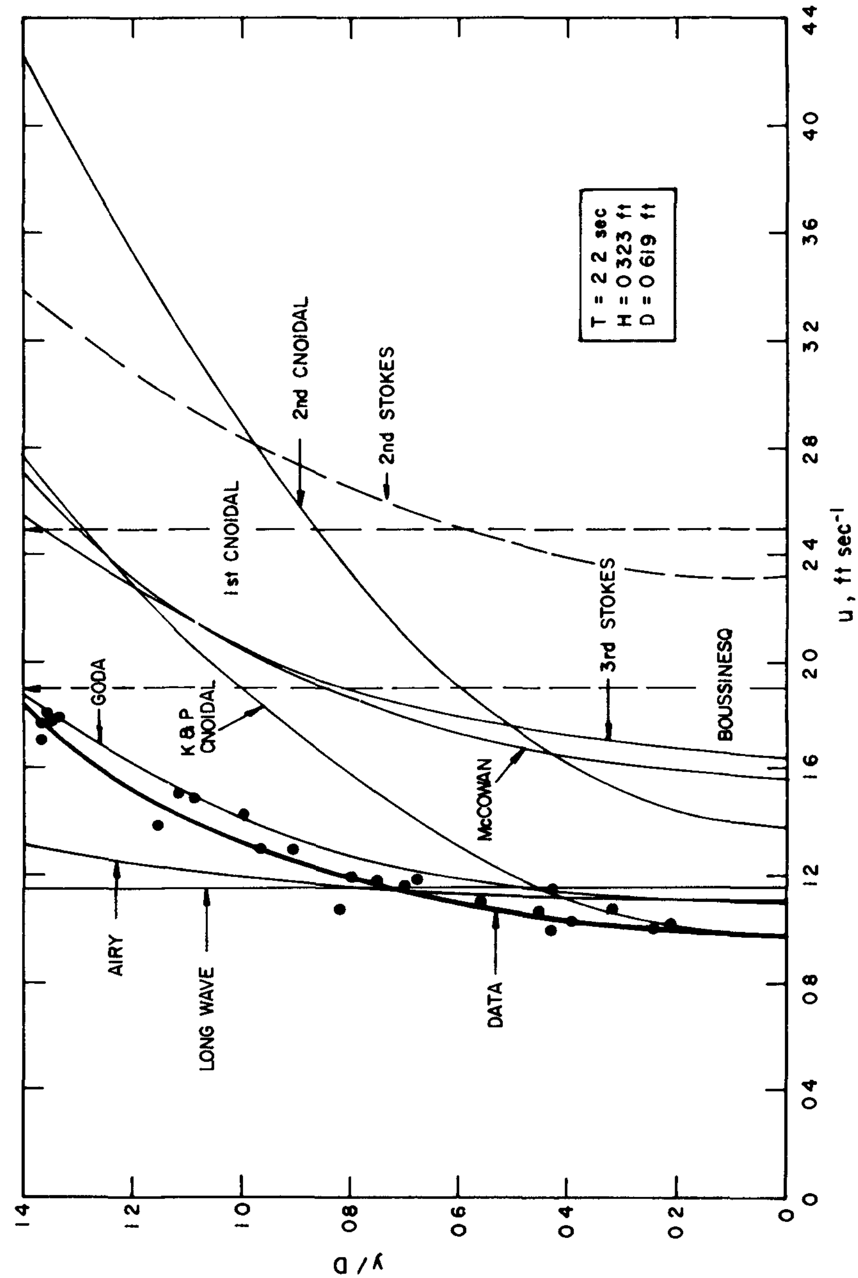

Figure 7. Horizontal Particle Velocity under the Great - NEAR-BREAKING WAVE.

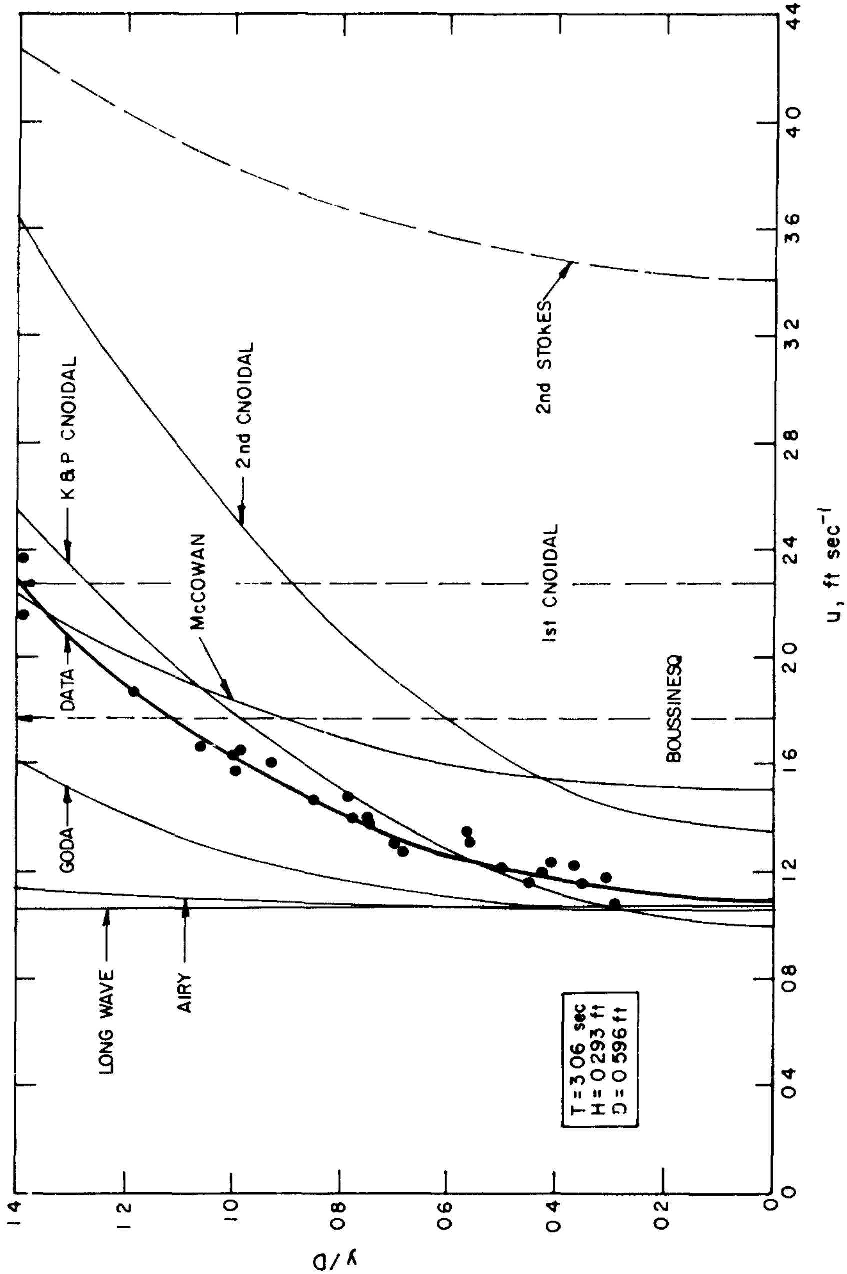

Figure 8. Horizontal Particle Velocity under the Crest - NEAR-BREAKING WAVE.

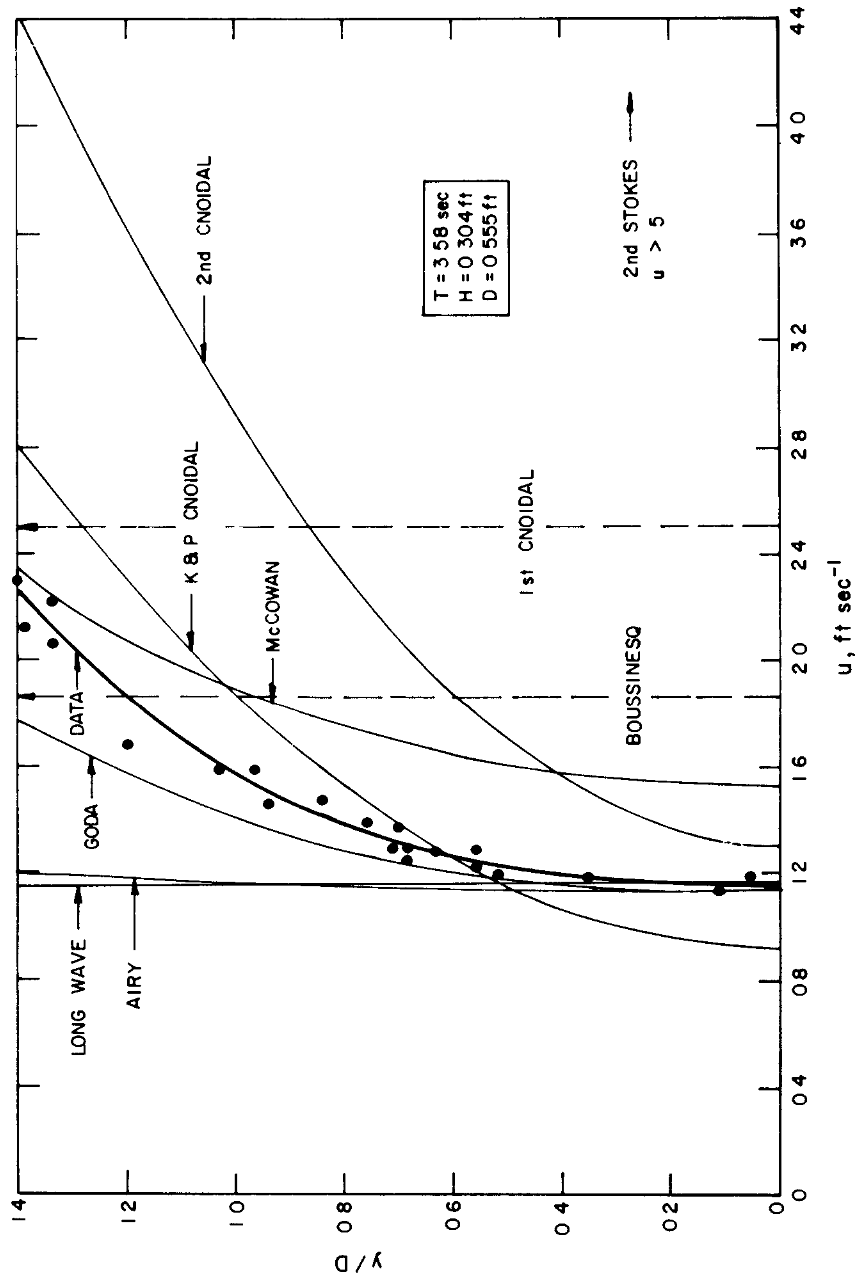

Figure 9. Horizontal Particle Velocity under the Crest - NEAR-BREAKING WAVE.

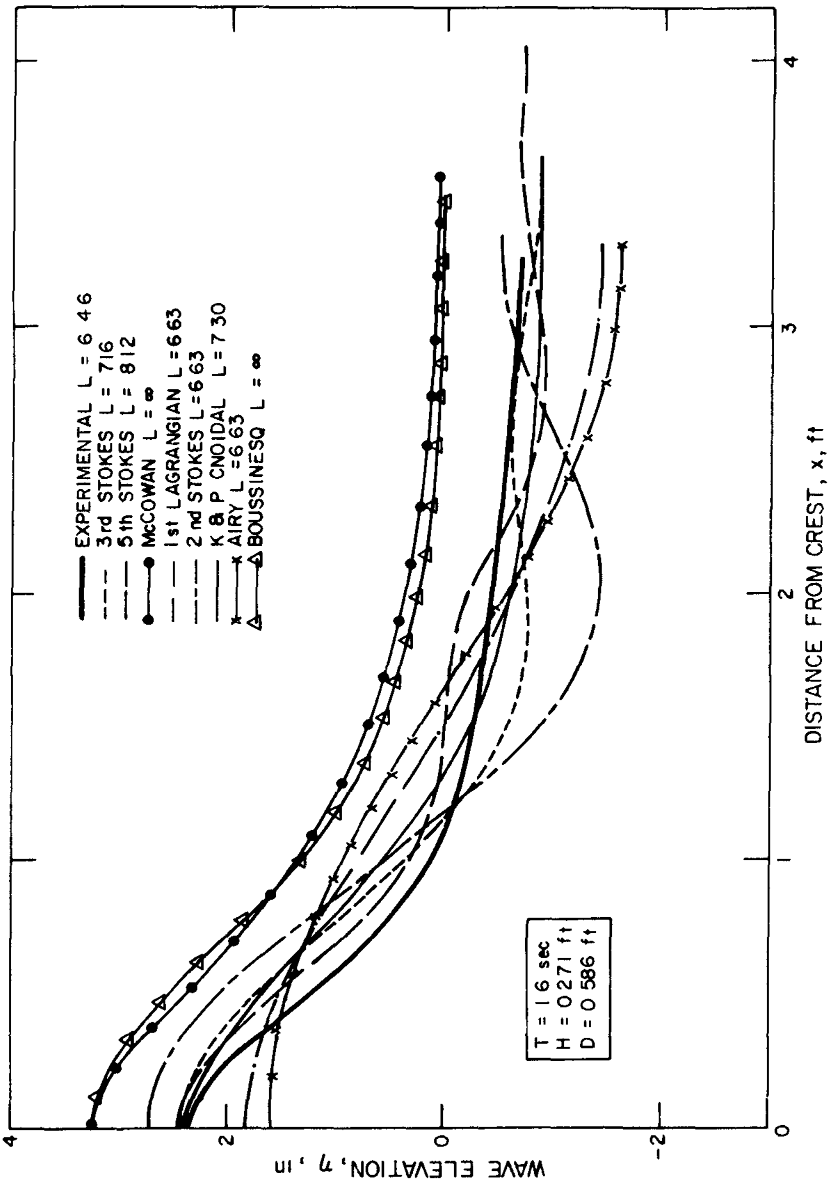

Figure 10. Free Surface Elevation.

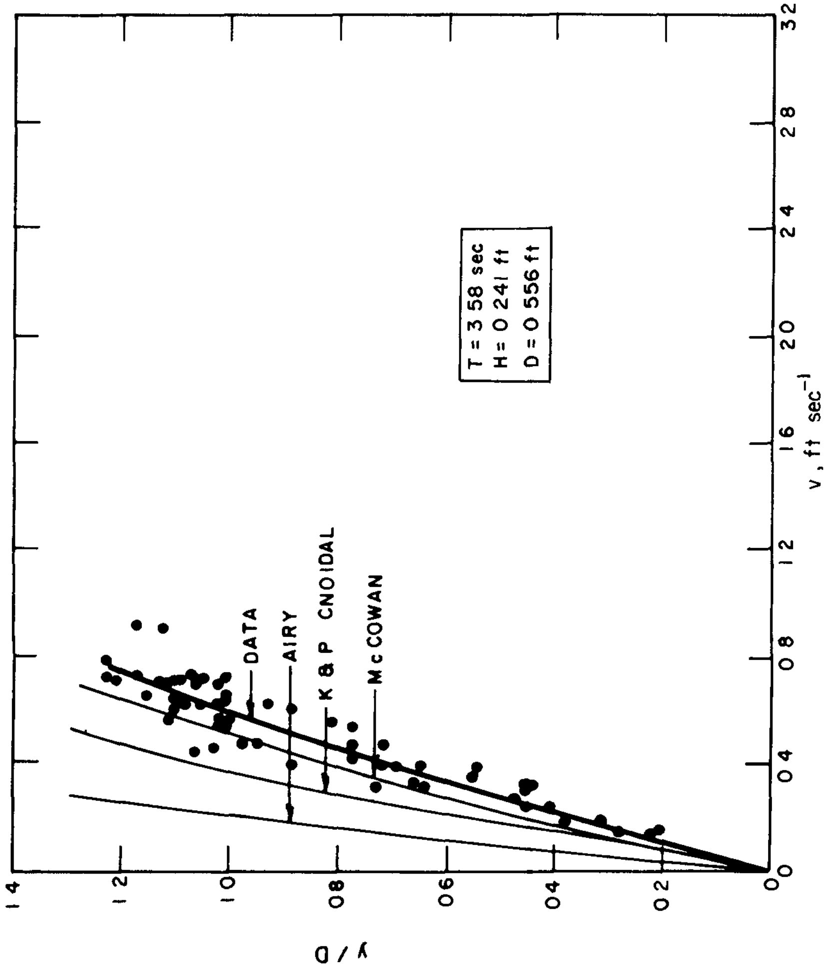

Figure 11 Vertical Particle Velocity (see text).