Shannon meets Tesla: Wireless information and power transfer

Pulkit Grover and Anant Sahai
Wireless Foundations, Department of EECS
University of California at Berkeley, CA-94720, USA
{pulkit, sahai} @eecs.berkeley.edu

Abstract— **The problem considered here is that of wireless information and power transfer across a noisy coupled-inductor circuit, which is a frequency-selective channel with additive white Gaussian noise. The optimal tradeoff between the achievable rate and the power transferred is characterized given the total power available. The practical utility of such systems is also discussed.**

I. INTRODUCTION

Ignoring the practical motivations for a moment, consider a hypothetical meeting of Claude Shannon and Nikola Tesla. They are both looking at the circuit in Fig. 1. While Tesla designed it to deliver power wirelessly to the load, Shannon wants to use it to send information.

Tesla observes the relation between the total available power P^{avail} and the delivered power P^{del} across frequency. Noticing that the the efficiency function, $\eta(f) = \frac{P^{del}(f)}{P^{avail}(f)}$ has a peak (see Fig. 2) at frequency of f_{peak} (≈ 5 MHz for the choice of circuit parameters in Fig. 2), Tesla wants to use just the one sinusoid at the max-efficiency fequency as the input (see Appendix I for detailed calculations).
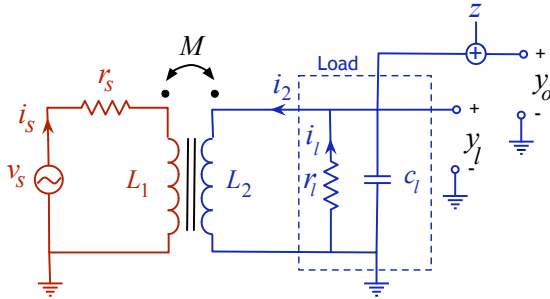

Fig. 1. A wireless power-transfer circuit. The coupled inductors can transmit power at short distances. M denotes the mutual inductance that decreases with distance.

This would not work for Shannon, since a sinusoid of fixed frequency has zero-bandwidth, and therefore zero communication rate. Shannon looks at the noisy output $y_o(t) = y_l(t) + z(t)$ (where $z(t)$ is additive white Gaussian noise of intensity N_0) and views the circuit as a frequency-selective channel[1], with the fading parameter $h(f)$ satisfying

[1]It is not clear to us if laws of thermodynamics allow us to observe *and* extract power simultaneously from the same signal. For this paper, we ignore this subtle issue and assume that there is no loss in power on observing the signal.
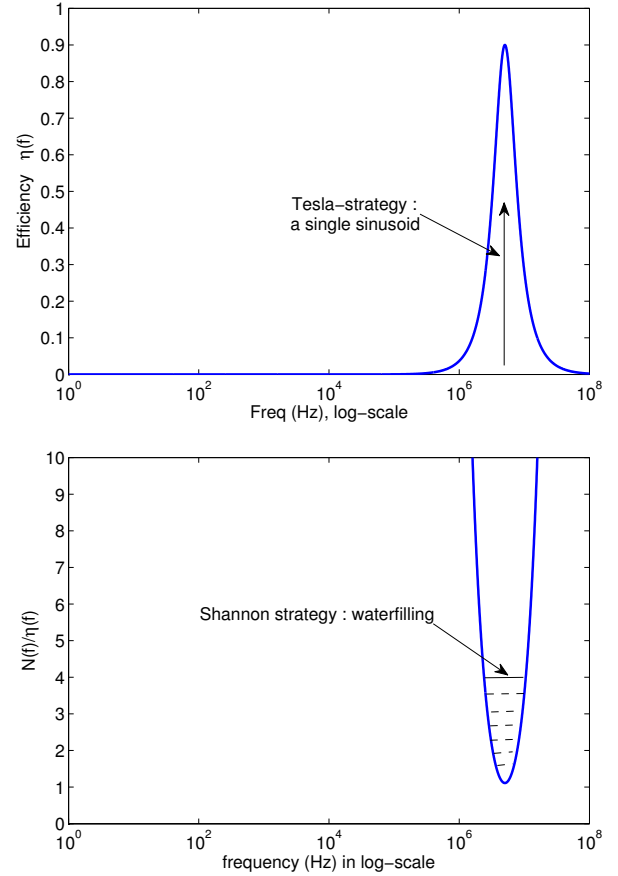

Fig. 2. The top plot shows the behavior of $\eta(f)$ with f. A peak is observed at the max-efficiency frequency $f_{peak} \approx 5$ MHz for $L_1 = L_2 = 0.1$mH, $M = 0.03$mH, $r_s = 100\Omega$, $r_l = 10$kΩ, $c_l = 10$pF. The Tesla-strategy (for maximum power efficiency) is to send a sinusoid at the max-efficiency frequency. The Shannon-strategy (for maximum communication rate), illustrated in bottom plot, is a waterfilling allocation, followed by random-Gaussian-coding. $N(f) = 1$ for all f.

$|h(f)|^2 = \eta(f)$. Therefore, Shannon wants to use the "waterfilling" allocation [1, Ch. 8] of Fig. 2.

This paper could simply be a fanciful investigation about how Shannon and Tesla would have arrived at a compromise. However, it has a much more practical interpretation as well. The coupled-inductor circuit shown in Fig. 1 is also the most common implementation of wireless power-transfer used, for example in many medical implants (e.g. the cochlear implant) [2, Ch. 16], futuristic wireless memories, electronic

toothbrushes [3], and even Tesla's famous Wardenclyff tower. Most such applications require, or can be enhanced, by simultaneous data-transfer as well. An understanding of the tradeoff between information and power transfer may therefore find utility in all these applications.

There is reason why many current implementations do not use the same link for information and power transfer. Sarpeshkar notes in his recent book [2], "... power efficiency is maximized for narrowband links that operate at low frequencies," whereas "data signals ... require larger link bandwidths, which are more easily obtained at higher operating frequencies." A separate coil for data transfer may therefore appear to be a good strategy. However, provisioning for an additional communication channel may be unnecessary[2], or even unwise because the infrastructure for this extra channel occupies chip-area and also consumes power. Further, recent progress in wireless power transfer for medical implants by Poon *et al* [6] shows that the optimal transmit frequency for human body is several hundreds of megahertz, increasing the available bandwidth significantly (though the motivation in [6] is reduction in the antenna size).

Even without taking the results of [6] into account, consider the current implementation of cochlear implants: even at small distances of a few millimeters (the coils are separated merely by the skin), the available (3-dB) bandwidth is on the order of a few MHz. The required data rate is about 1 Mbps, or smaller. Our example circuit (though for much lower frequencies and bandwidth, see Fig. 3) suggests that with less than 2% loss in power efficiency, communication rates of about half the waterfilling capacity can be attained in some cases. The power link may well suffice for transferring information as well.

But cochlear implants require very small data rates as compared to wireless memories. At the same time (as noted in [2, Fig. 16.7]), the bandwidth of a coupled-inductor circuit increases rapidly with the gap between the two coils. Therefore, while the required data rates will be much larger for wireless memories, so will be the available bandwidth. We believe that a deeper investigation is required to conclude if a separate data transfer link is required in all coupled-inductor-based implementations.

The organization of this paper is as follows. In Section III, an optimal tradeoff between the rate of information transfer R and the received power P^{del} is provided for a discrete version of the problem. A heuristic argument carries over the results to the continuous case in Section IV.

In the information-theoretic literature, the problem of simultaneous information and power transmission was first considered by Varshney [7], where using a general "capacity-energy function," tradeoffs between capacity and power delivered were characterized for some discrete channels, and an AWGN channel with an amplitude constraint on the input. Without fading in the average power-constrained AWGN case, the two goals of maximum rate and maximum efficiency of power transfer are aligned, and there is no non-

trivial tradeoff. The coupled-inductor circuit problem posed here is a special case of an AWGN channel with frequency-selective fading. In that respect, the contribution of this paper is to show that an AWGN channel with frequency-selective fading has nontrivial tradeoffs between the information and power transfer.

A related problem was considered by Gastpar [8], where received power constraints are imposed on devices so that the interference they cause is limited. While in Gastpar's case there is a limit on how *large* the received power can be (i.e. the power is constrained from above), here there is a *required* received power that is constrained from below.

While this paper is the first to consider the problem of information and power transfer on a coupled-inductor circuit, the techniques developed here can be applied more generally to any system with wireless power transfer over a slow frequency-selective fading channel.

II. PROBLEM STATEMENT

The power at the transmitter side (the left hand side of the coupled inductors in Fig. 1) is consumed at the source resistance r_s. At the receiver side (the right hand side in Fig. 1), it is consumed at the load resistance r_l. Power consumed by a resistance r is given by $I^2 r$ where I is the root mean square (rms) amplitude of the current [9, Ch. 10].

We first describe a discrete approximation to the continuous frequency problem. In the following, vectors are denoted in bold font, with a superscript to denote their size. For example, $\mathbf{X}^n$ is a vector of length n.

A. Discrete frequency bands

The observed output $\mathbf{Y}_o^n$ for the discrete problem is

$$Y_{o,i} = h_i \sqrt{r_l} I_{s,i} + Z_i \text{ for } i = \{1, 2, \ldots, n\}, \quad (1)$$

where $I_{s,i}$ is the (root mean square) amplitude of the input current in the i-th band, h_i is the transfer function of the current in the i-th band, so that the load current through resistance r_l is given by $I_{l,i} = h_i I_{s,i}$. The voltage $Y_{l,i}$ across the load is observed after addition of a white Gaussian noise $Z_i \sim \mathcal{N}(0, N)$ is iid across i and independent of the input $I_{s,i}$. The circuit parameters, and hence the h_i's, are known at the transmitter and the receiver.

The average power consumed at the input side in the i-th band is given by $P_{s,i} = \mathbb{E}\left[|I_{s,i}|^2\right] r_s$, where the expectation is over the messages (and the common randomness between the encoder and the decoder, if any). Similarly, the power consumed at the load resistance in i-th band is given by $P_{l,i} = \mathbb{E}\left[|I_{l,i}|^2 r_l\right] = \mathbb{E}\left[|I_{s,i}|^2\right] |h_i|^2 r_l$. Define

$$P_i := \mathbb{E}\left[|I_{s,i}|^2\right] (r_s + |h_i|^2 r_l) \quad (2)$$

as the power consumed at the transmitting end in the i-th band (note that power delivered to the load is also provided by the transmitter). The total power is bounded by P^{avail}, that is,

$$\sum_{i=1}^n P_i \leq P^{avail}. \quad (3)$$

---

[2]In distributed control systems, the same observation is made in [4], [5].

At the same time, the total power delivered is required to be P^{del}, i.e. $\sum_{i=1}^{n} \mathbb{E}\left[|I_{s,i}|^2\right]|h_i|^2 r_l \geq P^{del}$. Thus, the constraint on the receiver side is

$$\sum_{i=1}^{n} \frac{|h_i|^2 r_l P_i}{r_s + |h_i|^2 r_l} \geq P^{del}. \tag{4}$$

Define the *efficiency* of the power transfer in band i as

$$\eta_i = \frac{|h_i|^2 r_l}{r_s + |h_i|^2 r_l}. \tag{5}$$

The objective is to convey a message $M \in \{1, 2, \ldots, 2^{nR}\}$ to the receiver with error probability $P_e = \Pr(\widehat{M} \neq M) < \epsilon$. In the usual capacity formulation, we want to understand what is the maximum rate R for reliable communication ($\epsilon \to 0$) under the power constraints (3) and (4).

B. Continuous frequency bands

The continuous frequency band problem is defined analogously. The total power in frequency f is given by $P(f) = \mathbb{E}\left[|I_s(f)|^2\right](r_s + |H(f)|^2 r_l)$, where $I_s(\cdot)$ is power spectral density of the source current $i_s(\cdot)$. The constraints are given by

$$\int_f P(f)df \leq P^{avail}, \tag{6}$$

and

$$\int_f \eta(f)P(f)df \geq P^{del}. \tag{7}$$

The efficiency function $\eta(f)$ is defined as follows

$$\eta(f) = \frac{|H(f)|^2 r_l}{r_s + |H(f)|^2 r_l}. \tag{8}$$

For rigorous definition of capacity of a continuous-time channel such as this, we refer the reader to [1, Ch. 8].

III. SOLUTION TO THE DISCRETE VERSION

The maximum rate across the channel without the constraint on the power delivered is given by the waterfilling solution [10, Pg. 252]. Denote the power delivered by the waterfilling solution by P_{water}^{del}. A system operating on the optimal $\left(P^{del}, C(P^{avail}, P^{del})\right)$-tradeoff curve for fixed P^{avail} would therefore deliver power that is lower bounded by P_{water}^{del}.

The following theorem provides the characterization of capacity $C(P^{avail}, P^{del})$ as a function of the power constraints P^{avail} and P^{del}.

Theorem 1: For the problem described in Section II-A, define λ_{water} as the solution to the following

$$\sum_{i=1}^{n} \left(\frac{\log_2(e)}{\lambda_{water}} - \frac{N}{\eta_i}\right)^+ = P^{avail}, \tag{9}$$

and let $P_{water}^{del} = \sum_{i=1}^{n} \left(\frac{\log_2(e)}{\lambda_{water}} - \frac{N}{\eta_i}\right)^+$. Then for $P^{del} \geq P_{water}^{del}$, the optimal power allocation that attains the maximum rate for $P \geq P_{water}^{del}$ is given by $P_i^* :=$

$\left(\frac{\log_2(e)}{\lambda^* - \eta_i \mu^*} - \frac{N}{\eta_i}\right)^+$, where λ^* and μ^* satisfy the following two conditions

$$\sum_{i=1}^{n} \left(\frac{\log_2(e)}{\lambda^* - \eta_i \mu^*} - \frac{N}{\eta_i}\right)^+ = P^{avail}, \tag{10}$$

and

$$\sum_{i=1}^{n} \eta_i \left(\frac{\log_2(e)}{\lambda^* - \eta_i \mu^*} - \frac{N}{\eta_i}\right)^+ = P^{del}. \tag{11}$$

For any $P^{del} \leq P_{water}^{del}$, the optimal power allocation is attained at $\mu = 0$ and $\lambda = \lambda_{water}$, and the delivered power is still P_{water}^{del}.

The optimal power allocation attains the following maximum rate $C(P^{avail}, P^{del}))$

$$C(P^{avail}, P^{del}) = \sum_{i=1}^{n} \log_2\left(1 + \frac{\eta_i P_i^*}{N}\right). \tag{12}$$

Further, μ^* lies in the interval $\left(0, \frac{\lambda}{\eta_{\max}} - \frac{\log_2(e)}{N + \eta_{\max} P^{avail}}\right)$ where $\eta_{\max} = \max_i \eta_i$.

Proof: See Appendix II. ∎

The parameter λ_{water} corresponds to the Lagrange multiplier value that solves the waterfilling problem, i.e. the problem of maximizing rate without any constraint of the power delivered.

IV. SOLUTION TO THE CONTINUOUS VERSION

When only the transmit power is constrained, the waterfilling solution extends naturally to the continuous case, as shown by Gallager [1, Ch. 8]. However, his rigorous derivation is long and tedious, and to provide insights he himself provides a heuristic argument to complement his rigorous proofs. Gallager starts by considering signals that are limited in time-duration by T. The problem is then discretized by decomposing the input and output waveforms into different orthogonal bases. In a manner that is reminiscent of discrete-time problems, he then lets $T \to \infty$ to obtain (heuristically) the capacity region. However [1, Pg. 387], the noises on the parallel channels are correlated. Even though asymptotic pairwise independence of noises is plausible in the limit of $T \to \infty$, existence of mutual independence is unclear. In [1, Sec. 8.4, 8.5], Gallager uses a different technique to obtain a rigorous proof. Here we present a heuristic derivation (similar to that of waterfilling problem in [11, Pg. 184-185]) which can be made rigorous using Gallager's techniques.

Theorem 2: For the problem described in Section II-B, define λ_{water} to be the solution to the following

$$\int_f \left(\frac{\log_2(e)}{\lambda_{water}} - \frac{N_0}{\eta(f)}\right)^+ df = P^{avail}, \tag{13}$$

and let $P_{water}^{del} = \int_f \left(\frac{\log_2(e)}{\lambda_{water}} - \frac{N}{\eta(f)}\right)^+ df$. Then for $P^{del} \geq P_{water}^{del}$, the optimal power allocation that attains the maximum rate for $P \geq P_{water}^{del}$ is given by $P^*(f) =$

$\left(\frac{\log_2(e)}{\lambda^* - \eta(f)\mu^*} - \frac{N}{\eta(f)} \right)^+$, where λ^* and μ^* satisfy the following two conditions

$$\int_f \left(\frac{\log_2(e)}{\lambda^* - \eta(f)\mu^*} - \frac{N}{\eta(f)} \right)^+ df = P^{avail}, \quad (14)$$

and

$$\int_f \eta(f) \left(\frac{\log_2(e)}{\lambda^* - \eta(f)\mu^*} - \frac{N}{\eta(f)} \right)^+ df = P^{del}. \quad (15)$$

For any $P^{del} \leq P^{del}_{water}$, the optimal power allocation is attained at $\mu = 0$ and $\lambda = \lambda_{water}$, and the delivered power is still P^{del}_{water}.

The optimal power allocation attains the following maximum rate $C(P^{avail}, P^{del}))$

$$C(P^{avail}, P^{del}) = \int_f \log_2 \left(1 + \frac{\eta(f)P^*(f)}{N_0} \right) df. \quad (16)$$

Proof: See Appendix III. ∎

The resulting tradeoff is plotted in Fig. 3 for a set of parameter values.

ACKNOWLEDGMENTS

APPENDIX I
CALCULATION OF THE POWER EFFICIENCY FUNCTION $\eta(f)$

For reproducibility of our results, we include an analysis of the efficiency function $\eta(f)$ for the circuit shown in Fig. 1.

Writing the equation on the right and side of the circuit.

$$L_2 \frac{di_2(t)}{dt} + M \frac{di_s(t)}{dt} + i_l(t)r_l = 0. \quad (17)$$

Taking a Fourier transform,

$$j\omega L_2 I_2(j\omega) + j\omega M I_s(j\omega) + I_l(j\omega)r_l = 0. \quad (18)$$

Further,

$$I_l(j\omega) = \frac{I_2(j\omega)\frac{1}{j\omega c_l}}{r_l + \frac{1}{j\omega c_l}} = \frac{I_2(j\omega)}{1 + j\omega r_l c_l}. \quad (19)$$

Algebraic manipulations of (18) and (19) yields

$$\frac{I_l(j\omega)}{I_s(j\omega)} = \frac{j\omega M}{j\omega L_2(1 + j\omega r_l c_l) + r_l}. \quad (20)$$

Power consumed at the load is $\int_\omega |I_l(j\omega)|^2 r_l d\omega$, and that consumed at the source resistance r_s is $\int_\omega |I_s(j\omega)|^2 r_s d\omega$. Thus,

$$\eta(\omega) = \frac{|I_l(j\omega)|^2 r_l}{|I_l(j\omega)|^2 r_l + |I_s(j\omega)|^2 r_s} \quad (21)$$

Finally, $\eta(f) = \eta\left(\frac{\omega}{2\pi}\right)$ yields the efficiency function. We note here that $\eta(f)$ *can* be larger than 0.5. Also, the condition for maximizing $\eta(f)$ is *not* the same as "impedance matching," which is optimal when the objective is to maximize P^{del} with no constraint on P^{avail}.
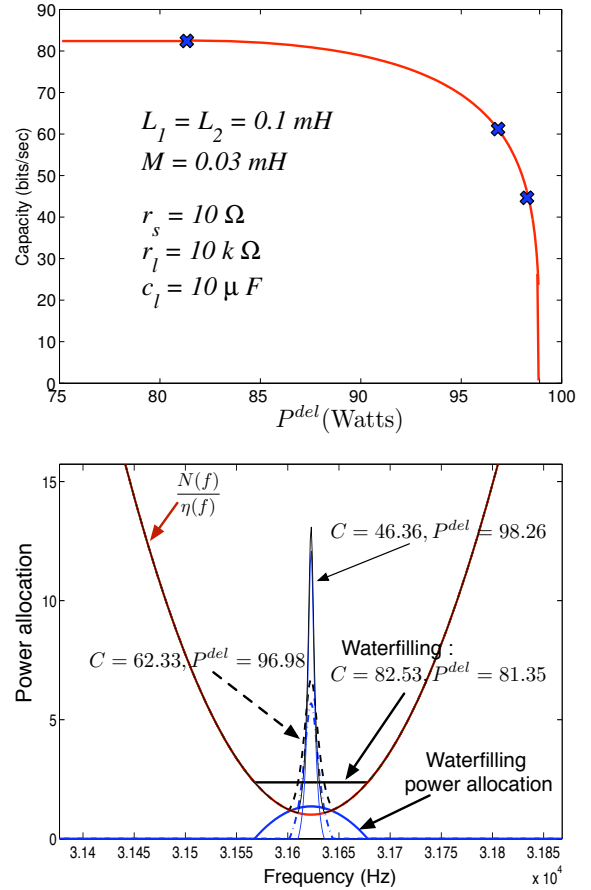

Fig. 3. Tradeoff between capacity and received power P^{del} for a set of parameter values for $P^{avail} = 100$ W. The maximum efficiency of 0.989 (corresponding to $P^{del} = 98.9$ W) is obtained at zero-capacity. The plot on the bottom shows the optimal power allocation at three points on the plot at the top. For each point, the lower curve is the power allocation, and the upper curve is the sum of power in the band and the ratio $\frac{N(f)}{\eta(f)}$. The power allocation becomes narrower, approaching the Tesla-strategy, as P^{del} increases to the maximum value of 98.9. We note that circuit parameters here are different from those in Fig. 2.

APPENDIX II
TRADEOFF BETWEEN INFORMATION RATE AND POWER IN THE DISCRETE CASE

For notational convenience, we use the notation $X_i := I_{s,i}r_l$. We first bound the mutual information across the channel.

$$\begin{aligned}
I(\mathbf{X}^n; \mathbf{Y}^n) &= h(\mathbf{Y}^n) - h(\mathbf{Y}^n|\mathbf{X}^n) \\
&= h(\mathbf{Y}^n) - h(\mathbf{Z}^n) \\
&\leq \sum_{i=1}^n h(Y_i) - h(Z_i) \\
&\leq \sum_{i=1}^n \log_2 \left(1 + \frac{|h_i|^2 r_l \mathbb{E}\left[|I_{s,i}|^2\right]}{N} \right).
\end{aligned}$$

Using (2),

$$I(\mathbf{X}^n; \mathbf{Y}^n) \leq \sum_{i=1}^{n} \log_2 \left(1 + \frac{|h_i|^2 r_l P_i}{(r_s + |h_i|^2 r_l) N} \right).$$

With the constraints (3) and (4), the capacity is upper bounded by

$$C(P^{avail}, P^{del})$$

$$\leq \max_{\substack{p(\mathbf{X}^n) \ s.t. \\ \sum_{i=1}^{n} P_i \leq P^{avail}, \\ \sum_{i=1}^{n} \frac{|h_i|^2 r_l P_i}{r_s + |h_i|^2 r_l} \geq P^{del}}} \sum_{i=1}^{n} \log_2 \left(1 + \frac{|h_i|^2 r_l P_i}{(r_s + |h_i|^2 r_l) N} \right)$$

$$= \max_{\substack{P_1, \ldots, P_n \ s.t. \\ \sum_{i=1}^{n} P_i \leq P^{avail}, \\ \sum_{i=1}^{n} \frac{|h_i|^2 r_l P_i}{r_s + |h_i|^2 r_l} \geq P^{del}}} \sum_{i=1}^{n} \log_2 \left(1 + \frac{|h_i|^2 r_l P_i}{(r_s + |h_i|^2 r_l) N} \right).$$

Observe that this rate is also achievable using independent random Gaussian codebooks in each sub-channel, and therefore the inequality above is actually an equality. Using $\eta_i = \frac{|h_i|^2 r_l}{r_s + |h_i|^2 r_l}$,

$$C(P^{avail}, P^{del}) = \max_{\substack{P_1, \ldots, P_n \ s.t. \\ \sum_{i=1}^{n} P_i \leq P^{avail}, \\ \sum_{i=1}^{n} \eta_i P_i \geq P^{del}}} \sum_{i=1}^{n} \log_2 \left(1 + \frac{\eta_i P_i}{N} \right). \quad (22)$$

The problem in (22) is one of convex optimization — the objective to be maximized is a concave-$\cap$ function, while the constraints are linear. Introducing non-negative Lagrange multipliers λ and μ,

$$\sum_{i=1}^{n} \log_2 \left(1 + \frac{\eta_i P_i}{N} \right) - \lambda \left(\sum_{i=1}^{n} P_i - P^{avail} \right)$$

$$+ \mu \left(\sum_{i=1}^{n} \eta_i P_i - P^{del} \right) = 0.$$

Differentiating w.r.t. P_i and equating to zero,

$$\log_2 (e) \times \left(\frac{1}{1 + \frac{\eta_i P_i}{N}} \right) \times \left(\frac{\eta_i}{N} \right) - \lambda + \mu \eta_i = 0,$$

which along with the constraint $P_i \geq 0$ (as in [10, Pg. 252]) results in,

$$P_i = \left(\frac{\log_2 (e)}{\lambda - \eta_i \mu} - \frac{N}{\eta_i} \right)^+. \quad (23)$$

The upper bound on μ follows from the fact that $P^{avail} > P_i$ for all i. The variables λ and μ can now be obtained using complementary-slackness conditions. Instead of stating the these conditions directly, we provide intuitive explanations for them.

For maximum rate, the first condition $\sum_{i=1}^{n} P_i \leq P^{avail}$ should be satisfied with with an equality, yielding one relationship between λ and μ.

For any $P^{del} \geq P^{del}_{water}$, the second inequality $\sum_{i=1}^{n} \eta_i P_i \geq P^{del}$ is also tight, and $\mu > 0$. Values of λ and μ can thus be calculated from these two equations for $P^{del} \geq P^{del}_{water}$. For $P^{del} < P^{del}_{water}$, the waterfilling solution is optimal and $\mu = 0$.

Appendix III
Tradeoff between information rate and power in the discrete case

Divide the frequency space into bands of uniform (and small) bandwidth df. The noise across the bands is iid distributed Gaussian with zero mean and variance $N_0 df$. The power allocated to the i-th band of center-frequency f is given by $P(f)df$. Rewriting (22),

$$\max_{\substack{\sum_f P(f)df \leq P^{avail}, \\ \sum_f \eta(f)P(f)df \geq P^{del}}} \sum_f df \log_2 \left(1 + \frac{\eta(f)P(f)df}{N_0 df} \right). \quad (24)$$

Following the derivation of (23),

$$P(f) = \left(\frac{\log_2 (e)}{\lambda - \eta(f)\mu} - \frac{N_0}{\eta(f)} \right)^+. \quad (25)$$

References

[1] Robert G. Gallager. *Information Theory and Reliable Communication*. John Wiley, New York, NY, 1971.

[2] Rahul Sarpeshkar. *Ultra Low Power Bioelectronics: Fundamentals, Biomedical Applications, and Bio-Inspired Systems*. Cambridge University Press, New York, NY, 2010.

[3] Tracy V Wilson. How wireless power works.

[4] Nuno C Martins. Witsenhausen's counter example holds in the presence of side information. *Proceedings of the 45th IEEE Conference on Decision and Control (CDC)*, pages 1111–1116, 2006.

[5] P Grover, SY Park, and A Sahai. The finite-dimensional Witsenhausen counterexample. *IEEE Transactions on Automatic Control, Submitted*, 2010.

[6] ASY Poon, S O'Driscoll, and TH Meng. Optimal frequency for wireless power transmission into dispersive tissue. *IEEE Trans. Antennas and Propagation*, to appear.

[7] Lav R. Varshney. Transporting information and energy simultaneously. *Proceedings of the 2008 IEEE International Symposium on Information Theory (ISIT)*, pages 1612–1616, July 2008.

[8] Michael Gastpar. On capacity under receive and spatial spectrum-sharing constraints. *IEEE Trans. Inform. Theory*, 53(2):471–487, February 2007.

[9] James William Nilsson and Susan A Riedel. *Electric circuits*. Prentice Hall, 8 edition, 2008.

[10] TM Cover and JA Thomas. *Elements of Information Theory*. Wiley, New York, 1st edition, 1991.

[11] David Tse and Pramod Viswanath. *Fundamentals of Wireless Communication*. Cambridge University Press, New York, 2005.