Commun. Math. Phys. 209, 437 – 476 (2000)

Communications in
Mathematical Physics

Shape Fluctuations and Random Matrices

Kurt Johansson

Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden.
E-mail: kurtj@math.kth.se

Received: 12 March 1999 / Accepted: 19 August 1999

Abstract: We study a certain random growth model in two dimensions closely related to the one-dimensional totally asymmetric exclusion process. The results show that the shape fluctuations, appropriately scaled, converges in distribution to the Tracy–Widom largest eigenvalue distribution for the Gaussian Unitary Ensemble (GUE).

1. Introduction and Results

The shape and height fluctuations in many 2-d random growth models are expected to be of order N^χ, with $\chi = 1/3$, if the mean of the linear size of the shape or the height is of order N. See [KS] for a review and [NP] for rigorous bounds on χ in first-passage percolation.

In this paper we will consider a specific model. It can be given several probabilistic interpretations, as a randomly growing Young diagram, a totally asymmetric one dimensional exclusion process, a certain zero-temperature directed polymer in a random environment or as a kind of first-passage site percolation model. The model has the advantage that we can prove that $\chi = 1/3$ and also compute the asymptotic distribution of the appropriately rescaled random variable. Interestingly, the limit distribution that occurs is the same as that of the scaled largest eigenvalue of an $N \times N$ random matrix from the Gaussian Unitary Ensemble (GUE) in the limit $N \to \infty$. The model in this paper has many similarities with the problem of the distribution of the length of the longest increasing subsequence in a random permutation where the same limiting distribution and $\chi = 1/3$ was found in [BDJ].

To define the model let $w(i, j)$, $(i, j) \in \mathbb{Z}_+^2$, be independent geometrically distributed random variables,

$$\mathbb{P}[w(i, j) = k] = (1 - q)q^k, \quad k \in \mathbb{N},$$

where $0 < q < 1$. Let $\Pi_{M,N}$ be the set of all up/right paths π in $\mathbb{Z}_+^2$ from $(1, 1)$ to (M, N), i.e. sequences (i_k, j_k), $k = 1, \dots, M + N - 1$, of sites in $\mathbb{Z}_+^2$ such that

$(i_1, j_1) = (1, 1)$, $(i_{M+N-1}, j_{M+N-1}) = (M, N)$ and $(i_{k+1}, j_{k+1}) - (i_k, j_k) = (1, 0)$ or $(0, 1)$. Define the random variable

$$G(M, N) = \max_{\pi \in \Pi_{M,N}} \sum_{(i,j)\in\pi} w(i, j). \tag{1.1}$$

We also define the closely related random variable

$$G^*(M, N) = \max_{\pi \in \Pi_{M,N}} \sum_{(i,j)\in\pi} w^*(i, j),$$

where $w^*(i, j) = w(i, j) + 1$, so that $\mathbb{P}[w^*(i, j) = k] = (1 - q)q^{k-1}$, $k \geq 1$. Clearly,

$$G^*(M, N) = G(M, N) + M + N - 1, \tag{1.2}$$

since all paths have the same length. Using this random variable we can define, for each $t \geq 0$, a random subset of the first quadrant by

$$A(t) = \{(M, N) \in \mathbb{Z}_+^2 ;\ G^*(M, N) \leq t\} + [-1, 0]^2. \tag{1.3}$$

From the definition of $G^*(M, N)$ and the fact that we consider up/right paths it follows that $A(t)$ has the form

$$\cup_{k=1}^{r} [k - 1, k] \times [0, \lambda_k]$$

for some integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$, so we can think of $A(t)$ as a Young diagram $\lambda = (\lambda_1, \ldots, \lambda_r)$. If we think of $t \in \mathbb{N}$ as a discrete time variable, $A(t)$ is a randomly growing Young diagram. Let $\partial^* A(t)$ be those unit cubes adjacent to $A(t)$ that can be added to $A(t)$ so that it is still a Young diagram, i.e. each cube in $\partial^* A(t)$ must have a cube in $A(t)$ or $\mathbb{R}^2 \setminus [0, \infty)^2$ immediately below and to the left of it. The fact that the $w^*(i, j)$'s are independent and geometrically distributed random variables implies that $A(t + 1)$ is obtained by picking each cube in $\partial^* A(t)$ independently with probability $p = 1 - q$ and adding those cubes that were picked to $A(t)$. (Recall that $\mathbb{P}[w^*(i, j) = k + l | w^*(i, j) \geq k] = \mathbb{P}[w(i, j) = l]$, $l \geq 0$, the lack of memory property.) The starting configuration is $A(0) = \emptyset$ and $\partial^* A(0) = [0, 1]^2$. In this model $G^*(M, N) = k$ means that the box $[M - 1, M] \times [N - 1, N]$ is added at time k. This growth model has been considered in [JPS].

This randomly growing Young diagram can also, equivalently, be thought of as a certain totally asymmetric exclusion process with discrete time, compare [Ro] or [Li, p. 412]. Let $C(t) = \partial([0, \infty)^2 \setminus A(t))$ and note that $C(t)$ consists of vertical and horizontal line segments of length 1. To each vertical line segment we associate a 1 and to each horizontal line segment a 0. If we read the numbers along $C(t)$, starting at infinity along the y-axis and ending at infinity along the x-axis, we get an infinite sequence $X(t) = (\ldots, x_{-1}(t), x_0(t), x_1(0), x_2(0), \ldots)$ of 0's and 1's, starting with infinitely many 1's and ending with infinitely many 0's; we let x_0 be the last number we have before passing through the line $x = y$. We can think of $X(t)$ as a configuration of particles, where $x_k = 1$ means that there is a particle at k, whereas $x_k = 0$ means that there is no particle at k. The stochastic growth of $A(t)$ described above corresponds to the following stochastic dynamics of the particle system. At time t each particle independently moves to the right with probability $1 - q$ provided there is no particle immediately to the right of it. Otherwise it does not move. The starting configuration is $x_k(0) = 1_{(-\infty,0]}(k)$. In this particle model $G^*(M, N) = k$ means that the particle initially at position $-(N - 1)$ has moved M steps at time k.

Our first result concerns the mean and large deviation properties of $G(M, N)$.

Theorem 1.1. *For each* $q \in (0,1)$ *and* $\gamma \geq 1$,

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}[G([\gamma N], N)] = \frac{(1+\sqrt{q\gamma})^2}{1-q} - 1 \doteq \omega(\gamma, q). \tag{1.4}$$

Also, $G([\gamma N], N)$ *has the following large deviation properties. There are functions* $i(\epsilon)$ *and* $\ell(\epsilon)$ *(which depend on* q *and* γ*), so that, for any* $\epsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}[G([\gamma N], N) \leq N(\omega(\gamma, q) - \epsilon)] = -\ell(\epsilon) \tag{1.5}$$

and

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}[G([\gamma N], N) \geq N(\omega(\gamma, q) + \epsilon)] = -i(\epsilon). \tag{1.6}$$

The functions $\ell(x)$ *and* $i(x)$ *are* > 0 *if* $x > 0$.

Note that the existence of the limit (1.4) follows by a subadditivity argument, so it is the explicit form of the constant that is interesting. The large deviation result (1.6) has been obtained in [Se2]. The theorem will be proved in Sect. 2.

The theorem implies that $\frac{1}{t}A(t)$ has an asymptotic shape A_0 as $t \to \infty$, in the sense that given any $\epsilon > 0$,

$$(1-\epsilon)A_0 \subseteq \frac{1}{t}A(t) \subseteq (1+\epsilon)A_0$$

for all sufficiently large t. It follows from the definition of $A(t)$, (1.3), and Theorem 1.1 that

$$A_0 = \{(x, y) \in [0, \infty)^2 ;\ y + 2\sqrt{qxy} + x \leq 1 - q\}.$$

The boundary of A_0 consists of two line segments from the origin to $(1-q, 0)$ and $(0, 1-q)$ and part of an ellipse that is tangent to the x- and y-axes.

We now want to understand the fluctuations of $A(t)$ around its asymptotic shape A_0, i.e. the fluctuations of $G([\gamma N], N)$ around $N\omega(\gamma, q)$. Before we can formulate the result we need some preliminaries. Let $\mathrm{Ai}\,(x)$ be the Airy function defined by

$$\mathrm{Ai}\,(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(t+is)^3/3 + ix(t+is)} dt,$$

where $s > 0$ is arbitrary. Consider the *Airy kernel*

$$A(x, y) = \frac{\mathrm{Ai}\,(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}\,(y)}{x - y}, \tag{1.7}$$

as an integral kernel on $L^2[s, \infty)$. The Fredholm determinant

$$F(s) = \det(I - A)\ |_{L^2[s,\infty)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[s,\infty)^k} \det(A(x_i, x_j))_{i,j=1}^k d^k x \tag{1.8}$$

is a distribution function. It is the distribution function of the appropriately scaled largest eigenvalue of an $N \times N$ random matrix from the Gaussian Unitary Ensemble (GUE) in the limit $N \to \infty$, the Tracy–Widom distribution, see [TW1]. The distribution function $F(s)$ can also be defined using a certain Painlevé II function,

$$F(s) = \exp[-\int_s^{\infty} (x - s)u(x)^2 dx], \tag{1.9}$$

where $u(x)$ is the unique solution of the Painlevé II equation

$$u'' = 2u^3 + xu,$$

with the asymptotics $u(x) \sim \mathrm{Ai}\,(x)$ as $x \to \infty$. The fact that the expressions (1.8) and (1.9) are equal is proved in [TW1].

Theorem 1.2. *For each $q \in (0,1)$, $\gamma \geq 1$ and $s \in \mathbb{R}$,*

$$\lim_{N\to\infty} \mathbb{P}[\frac{G([\gamma N], N) - N\omega(\gamma, q)}{\sigma(\gamma, q)N^{1/3}} \leq s] = F(s), \tag{1.10}$$

where

$$\sigma(\gamma, q) = \frac{q^{1/6}\gamma^{-1/6}}{1-q}(\sqrt{\gamma} + \sqrt{q})^{2/3}(1+\sqrt{q\gamma})^{2/3}. \tag{1.11}$$

The theorem will be proved in Sect. 3. We have not proved convergence of the moments of the rescaled random variable, see Remark 2.5. This theorem should be compared with the result obtained in [BDJ], that if $\ell_N(\sigma)$ is the length of a longest increasing subsequence in a random permutation $\sigma \in S_N$ (all $N!$ permutations have the same probability), then

$$\lim_{N\to\infty} \mathbb{P}[(\sqrt{N})^{-1/3}(\ell_N(\sigma) - 2\sqrt{N}) \leq s] = F(s). \tag{1.12}$$

Note that in both cases we have the same exponent $1/3$, the standard deviation is $\sim$ (mean)$^{1/3}$

The proofs of Theorems 1.1 and 1.2 are based on the following result which will be proved in Sect. 2.

Proposition 1.3. *For any $M \geq N \geq 1$,*

$$\mathbb{P}[G(M,N) \leq t] = \frac{1}{\mathcal{Z}_{M,N}} \sum_{\substack{h\in\mathbb{N}^N \\ \max\{h_i\}\leq t+N-1}} \prod_{1\leq i<j\leq N} (h_i - h_j)^2 \cdot \prod_{i=1}^{N} \binom{h_i + M - N}{h_i} q^{h_i}, \tag{1.13}$$

where $\mathcal{Z}_{M,N}$ is the normalization constant (partition function).

This remarkable formula should be compared with the formula for the distribution function for the largest eigenvalue, $\lambda_{\max}$, of an $N \times N$ random matrix from GUE,

$$\mathbb{P}[\lambda_{\max} \leq t] = \frac{1}{Z_N} \int_{(-\infty,t]^N} \prod_{1\leq i<j\leq N} (x_i - x_j)^2 \prod_{j=1}^{N} e^{-2Nx_j^2} d^N x. \tag{1.14}$$

There is a clear similarity between the two expressions, so we can use the ideas developed to investigate (1.14). Just as the right-hand side of (1.14) can be written as a Fredholm determinant, so can the right-hand side of (1.13). The kernel for (1.13) is the *Meixner kernel.*

$$K_{M,N}(x,y) = \frac{\kappa_{N-1}}{\kappa_N}\frac{M_N(x)M_{N-1}(y)-M_{N-1}(x)M_N(y)}{x-y}(w_K^q(x)w_K^q(y))^{1/2}, \tag{1.15}$$

where $M_N(x)=\kappa_N x^N+\dots$ are the normalized orthogonal polynomials with respect to the discrete weight, $K=M-N+1$,

$$w_K^q(x)=\binom{x+K-1}{x}q^x, \quad x\in\mathbb{N}. \tag{1.16}$$

This Meixner kernel also appears in the recent paper [BO]. The polynomial $M_N(x)$ is a multiple of the classical Meixner polynomials $m_N^{K,q}(x)$. Using the explicit generating function for the Meixner polynomials, see [Ch], the appropriate asymptotics of the kernel (1.15) can be analyzed. This will be done in Sect. 5.

Let $u(i,j)$, $(i,j)\in\mathbb{Z}_+^2$, be independent exponentially distributed random variables with parameter 1. Let $H(M,N)$ be the analogue of $G(M,N)$ for these random variables, i.e.

$$H(M,N)=\max\{\sum_{(i,j)\in\pi}u(i,j)\,;\,\pi\in\Pi_{M,N}\}. \tag{1.17}$$

We can consider the related stochastically growing Young diagram and totally asymmetric exclusion process just as in the geometric case, where we now have continuous time. This simple exclusion process is exactly the one considered by Rost, [Ro], see also [Li]. In this process $X(t)=(\eta_k(t))_{k=-\infty}^{\infty}\in\{0,1\}^{\mathbb{Z}}$ the initial configuration is $1_{(-\infty,0]}(k)$ and a particle ($\eta_k=1$) jumps with exponential rate to the right one step provided there is no particle at $k+1$ ($\eta_{k+1}=0$). By taking the $q\to 1$ limit in (1.13) we obtain

Proposition 1.4. *For any $M\geq N\geq 1$, $t\geq 0$,*

$$\mathbb{P}[H(M,N)\leq t]=\frac{1}{Z'_{M,N}}\int_{[0,t]^N}\prod_{1\leq i<j\leq N}(x_i-x_j)^2\prod_{j=1}^N x_j^{M-N}e^{-x_j}d^Nx. \tag{1.18}$$

Proof. If X_L is geometrically distributed with parameter $1-1/L$, then $L^{-1}X_L$ converges in distribution to an exponential random variable with parameter 1. Since $G(M,N)$ is a continuous function of the $w(i,j)$'s, Proposition 1.3 gives

$$\begin{aligned}
&\mathbb{P}[H(M,N)\leq t]\\
&=\lim_{L\to\infty}\frac{1}{\mathcal{Z}_{M,N}}\sum_{(*)}\prod_{1\leq i<j\leq N}(h_i-h_j)^2\prod_{i=1}^N\binom{h_i+M-N}{h_i}(1-1/L)^{h_i}\\
&=\lim_{L\to\infty}\frac{L^{N^2}}{\mathcal{Z}_{M,N}(M-N)!}\sum_{(*)}\prod_{1\leq i<j\leq N}(\frac{h_i-h_j}{L})^2\prod_{i=1}^N e^{-\frac{h_i}{L}+o(\frac{1}{L})}\prod_{k=1}^{M-N}(\frac{h_i+k}{L})\\
&=\frac{1}{Z'_{M,N}}\int_{[0,t]^N}\prod_{1\leq i<j\leq N}(x_i-x_j)^2\prod_{j=1}^N x_j^{M-N}e^{-x_j}d^Nx,
\end{aligned}$$

where $(*)$ means summation over all $h\in\mathbb{N}^N$ such that $\max\{h_i\}\leq[Lt]+N-1$. $\square$

Remark 1.5. The right-hand side in (1.18) is the probability that the largest eigenvalue in the Laguerre ensemble is $\le t$. It occurs in the following way. Let A be an $N \times M$ rectangular matrix ($N \le M$) with entries that are complex Gaussian random variables with mean zero and variance 1/2. Then the right-hand side in (1.18) is the distribution function for the largest eigenvalue of AA^*, see [Ja].

Theorem 1.6. *For each* $\gamma \ge 1$,

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}[H([\gamma N], N)] = (1+\sqrt{\gamma})^2, \tag{1.19}$$

and there are functions $i_*(\epsilon)$ *and* $\ell_*(\epsilon)$ *(which depend on* γ*), so that for any* $\epsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}[H([\gamma N], N) \le N((1+\sqrt{\gamma})^2 - \epsilon)] = -\ell_*(\epsilon) \tag{1.20}$$

and

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}[H([\gamma N], N) \ge N((1+\sqrt{\gamma})^2 + \epsilon)] = -i_*(\epsilon). \tag{1.21}$$

Furthermore, assume that $a_N = O(N^{1/3})$ *as* $N \to \infty$ *and pick* d_N *so that* $d_N - (1 + 1/\sqrt{\gamma})a_N = o(N^{1/3})$ *as* $N \to \infty$. *Then, for each* $\gamma \ge 1$,

$$\lim_{N\to\infty} \mathbb{P}[\frac{H(\gamma N + a_N, N) - (1+\sqrt{\gamma})^2 N - d_N}{\gamma^{-1/6}(1+\sqrt{\gamma})^{4/3}N^{1/3}} \le s] = F(s). \tag{1.22}$$

Proof. For the proof of (1.19) to (1.21) see Remark 2.3. Write $c = (1+\sqrt{\gamma})^2$ and $\rho = \gamma^{-1/6}(1+\sqrt{\gamma})^{4/3}$. Then, by Proposition 1.4,

$$\begin{aligned}&\mathbb{P}[H(\gamma N + a_N, N) \le cN + d_N + \rho N^{1/3} s] \\ &= \frac{1}{Z'_{\gamma N + a_N, N}} \int_{[0, cN+d_N+\rho N^{1/3}s]^N} \Delta(x)^2 \prod_{j=1}^{N} x_j^{\alpha_N} e^{-x_j} d^N x,\end{aligned}$$

where $\Delta(x) = \prod_{1\le i<j\le N}(x_j - x_i)$ and $\alpha_N = (\gamma - 1)N + a_N$. By a standard argument, see [Me, Ch. 5], [TW3] or Sect. 3, this equals the Fredholm determinant

$$\sum_{k=0}^{N} \frac{(-1)^k}{k!} \int_{[s,\infty)^k} \det(\rho N^{1/3} K_N^{\alpha_N}(cN + d_N + \rho N^{1/3}\xi_i, cN + d_N + \rho N^{1/3}\xi_j))_{i,j=1}^k d^k\xi, \tag{1.23}$$

where

$$K_N^{\alpha}(x, y) = \frac{\kappa_{N-1}}{\kappa_N} \frac{\ell_N^{\alpha}(x)\ell_{N-1}^{\alpha}(y) - \ell_N^{\alpha}(y)\ell_{N-1}^{\alpha}(x)}{x - y} \left(x^{\alpha}e^{-x}y^{\alpha}e^{-y}\right)^{1/2}$$

is the Laguerre kernel. Here,

$$\ell_n^{\alpha}(x) = \left(\frac{n!}{(\alpha+n)!}\right)^{1/2} (-1)^n L_n^{\alpha}(x) = \kappa_n x^n + \dots$$

are the normalized associated Laguerre polynomials,

$$\int_0^\infty \ell_n^\alpha(x)\ell_m^\alpha(x)x^\alpha e^{-x}dx = \delta_{nm}.$$

From asymptotic formulas for these polynomials it follows that

$$\lim_{N\to\infty} K_N^{\alpha_N}(cN + d_N + \rho N^{1/3}\xi, cN + d_N + \rho N^{1/3}\eta) = A(\xi, \eta). \tag{1.24}$$

This can be proved in the same way as the corresponding results for Meixner polynomials, see Sects. 3 and 4, by using the integral representation

$$L_n^\alpha(x) = \frac{e^x}{2\pi i}\int_C \frac{e^{-xz}z^{n+\alpha}}{(z-1)^{n+1}}dz,$$

where C is a circle surrounding $z = 1$. Using (1.23), (1.24) and some estimates (compare Lemma 3.1) we obtain

$$\begin{aligned}&\lim_{N\to\infty} P[H(\gamma N + a_N, N) \le cN + d_N + \rho N^{1/3}s]\\ &= \sum_{k=0}^\infty \frac{(-1)^k}{k!}\int_{[s,\infty)^k} \det(A(\xi_i, \xi_j))_{i,j=1}^k d^k\xi = F(s).\end{aligned}$$

We will not present all the details since they are similar to the proof of Theorem 1.2. □

Using this result we can get a fluctuation theorem for Rost's totally asymmetric simple exclusion process defined above. The random variable $H(N, M)$ is the first time at which the particle starting at $-(N-1)$ has moved exactly M steps to the right. If we define $Y(k, t) = \sum_{j>k}\eta_j(t)$ to be the number of particles to the right of k at time t, then $Y(k, t) > m$ means that the particle that starts at $-m$ has moved $\ge m + k + 1$ steps at time t. Hence

$$\mathbb{P}[Y(k, t) \le m] = 1 - \mathbb{P}[H(m + k + 1, m + 1) \le t].$$

Using this relation and (1.19) to (1.21) we obtain the following result first proved by Rost, [Ro],

$$\frac{1}{t}Y([ut], t) \to \frac{1}{4}(1-u)^2$$

almost surely as $t \to \infty$, $|u| \le 1$. Now, using (1.22) it is fairly straightforward to show the following result.

Corollary 1.7. *For each* $u \in [0, 1)$,

$$\lim_{t\to\infty} \mathbb{P}[Y([ut], t) \le \frac{t}{4}(1-u)^2 + \frac{(1-u)^{2/3}}{(1+u)^{1/3}}\xi t^{1/3}] = 1 - F(-\xi).$$

Remark 1.8. We can interpret Theorems 1.1 and 1.2 (and analogously Theorem 1.6) as a result for a kind of zero-temperature directed polymer or equivalently a directed first-passage site percolation model in the following way.

Let S_k be the simple random walk in $\mathbb{Z}$ starting at 0 at time 0 and ending at 0 at time $2N + 2$. Denote the set of all possible paths by $\mathcal{P}_N$. Let $v(i, j)$, $(i, j) \in \mathbb{Z}^2$ be independent, identically distributed random variables, and let $\beta > 0$. On $\mathcal{P}_N$ we put the random path probability measure

$$Q_N^\beta[S] = \frac{1}{C_N^\beta} \exp(-\beta \sum_{k=1}^{2N} v(k, S_k)),$$

$S \in \mathcal{P}_N$, where C_N^β is the normalization constant. This measure describes a directed polymer (S) fixed at both endpoints at inverse temperature β in the random environment given by the $v(i, j)$'s, see [Pi]. The *free energy* is $-\beta^{-1} \log C_N^\beta$, and in the zero temperature limit $\beta \to \infty$ this becomes

$$F_N^{GS} = \min_{Z \in \mathcal{P}_N} \sum_{k=1}^{2N} v(k, S_k), \tag{1.25}$$

the ground state energy. By rotating the coordinate system by the angle $-\pi/4$ it is seen that (1.25) can, equivalently, be thought of as a first-passage time in a directed first passage site percolation model. Let $u(i, j)$, $(i, j) \in \mathbb{R}_+^2$, be independent, identically distributed random variables (with the same distribution as the $v(i, j)$'s). Then F_N^{GS} has the same distribution as $F(N, N)$, where

$$F(M, N) = \min_{\pi \in \Pi_{M,N}} \sum_{(i,j) \in \pi} u(i, j).$$

(The $u(i, j)$'s are usually thought of as passage times and $F(M, N)$ is the minimal flow time from (1, 1) to (M, N). Hence it is natural to assume that $u(i, j) \geq 0$, but this will not be the case below.) We can define a random shape

$$B(t) = \{(M, N) \in \mathbb{Z}_+^2; F(M, N) \leq t\} + [-1, 0]^2.$$

Set $u(i, j) = \alpha - w(i, j)$, where $\alpha > \alpha_{\min} = (1 - q)^{-1}(q + \sqrt{q})$ (this condition on α ensures that $B(t)$ will grow); $w(i, j)$ are the geometrically distributed random variables considered above. Then clearly,

$$F(M, N) = \alpha(M + N - 1) - G(M, N). \tag{1.26}$$

Let $\gamma \geq 1$, set $\hat{x}(\gamma) = (1 + \gamma^2)^{-1/2}(\gamma, 1)$, a unit vector and $[n\hat{x}(\gamma)] = ([N\gamma], N)$, ($[\cdot]$ the integer part), where $N = [(1 + \gamma^2)^{-1/2} n]$, so that $[n\hat{x}(\gamma)]$ is a lattice site near $n\hat{x}(\gamma)$. Let $T_n(\gamma)$ be the first time $s \geq 0$ for which $B(s)$ reaches $[n\hat{x}(\gamma)]$,

$$T_n(\gamma) = \inf\{s \geq 0; [n\hat{x}(\gamma)] \in B(s)\}.$$

Clearly, by the definition of $B(s)$ and Eq. (1.26),

$$T_n(\gamma) = \alpha([\gamma N] + N - 1) - G([\gamma N], N),$$

where $N = [(1 + \gamma^2)^{-1/2} n]$.

Theorem 1.1 implies that for each $q \in (0, 1)$ and $\gamma \geq 1$,

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[T_n(\gamma)] = \frac{1}{\sqrt{1 + \gamma^2}} [\alpha(\gamma + 1) - \frac{(1 + \sqrt{q\gamma})^2}{1 - q} + 1] \doteq \mu(\gamma).$$

Also, $T_n(\gamma)$ has large deviation properties similar to those for $G([\gamma N], N)$. Using this result we can compute the asymptotic shape of $B(t)$. It follows from Theorem 1.2 that

$$\mathbb{P}[\frac{T_n(\gamma) - n\mu(\gamma)}{(1+\gamma^2)^{-1/6}\rho(q,\gamma)n^{1/3}} \leq s] \to 1 - F(-s),$$

as $n \to \infty$.

Conjecture 1.9. Is the result for $G([\gamma N], N)$ limited to geometric and exponential random variables? Normally, we expect limit laws for appropriately scaled random variables to be independent of the details. It is therefore natural to conjecture that if the $w(i, j)$'s are i.i.d. random variables with some suitable asumptions on their distribution, then there are constants a and b so that $(G([\gamma N], N) - aN)/bN^{1/3}$ converges to a random variable with distribution $F(s)$. By Remark 1.8 this leads to a related conjecture for directed first-passage site percolation.

2. The Coulomb Gas

2.1. Combinatorics. The key combinatorial ingredient is the Knuth correspondence introduced in [Kn]. It generalizes the Schensted correspondence [Sc] which is used in [BDJ]. Write $[N] = \{1, \dots, N\}$. Let $\mathcal{M}_{M,N}$ denote the set of all $M \times N$ matrices $A = (a_{ij})$ with non-negative integer elements, and let $\mathcal{M}^k_{M,N}$ be the subset of those matrices that satisfy $\sum_{i=1}^M \sum_{j=1}^N a_{ij} = k$. A two-rowed array

$$\sigma = \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{pmatrix}$$

is called a generalized permutation if the columns $\binom{i_r}{j_r}$ are lexicographically ordered, i.e. either $i_r < i_{r+1}$ or $i_r = i_{r+1}$, $j_r \leq j_{r+1}$. There is a one-to-one correspondence between the set $\mathcal{S}^k_{M,N}$ of all generalized permutations of length k, where the elements in the upper row come from $[M]$ and the elements in the lower row from $[N]$, and $\mathcal{M}^k_{M,N}$ defined by $\sigma \to f(\sigma) = A = (a_{ij})$, where

$$a_{ij} = \#\text{times} \begin{pmatrix} i \\ j \end{pmatrix} \text{ occurs in } \sigma.$$

We say that $\binom{i_{r_1}}{j_{r_1}}, \dots, \binom{i_{r_m}}{j_{r_m}}$, $r_1 < r_2 < \dots < r_m$ is an increasing subsequence in σ if $j_1 \leq j_2 \leq \dots \leq j_{r_m}$. Let $\ell(\sigma)$ denote the length of a longest increasing subsequence in σ.

Example. The generalized permutatation

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 1 & 3 \end{pmatrix}$$

corresponds to

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

A longest increasing subsequence is 1 2 2 2 2 2 2 3 so $\ell(\sigma) = 8$.

Recall from Sect. 1 that $\Pi_{M,N}$ denotes the set of all up/right paths π from $(1, 1)$ to (M, N) through the sites (i, j) with $1 \le i \le M$, $1 \le j \le N$.

Lemma 2.1. *For each* $A \in \mathcal{M}^k_{M,N}$,

$$\max\{\sum_{(i,j)\in\pi} a_{ij}\ ;\ \pi \in \Pi_{M,N}\} = \ell(f^{-1}(A)). \tag{2.1}$$

Proof. This is clear from the definitions. That we go to the right corresponds to the fact that $i_{r_1} \le \cdots \le i_{r_m}$ and that we go up corresponds to $j_{r_1} \le \cdots \le j_{r_m}$ (the upper row gives row indices whereas the lower row gives column indices in the matrix). □

Now, Knuth has defined a one-to-one mapping from the set $\mathcal{S}^k_{M,N}$ to pairs (P, Q) of semi-standard Young tableaux of the same shape λ, which is a partition of k, $\lambda \vdash k$, where P has elements in $[N]$ and Q has elements in $[M]$. (More information on Young tableaux can be found in [Sa] and [Fu].) This correspondence has the property that if $\sigma \to (P, Q)$ and P, Q have shape λ, then $\ell(\sigma) =$ the length of the first row, λ_1, in λ. Consider $G(M, N)$ defined by (1.1). The $M \times N$ matrix $W = (w(i, j))$ is a random element in $\mathcal{M}_{M,N}$. Let

$$S(M, N) = \sum_{i=1}^{M}\sum_{j=1}^{N} w(i, j)$$

and

$$p_{M,N}(t) = \mathbb{P}[G(M, N) \le t].$$

Then,

$$p_{M,N}(t) = \sum_{k=0}^{\infty} \mathbb{P}[G(M, N) \le t | S(M, N) = k]\mathbb{P}[S(M, N) = k]. \tag{2.2}$$

For a fixed $A \in \mathcal{M}^k_{M,N}$ we have

$$\mathbb{P}[\{A\}] = \prod_{i,j}(1 - q)q^{a_{ij}} = (1 - q)^{MN} q^k,$$

since $\sum_{i,j} a_{ij} = k$. We have proved

Lemma 2.2. *The conditional probability* $\mathbb{P}[\cdot | S(M, N) = k]$ *is the uniform distribution on* $\mathcal{M}^k_{M,N}$.

This lemma is the reason that we choose the $w(i, j)$'s to be independent and geometrically distributed. Note that

$$\mathbb{P}[S(M, N) = k] = \#\mathcal{M}^k_{M,N}(1 - q)^{MN} q^k. \tag{2.3}$$

Let $L(\lambda, M, N)$ denote the number of pairs (P, Q) of semi-standard Young tableaux of shape λ, such that P has elements in $[N]$ and Q has elements in $[M]$. Combining Lemma 2.1, Lemma 2.2 and the Knuth correspondence we see that

$$\mathbb{P}[G(M, N) \le t | S(M, N) = k] = \frac{1}{\#\mathcal{M}^k_{M,N}} \sum_{\lambda \vdash k, \lambda_1 \le t} L(\lambda, M, N). \tag{2.4}$$

To compute $L(\lambda, M, N)$ we use

Lemma 2.3. *The number of semi-standard tableaux of shape* λ *and elements in* $[N]$ *equals*

$$\prod_{1\le i<j\le N} \frac{\lambda_i - \lambda_j + j - i}{j-i}.$$

Proof. We have two formulas for the Schur polynomial in N variables associated with the partition λ, [Sa, Fu],

$$s_\lambda(x) = \sum_T x^T = \frac{\det(x_j^{\lambda_i+N-i})_{1\le i,j\le N}}{\det(x_j^{N-i})_{1\le i,j\le N}},$$

where the sum is over all semi-standard λ-tableaux T with elements in $[N]$ and $x^T = x_1^{m_1}\dots x_N^{m_N}$ with m_j equal to the number of times j occurs in T. Hence, evaluating the Vandermonde determinants,

$$s_\lambda(1, x, \dots, x^{N-1}) = x^r \prod_{1\le i<j\le N} \frac{x^{\lambda_i-\lambda_j+j-i}-1}{x^{j-i}-1},$$

where $r = \sum_{i=1}^N (i-1)\lambda_i$. The number of semi-standard tableaux with elements in $[N]$ equals

$$s_\lambda(1,1,\dots,1) = \lim_{x\to 1} s_\lambda(1,x,\dots,x^{N-1}) = \prod_{1\le i<j\le N} \frac{\lambda_i-\lambda_j+j-i}{j-i}.$$

This completes the proof of the lemma. $\square$

It follows from Lemma 2.3 that

$$L(\lambda, M, N) = \prod_{1\le i<j\le M} \frac{\lambda_i-\lambda_j+j-i}{j-i} \prod_{1\le i<j\le N} \frac{\lambda_i-\lambda_j+j-i}{j-i}. \tag{2.5}$$

We assume from now on that $M \ge N$, the other case is analogous by symmetry. Since the numbers in the columns in P and Q are strictly increasing we must have $\lambda_i = 0$ if $N < i \le M$. Hence

$$L(\lambda, M, N) = \prod_{1\le i<j\le M} \left(\frac{\lambda_i-\lambda_j+j-i}{j-i}\right)^2 \prod_{i=1}^{N} \prod_{j=N+1}^{M} \left(\frac{\lambda_i+j-i}{j-i}\right).$$

Let $h_j = \lambda_j + N - j$, $j = 1, \dots, N$, so that $h_1 = \lambda_1 + N - 1$, $h_N = \lambda_N \ge 0$ and $h_1 > h_2 > \dots > h_N$. Then

$$\begin{aligned} L(\lambda, M, N) &= \prod_{1\le i<j\le N} \frac{(h_i-h_j)^2}{(j-i)^2} \prod_{i=1}^{N}\prod_{j=N+1}^{M} \frac{h_i+j-N}{j-i} \\ &= \prod_{j=0}^{N-1} \frac{1}{j!(M-N+j)!} \prod_{1\le i<j\le N} (h_i-h_j)^2 \prod_{i=1}^{N} \frac{(h_i+M-N)!}{h_i!}. \end{aligned} \tag{2.6}$$

The condition $\sum_{j=1}^{N}\lambda_j = k$ translates into $\sum_{j=1}^{N} h_j = k + N(N-1)/2$ and $\lambda_1 \le t$ to $h_1 \le t + N - 1$. By (2.2), (2.3) and (2.4) we have

$$p_{M,N}(t) = \sum_{k=0}^{\infty}(1-q)^{MN}q^k \sum_{\lambda\vdash k,\lambda_1\le k} L(\lambda, M, N),$$

and inserting (2.6) yields

$$p_{M,N}(t) = \frac{(1-q)^{MN}}{N!}q^{-N(N-1)/2}\prod_{j=0}^{N-1}\frac{1}{j!(M-N+j)!}$$
$$\times\sum_{k=0}^{\infty}\sum_{\substack{h\in\mathbb{N}^N\\ \sum h_i = k+N(N-1)/2\\ \max\{h_i\}\le t+N-1}}\prod_{1\le i<j\le N}(h_i-h_j)^2\prod_{i=1}^{N}\frac{(h_i+M-N)!}{h_i!}q^{\sum_{i=1}^{N}h_i},$$

where we have used the symmetry under permutation of the h_i's. Summing over k gives all the possible values of $\sum h_i$, so we obtain

$$p_{M,N}(t) = \frac{1}{\mathcal{Z}_{M,N}}\sum_{\substack{h\in\mathbb{N}^N\\ \max\{h_i\}\le t+N-1}}\prod_{1\le i<j\le N}(h_i-h_j)^2\prod_{i=1}^{N}w_{M-N+1}^{q}(h_i), \qquad (2.7)$$

where $w_K^q(x)$ is given by (1.16) and

$$\mathcal{Z}_{M,N} = q^{N(N-1)/2}(1-q)^{-MN}\prod_{j=0}^{N-1}j!(M-N+j)!. \qquad (2.8)$$

This proves Proposition 1.3. □

2.2. The large deviation estimate. In order to investigate the location of the rightmost charge in (2.7) and prove large deviation formulas we rescale the discrete Coulomb gas (2.7). Let $M=[\gamma N]$, $\gamma\ge 1$ fixed, and $K=K(N)=[\gamma N]-M+1$. Set $\mathbb{A}_N=\frac{1}{N}\mathbb{N}$, $\mathbb{A}_N(s)=\{x\in\mathbb{A}_N;\ x\le s\}$ and

$$V_N^{\gamma,q}(t) = -\frac{1}{N}\log w_{K(N)}^{q}(Nt), \quad t\ge 0.$$

Using Stirling's formula we see that

$$\begin{aligned}\lim_{N\to\infty} V_N^{\gamma,q}(t) = {}& t\log\frac{1}{q} - (t+\gamma-1)\log(t+\gamma-1)\\ &+ t\log t + (\gamma-1)\log(\gamma-1) \doteq V^{\gamma,q}(t)\end{aligned} \qquad (2.9)$$

uniformly on compact subsets of $[0,\infty)$. (We will often omit the superscripts γ and q.) Rescaling the variables in (2.7) by setting $h_i = Nx_i$, $x_i\in\mathbb{A}_N$ we see that (2.7) can be written

$$p_N(t) \doteq p_{M(N),N}(t) = \frac{Z_N(\frac{t}{N}+1-\frac{1}{N})}{Z_N}, \qquad (2.10)$$

where

$$Z_N(s) = \sum_{x \in \mathbb{A}_N(s)^N} \Delta_N(x)^2 \exp\bigl(-N \sum_{j=1}^{N} V_N(x_j)\bigr) \tag{2.11}$$

and $Z_N = Z_N(\infty)$. Here $\Delta_N(x) = \prod_{1 \le i < j \le N}(x_j - x_i)$ is the Vandermonde determinant.

When investigating the large deviation properties of $p_N(t)$ we may just as well consider more general confining potentials V_N. Assume that $V_N : [0, \infty) \to \mathbb{R}$, $N \ge 1$, satisfy

(i) V_N is continuous, $N \ge 1$.
(ii) There are constants $\xi > 0$, $T \ge 0$ and $N_0 \ge 1$ such that

$$V_N(t) \ge (1 + \xi) \log(t^2 + 1) \tag{2.12}$$

for $t \ge T$ and $N \ge N_0$.
(iii) $V_N(t) \to V(t)$ uniformly on compact subsets of $[0, \infty)$.

Set for $x \in \mathbb{A}_N^M$ and $\beta > 0$,

$$Q_{M,N}(x) = |\Delta_M(x)|^\beta \prod_{j=1}^{M} \exp(-\frac{\beta N}{2} V_N(x_j)).$$

(This M is not the same as the previous M.) Define the partition functions

$$Z_{M,N}(t) = \sum_{x \in \mathbb{A}_N(t)^M} Q_{M,N}(x),$$

$Z_{M,N} = Z_{N,M}(\infty)$ and the probability measure

$$P_{M,N}[B] = \frac{1}{Z_{M,N}} \sum_{x \in B} Q_{M,N}(x),$$

$B \subseteq \mathbb{N}^M$. We are interested in the distribution of the position of the rightmost charge, $\max_{1 \le k \le M} x_k$. Its distribution function is given by

$$F_{M,N}(t) = P_{M,N}[\max x_k \le t] = \frac{Z_{M,N}(t)}{Z_{M,N}}. \tag{2.13}$$

(If $M = N$ we write $F_N(t)$.)

In order to formulate the large deviation results for $F_N(t)$ we need some results from weighted potential theory, [ST]. The results we need differ from the usual ones since we are interested in the continuum limit of a discrete Coulomb gas, so that the particle density of the rescaled gas is always ≤ 1. Hence, the equilibrium measures will be absolutely continuous with a density ϕ satisfying $0 \le \phi \le 1$. Let $\mathcal{A}_s$ denote the set of all $\phi \in L^1[0, s)$ such that $0 \le \phi \le 1$ and $\int_0^s \phi = 1$, $1 \le s \le \infty$. Given $V : [0, \infty) \to \mathbb{R}$, continuous and such that there is a $\delta > 0$ and a $T \ge 0$ such that

$$V(t) \ge (1 + \delta) \log(t^2 + 1) \tag{2.14}$$

for $t \geq T$, we set

$$k_V(x, y) = \log|x-y|^{-1} + \frac{1}{2}V(x) + \frac{1}{2}V(y)$$

and

$$I_V[\phi] = \int_0^s \int_0^s k_V(x, y)\phi(x)\phi(y)dxdy,$$

for $\phi \in \mathcal{A}_s$.

The proof of the next proposition is similar to the corresponding result in weighted potential theory. See [DS] and also [LL] where a very similar problem is treated.

Proposition 2.1. *For each $s \in [1, \infty]$ there is a unique $\phi_V^s \in \mathcal{A}_s$ such that*

$$\inf_{\phi \in \mathcal{A}_s} I_V[\phi] = I_V[\phi_V^s] = F_V^s.$$

The extremal function ϕ_V^s has compact support. (If $s = \infty$ we will drop the superscript.)

Let $b_V = \sup(\operatorname{supp}\phi_V)$ be the right endpoint of the support of ϕ_V. Set $J(t) = 0$ for $t \leq b_V$ and

$$J(t) = \inf_{\tau \geq t} \int_0^\infty k_V(\tau, x)\phi_V(x)dx - F_V \tag{2.15}$$

for $t \geq b_V$. Also, set

$$L(t) = \frac{1}{2}(F_V^t - F_V)$$

for $t \geq 1$. The next theorem gives the large deviations for the distribution function $F_N(t)$ defined by (2.13)

Theorem 2.2. *Assume that $V_N(t)$ satisfies the assumptions (i)–(iii) above. Then*

$$\lim_{N \to \infty} \frac{1}{N^2} \log F_N(t) = -\beta L(t) \tag{2.16}$$

for any $t \geq 1$ and $L(t) > 0$ if $t < b_V$. Assume furthermore that $J(t) > 0$ for $t > b_V$. Then

$$\lim_{N \to \infty} \frac{1}{N} \log(1 - F_N(t)) = -\beta J(t) \tag{2.17}$$

for all t.

We postpone the proof to Sect. 4.

Remark 2.3. The same result is true for a continuous Coulomb gas on $\mathbb{R}$ with density

$$\frac{1}{Z_N^\beta}|\Delta_N(x)|^\beta \exp(-\frac{\beta N}{2}\sum_{j=1}^N V(x_j)), \tag{2.18}$$

on $\mathbb{R}^N$, which occur in random matrix theory. The choice $\beta = 2$ and $V(t) = 2t^2$ corresponds to the Gaussian Unitary Ensemble (GUE), compare (1.14). We assume that V is continuous and satisfies (2.14). In this case $\mathcal{A}_s$ is replaced by $\mathcal{M}_1(s)$, the set of all probability measures on $(-\infty, s)$, and $\phi_V(x)dx$ is replaced by the equilibrium measure $d\mu_V(t)$, see [Jo]. The proof is essentially the same. The formula (2.16) for certain V is a consequence of the result in [BG], see also [HP]. Also, (2.17) has been proved in the case $V(t) = t^2/2$ in [BDG]. If we take (2.18) on $[0, \infty)^N$ with $\beta = 2$ and $V(t) = -(M/N - 1)\log t + t$ we get the measure in (1.18), and in this way we can prove (1.19) to (1.21).

We can now apply Theorem 2.2 to the model we are interested in. It is straightforward to verify that $V_N^{\gamma,q}$ satisfies the conditions (i) - (iii) with limiting external potential $V^{\gamma,q}(t)$. Write $b_{V^{\gamma,q}} = b(\gamma, q)$. The computation of $\phi_{V^{\gamma,q}}$ will be outlined in Sect. 6. We have

$$b(\gamma, q) = \frac{(1+\sqrt{q\gamma})^2}{1-q}.$$

If $\gamma \geq 1/q$, then

$$\phi_{V^{\gamma,q}}(t) = v(\frac{2}{c}(t-a)-1), \quad a \leq t \leq b,$$

where $a = \frac{(1-\sqrt{q\gamma})^2}{1-q}$, $c = b(\gamma, q) - a$ and

$$v(x) = \frac{1}{2\pi}[\arctan(\frac{Dx+1}{\sqrt{1-x^2}\sqrt{D^2-1}}) - \arctan(\frac{Bx+1}{\sqrt{1-x^2}\sqrt{B^2-1}})], \tag{2.19}$$

$B = (\gamma+q)/2\sqrt{q\gamma}$, $D = (1+q\gamma)/2\sqrt{q\gamma}$. If $\gamma < 1/q$, then,

$$\phi_{V^{\gamma,q}}(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq a \\ v(\frac{2}{c}(t-a)-1), & \text{if } a \leq t \leq b, \end{cases}$$

where

$$v(x) = \frac{1}{2\pi}[\pi - \arctan(\frac{Dx+1}{\sqrt{1-x^2}\sqrt{D^2-1}}) - \arctan(\frac{Bx+1}{\sqrt{1-x^2}\sqrt{B^2-1}})] \tag{2.20}$$

with a, c, B, D as before.

We will not discuss the explicit form of the lower tail rate function. The upper tail rate function is given by

$$J(t) = \frac{c}{8\sqrt{q\gamma}} \int_1^x (x-y)[\frac{\gamma-q}{y+B} + \frac{1-q\gamma}{y+D}]\frac{dy}{\sqrt{y^2-1}}, \tag{2.21}$$

with c, B, D as above and $x = 2(t-a)/c - 1$. Using this formula we can show that (see Sect. 6) there are constants $c_1 > 0$ and $c_2 > 0$ so that

$$J(b+\delta) \geq \begin{cases} c_1\delta^{3/2} & \text{if } 0 \leq \delta \leq 1 \\ c_2\delta & \text{if } \delta \geq 1 \end{cases} \tag{2.22}$$

and

$$J(b+\delta) = \frac{2(1-q)^{3/2}\gamma^{1/4}}{3q^{1/4}(\sqrt{q}+\sqrt{\gamma})(1+\sqrt{q\gamma})}\delta^{3/2} + O(\delta^{5/2}). \tag{2.23}$$

In particular $J(t) > 0$ if $t > b(\gamma, q)$.

From (2.10), (2.13) and Theorem 2.2 we obtain

$$\lim_{N\to\infty} \frac{1}{N^2} \log p_N(Nt) = -2L(t+1) \tag{2.24}$$

and

$$\lim_{N\to\infty} \frac{1}{N} \log(1 - p_N(Nt)) = -2J(t+1) \tag{2.25}$$

for each $t \geq 0$. These formulas imply Theorem 1.1 with $\ell(\epsilon) = 2L(b_V - \epsilon)$ and $i(\epsilon) = 2J(b_V + \epsilon)$. By Theorem 2.2 and (2.22) we have $i(\epsilon) > 0$ and $\ell(\epsilon) > 0$ if $\epsilon > 0$.

By a superadditivity argument, the limit (2.25) actually gives a large deviation estimate for all N, compare [Se1].

Corollary 2.4. *For all $t \geq 0$ and $N \geq 1$,*

$$1 - p_N(Nt) \leq \exp(-2NJ(t+1)). \tag{2.26}$$

Proof. For $1 \leq M_1 \leq M_2$ and $1 \leq N_1 \leq N_2$ we let $G[(M_1, N_1), (M_2, N_2)]$ denote the maximum of $\sum_{(i,j)\in\pi} w(i, j)$ over all up/right paths from (M_1, N_1) to (M_2, N_2). Note that if $1 \leq M_1 < M_2$ and $1 \leq N_1 < N_2$, then

(i) $G[(M_1+1, N_1+1), (M_2, N_2)]$ and $G[(1, 1), (M_2-M_1, N_2-N_1)]$ are identically distributed.

(ii) $G[(1, 1), (M_1, N_1)]$ and $G[(M_1+1, N_1+1), (M_2, N_2)]$ are independent. Since $[2\gamma N] \geq 2[\gamma N]$, we have

(iii) $G[([\gamma N]+1, N+1), ([2\gamma N], 2N)] \geq G[([\gamma N]+1, N+1), (2[\gamma N], 2N)]$. Write $a_N(t) = 1 - p_N(Nt) = \mathbb{P}[G((1, 1), ([\gamma N], N)) > Nt]$. Then, by (i) and (iii),

$$a_N(t) \leq \mathbb{P}[G(([\gamma N]+1, N+1), ([2\gamma N], 2N)) > Nt]$$

and hence, by (ii), $a_N(t)^2 \leq a_{2N}(t)$. Repeated use of this inequality yields $N^{-1} \log a_N(t) \leq (2^k N)^{-1} \log a_{2^k N}(t)$, and by letting $k \to \infty$ and using (2.25) we find $N^{-1} \log a_N(t) \leq -2J(t+1)$. □

Remark 2.5. We cannot prove convergence of the moments of the rescaled random variable in Theorem 1.2 since we have no finite N estimate of $\mathbb{P}[G([\gamma N], N) - \omega N \leq -sN^{1/3}]$ for $s > 0$ large. This would require an estimate of the finite N Fredholm determinant. In the other direction we can use the estimate in Corollary 2.4. The same remark applies to Theorem 1.6.

Remark 2.6. In [BR] it is proved by Baik and Rains that if we consider permutations with certain restrictions we can get the Tracy–Widom distributions for GOE and GSE as limiting laws for longest increasing and decreasing subsequences. By considering a restricted geometry we can obtain the Tracy–Widom distribution for GOE, [TW2], also in the present setting. Let $w(i, j)$, $1 \leq i \leq j$ be independent geometrically distributed random variables, $\mathbb{P}[w(i, j) = k] = (1-q)q^k$ for $1 \leq i < j$ and $\mathbb{P}[w(i, i) = k] = (1-\sqrt{q})q^{k/2}$ for $i \geq 1$. Set $w(i, j) = w(j, i)$, if $i > j \geq 1$, so that $A = (w(i, j))$ is a symmetric matrix. The Knuth correspondence maps A to a pair of semistandard Young tableaux (P, Q) with $Q = P$, i.e. A maps to a single semistandard Young tableaux, see [Kn] or [Fu]. Let $\Pi^{\text{sym}}_{N,N}$ be the set of all up/right paths from $(1, 1)$ to (N, N) in $\{(i, j) \in \mathbb{Z}^2_+ ;\ 1 \leq i \leq j\}$, i. e. in a triangle, and set

$$F(N) = \max\{\sum_{(i,j)\in\pi} w(i, j) ;\ \pi \in \Pi^{\text{sym}}_{N,N}\}.$$

Now, we also have

$$F(N) = \max\{\sum_{(i,j)\in\pi} w(i, j) ;\ \pi \in \Pi_{N,N}\},$$

which equals the length of the first row in P, because those parts of a maximal path in $\Pi_{N,N}$ which goes below the diagonal can be reflected in the diagonal to give a path in $\Pi^{\text{sym}}_{N,N}$ without changing the sum $\sum w(i, j)$ since $w(i, j)$ is symmetric.

The same argument as above now gives

$$\mathbb{P}[F(N) \le t] = \frac{1}{Z_N^{(1)}} \sum_{\substack{h \in \mathbb{N}^N \\ \max\{h_j\} \le t+N-1}} \prod_{1 \le i < j \le N} |h_i - h_j| \prod_{i=1}^{N} q^{h_i/2}.$$

This corresponds to $\beta = 1$, $\gamma = 1$ in Theorem 2.2. It should be possible to analyze the asymptotics in this case analogously to GOE, see [TW2], to show that we can find constants a and b so that $\mathbb{P}[F(N) \le aN + sbN^{1/3}]$ converges to $F_1(t)$, the Tracy–Widom distribution for GOE. However it is not immediate to generalize the techniques of [TW2], so this remains to be done. Note that again we can take the limit $q \to 1$ to get the case of exponentially distributed random variables.

3. The Fredholm Determinant

From the identity (2.7) we have

$$p_N(t) = \psi_N(t + N - 1), \tag{3.1}$$

where

$$\psi_N(s) = \mathcal{E}_N[\prod_{j=1}^{N}(1 - \chi_s(h_j))]. \tag{3.2}$$

Here

$$\mathcal{E}_N[\cdot] = \frac{1}{\mathcal{Z}_{M(N),N}} \sum_{h \in \mathbb{N}^N} (\cdot)\Delta_N(h)^2 \prod_{j=1}^{N} w^q_{K(N)}(h_j),$$

$K(N) = M(N) - N + 1$, $M(N) = [\gamma N]$ and $\chi_s(t)$ is the indicator function for the interval (s, ∞). We will take s in (3.2) to be an integer.

Let $M_j^{K,q}(x)$, $j = 0, 1, \dots$ be the normalized orthogonal polynomials with respect to the weight $w_K^q(x)$ on $\mathbb{N}$,

$$\sum_{x=0}^{\infty} M_i^{k,q}(x) M_j^{K,q}(x) w_K^q(x) = \delta_{ij},$$

and $M_j^{K,q}(x) = \kappa_j x^j + \dots$ with $\kappa_j > 0$. Set

$$K_N(x, y) = \sum_{j=0}^{N-1} M_j^{K,q}(x) M_j^{K,q}(y) w_K^q(x)^{1/2} w_K^q(y)^{1/2},$$

so that $K_N(x, y)$ is a reproducing kernel on $\ell^2(\mathbb{N})$.

The polynomials $M_n^{K,q}$ are multiples of the standard Meixner polynomials, [NSU, Ch],

$$M_n^{K,q}(x) = \frac{(-1)^n}{d_n} m_N^{K,q}(x),$$

where

$$d_n^2 = \frac{n!(n+K-1)!}{(1-q)^K q^n (K-1)!}.$$

The leading coefficient in $m_n^{K,q}$ is $(\frac{q-1}{q})^n$ and consequently

$$\kappa_n = \frac{1}{d_n}\left(\frac{1-q}{q}\right)^n.$$

The Meixner polynomials have the generating function, [Ch],

$$\sum_{n=0}^{\infty} m_n^{K,q}(x)\frac{t^n}{n!} = (1-\frac{t}{q})^x(1-t)^{-x-K}. \tag{3.3}$$

The Christoffel–Darboux formula, [Sz], gives

$$\begin{aligned} K_N(x,y) &= \frac{\kappa_{N-1}}{\kappa_N}\frac{M_N(x)M_{N-1}(y)-M_N(y)M_{N-1}(x)}{x-y}w_K^q(x)^{1/2}w_K^q(y)^{1/2} \\ &= -\frac{q}{(1-q)d_{N-1}^2}\frac{m_N(x)m_{N-1}(y)-m_N(y)m_{N-1}(x)}{x-y}w_K^q(x)^{1/2}w_K^q(y)^{1/2}, \end{aligned} \tag{3.4}$$

where we have omitted the upper indices. Standard computations from random matrix theory, [Me], Ch. 5 and [TW2], show that ψ_N can be written as a Fredholm determinant,

$$\psi_N(s) = \sum_{k=0}^{N}\frac{(-1)^k}{k!}\sum_{h\in\{s+1,s+2,\dots\}^k}\det(K_N(h_i,h_j))_{1\le i,j\le k}. \tag{3.5}$$

The proof of Theorem 1.2 is based on taking the appropriate limit in (3.5).

The next lemma will allow us to compute the asymptotics of the right-hand side of (3.5).

Lemma 3.1. *Let $b \ge 0$ be a constant and assume that $\rho_N \to \infty$ as $N \to \infty$. Suppose furthermore that $K_N : \mathbb{N}\times\mathbb{B}\to\mathbb{R}$, $N \ge 1$, satisfies the following properties.*

(i) *Let $M_1 > 0$ be a given constant. There is a constant C such that*

$$\sum_{m=1}^{\infty} K_N(bN+\rho_N\tau+m, bN+\rho_N\tau+m) \le C \tag{3.6}$$

for all $N \ge 1$, $\tau \ge -M_1$.

(ii) *Given $\epsilon > 0$, there is an $L > 0$ so that*

$$\sum_{m=1}^{\infty} K_N(bN+\rho_N L+m, bN+\rho_N L+m) \le \epsilon, \tag{3.7}$$

for all $N \ge 1$.

(iii) *Let $M_0 > 0$ be a given constant. If $A(\xi,\eta)$ is the Airy kernel defined by (1.7), then*

$$\lim_{N\to\infty}\rho_N K_N(bN+\rho_N\xi, bN+\rho_N\eta) = A(\xi,\eta) \tag{3.8}$$

uniformly for $\xi,\eta \in [-M_0, M_0]$.

(iv) *The matrix* $(K_N(x_i, x_j))_{i,j=1}^k$ *is positive definite for any* $x_i, x_j \in [0, \infty)$, $k \geq 1$
Then, for each fixed $t \in \mathbb{R}$,

$$\lim_{N\to\infty} \sum_{k=0}^{N} \frac{(-1)^k}{k!} \sum_{h\in\mathbb{N}^k} \det(K_N(bN + \rho_N t + h_i, bN + \rho_N t + h_j))_{i,j=1}^k = F(t), \quad (3.9)$$

where $F(t)$ *is given by (1.8).*

Proof. It follows from (iv) that

$$|\det(K_N(x_i, x_j))_{1\leq i,j\leq k}| \leq \prod_{j=1}^{k} K_N(x_j, x_j), \quad (3.10)$$

see for example [HJ]. Consequently,

$$|\sum_{h\in\mathbb{N}^k} \det(K_N(a_N + h_i, a_N + h_j))_{1\leq i,j\leq k}| \leq \left(\sum_{m=1}^{\infty} K_N(m, m)\right)^k, \quad (3.11)$$

where we have written $a_N = bN + \rho_N t$.

Choose M_1 so that $|t| \leq M_1$. Let $\epsilon > 0$ be given. It follows from the estimates (3.6) and (3.11) that we can choose ℓ so that

$$|\sum_{k=\ell+1}^{N} \frac{(-1)^k}{k!} \sum_{h\in\mathbb{N}^k} \det(K_N(a_N + h_i, a_N + h_j))_{i,j=1}^k| \leq \sum_{k=\ell+1}^{\infty} \frac{C^k}{k!} \leq \epsilon, \quad (3.12)$$

for all $N \geq 1$. Choose L_0 so that (3.11) holds with $L = L_0 - M_0$. Then, by the estimates (3.6), (3.7) and (3.10),

$$\begin{aligned}
&\left|\left(\sum_{h\in\mathbb{N}^k} - \sum_{h\in([L_0\rho_N]^c)^k}\right) \det(K_N(a_N + h_i, a_N + h_j))_{1\leq i,j\leq k}\right| \\
&\leq \sum_{\substack{h\in\mathbb{N}^k \\ \text{some } h_j > L_0\rho_N}} \prod_{i=1}^{k} K_N(a_N + h_i, a_N + h_i) \\
&\leq \sum_{j=1}^{k} \sum_{\substack{h\in\mathbb{N}^k \\ h_j > L_0\rho_N}} \prod_{i=1}^{k} K_N(a_N + h_i, a_N + h_i) \\
&\leq k\left(\sum_{m=1}^{\infty} K_N(a_N + m, a_N + m)\right)^{k-1} \left(\sum_{m=1}^{\infty} K_N(bN + L\rho_N + m, bN + L\rho_N + m)\right) \\
&\leq kC^{k-1}\epsilon.
\end{aligned} \quad (3.13)$$

Denote the Fredholm determinant in the right-hand side of (3.9) by $D_N(t)$. Inserting the estimates (3.12) and (3.13) into the formula (3.9) we obtain

$$\begin{aligned}&\left|D_N(t)-\sum_{k=0}^{\ell}\frac{(-1)^k}{k!}\sum_{h\in[L_0\rho_N]^k}\det(\mathcal{K}_N(\sigma+\frac{h_i}{\rho_N},\sigma+\frac{h_j}{\rho_N}))_{1\le i,j\le k}\frac{1}{\rho_N^k}\right|\\&\le\left(\sum_{k=0}^{\ell}\frac{kC^{k-1}}{k!}+1\right)\epsilon\le(1+e^C)\epsilon,\end{aligned}\tag{3.14}$$

where

$$\mathcal{K}_N(\xi,\eta)=\rho_N K_N(bN+\rho_N\xi,bN+\rho_N\eta).$$

By assumption (iii), with $M_0=L_0+M_1$, we can chose N_0 so that if $N\ge N_0$, then

$$|\det(\mathcal{K}_N(\sigma+\frac{x}{\rho_N},\sigma+\frac{y}{\rho_N}))-\det(A(\sigma+\frac{x}{\rho_N},\sigma+\frac{y}{\rho_N}))|\le\frac{\epsilon}{L_0^k}$$

for all $x,y\in[L_0\rho_N]$. Thus,

$$\begin{aligned}&\left|\sum_{k=0}^{\ell}\frac{(-1)^k}{k!}\sum_{h\in[L_0\rho_N]^k}\left[\det(\mathcal{K}_N(t+\frac{h_i}{\rho_N},t+\frac{h_j}{\rho_N}))-\det(A(t+\frac{h_i}{\rho_N},t+\frac{h_j}{\rho_N}))\right]\frac{1}{\rho_N^k}\right|\\&\le\sum_{k=0}^{\ell}\frac{1}{k!}\left(\frac{L_0\rho_N+1}{L_0\rho_N}\right)^k\epsilon\le C'\epsilon.\end{aligned}\tag{3.15}$$

Combining the estimates (3.14) and (3.15) we find

$$\left|D_N(t)-\sum_{k=0}^{\ell}\frac{(-1)^k}{k!}\sum_{h\in[L_0\rho_N]^k}\det(A(\sigma+\frac{h_i}{\rho_N},\sigma+\frac{h_j}{\rho_N}))_{i,j=1}^k\frac{1}{\rho_N^k}\right|\le C''\epsilon.\tag{3.16}$$

The Airy kernel can be written, [TW1],

$$A(x,y)=\int_0^\infty \mathrm{Ai}\,(x+s)\mathrm{Ai}\,(y+s)ds.\tag{3.17}$$

Using the formula, see for example [Hö], p. 214,

$$\mathrm{Ai}\,(x)=e^{-\frac{2}{3}x^{3/2}}\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-\xi^2\sqrt{x}+i\xi^3/3}d\xi,$$

valid for $x>0$, we see that

$$|\mathrm{Ai}\,(x)|\le\frac{1}{2\sqrt{\pi}x^{1/4}}e^{-\frac{2}{3}x^{3/2}},\quad x>0.$$

This estimate can be used to show that the Airy kernel satisfies (i) and (ii) above. Since the matrix $(A(\xi_i,\xi_j))_{1\le i,j\le k}$ is positive definite, we can use the same argument as above to show that

$$\left|\left(\sum_{k=0}^{\infty}\int_{[t,\infty)^k}-\sum_{k=0}^{\ell}\int_{[t,L_0]^k}\right)\frac{(-1)^k}{k!}\det(A(\xi_i,\xi_j))_{i,j=1}^k d^k\xi\right|\le\epsilon\tag{3.18}$$

provided ℓ and L_0 are sufficiently large. From (3.17) we see that choosing $N_1 \geq N_0$ large enough we have

$$\left| D_N(t) - \sum_{k=0}^{\ell} \frac{(-1)^k}{k!} \int_{[t,L_0]^k} \det(A(\xi_i, \xi_j))_{1\leq i,j\leq k} d^k\xi \right| \leq C'''\epsilon \tag{3.19}$$

for all $N \geq N_1$. If we combine the estimates (3.18) and (3.19) we have proved the lemma. □

To apply this lemma to the Meixner kernel (3.4) we need

Lemma 3.2. *The Meixner kernel satisfies the properties (i) to (iv) in Lemma 3.1 with $b = b(\gamma, q)$ as before and $\rho_N = \sigma N^{1/3}$, where σ is given by (1.11).*

This lemma will be proved in Sect. 5. We can now combine (3.1), (3.5) and (3.9) to get

$$\lim_{N\to\infty} p_N((b-1)N + \sigma N^{1/3}t) = F(t), \tag{3.20}$$

which is (1.10) and Theorem 1.2 is proved. □

4. Proof of the Large Deviation Theorem

In this section we will prove Theorem 2.2. Set

$$K_{N,V}(x) = \sum_{1\leq i\neq j\leq N} k_V(x_i, x_j).$$

By adding a constant C to V_N, which does not alter the problem we can, by assumption (ii) on V_N, assume that

$$V_N(t) - \log(t^2+1) \geq \xi \log(t^2+1) \tag{4.1}$$

for all $t \geq 0$. Since $|t-s|^2 \leq (t^2+1)(s^2+1)$, this implies

$$-K_{M,V_N}(x) \leq -\xi(M-1) \sum_{j=1}^{M-1} \log(1+x_j^2) \tag{4.2}$$

for all $x \in [0,\infty)^M$. Note that

$$\sum_{1\leq j\neq k\leq N-1} \log|x_j - x_k| - N \sum_{j=1}^{N-1} V_N(x_j) = -K_{N-1,V_N}(x) - \sum_{j=1}^{N-1} V_N(x_j). \tag{4.3}$$

The next lemma is analogous to Lemma 4.2 in [Jo].

Lemma 4.1. *Let $\{s_N\}$ be a sequence in $[0,\infty)$ such that $s_N \to s > 0$ as $N \to \infty$, or $s_N \equiv \infty$. Set, for a given $\alpha > 0$,*

$$\Omega_{N,\alpha}(s) = \{x \in \mathbb{A}_N(s)^{N-1}\,;\ \frac{1}{N^2} K_{N-1,V_N}(x) \leq F_V^{\sigma} + \alpha\}.$$

Let $0 \le \lambda \le 1$ *and let* $\sigma_N \in \mathbb{A}_N$, $N \ge 1$, *be a sequence converging to* $\sigma > 0$. *Define a probability measure on* $\mathbb{A}_N(s_N)^{N-1}$ *by*

$$P^{\lambda,\sigma_N}_{N-1,N}(\Omega; s_N) = \frac{1}{Z^{\lambda,\sigma_N}_{N-1,N}(s_N)} \sum_{x\in\Omega} \prod_{j=1}^{N-1} |\sigma_N - x_j|^{\lambda\beta} Q_{N-1,N}(x), \tag{4.4}$$

where $Z^{\lambda,\sigma_N}_{N-1,N}(s_N)$ *is a normalization constant. (*$E^{\lambda,\sigma_N}_{N-1,N}[\cdot; s_N]$ *denotes the corresponding expectation and if* $s_N \equiv \infty$ *or* $\lambda = \sigma_N = 0$ *we omit them in the notation.) Fix* $\eta > 0$. *Then there is an* N_1 *such that for all* $a \ge 0$ *and* $N \ge N_1$,

$$P^{\lambda,\sigma_N}_{N-1,N}(\Omega_{N,\eta+a}(s_N)^c; s_N) \le e^{-\frac{\beta}{4}aN^2}. \tag{4.5}$$

Proof. We first prove the following claim.

Claim 4.2. *Let* $\sigma_N \in \mathbb{A}_N$, $\sigma_N \to \sigma$ *as* $N \to \infty$ *and* $s \in (0,\infty]$. *For each* $N \ge 2$ *we can choose* $(x_1^N, \dots, x_{N-1}^N) \in \mathbb{A}_N(s)^{N-1}$ *so that*

$$\frac{1}{N^2} \sum_{1\le j\ne k\le N-1} \log|x_j^N - x_k^N|^{-1} + \frac{1}{N}\sum_{j=1}^{N-1} V_N(x_j^N) \\ - \frac{1}{N^2}\sum_{j=1}^{N-1} \log|\sigma_N - x_j^N| \to F_V^s \tag{4.6}$$

as $N \to \infty$.

To see this set

$$y_k^N = \max\{\frac{j}{N};\ j \in \mathbb{N} \text{ and } \int_0^{j/N} \phi_V^s(t)dt < \frac{k}{N}\}.$$

If $y_k^N \ne \sigma_N$ for $k = 1, \dots, N-1$, we set $x_k^N = y_k^N$. If $y_{k_0}^N = \sigma_N$, we set $x_k^N = y_k^N$ for $k < k_0$ and $x_k^N = y_k^N + 1/N$ for $k = k_0, \dots, N-1$. Using the fact that $0 \le \phi_V^s \le 1$ it is not difficult to see that $x_1^N < x_2^N < \dots < x_{N-1}^N \le L$ for all N and some fixed L. Furthermore

$$\frac{1}{N-1}\sum_{k=1}^{N-1} \delta_{x_k^N} \to \phi_V^s(x)dx \tag{4.7}$$

weakly as $N \to \infty$. The property (iii) in the assumptions on V_N implies

$$\frac{1}{N}\sum_{j=1}^{N-1} V_N(x_j^N) \to \int_0^\infty V(t)\phi_V^s(t)dt. \tag{4.8}$$

Clearly,

$$\frac{1}{N^2}\sum_{j=1}^{N-1} \log|\sigma_N - x_j^N|^{-1} \le \frac{2}{N^2}\sum_{j=1}^{N-1} \log\frac{N}{j} = \frac{2}{N^2}\log\frac{N^{N-1}}{(N-1)!}, \tag{4.9}$$

which $\to 0$ as $N \to \infty$. Also, since $\sigma_N \to \sigma$ and the x_j^N belong to a bounded set, we get a bound in the other direction which goes to 0 as $N \to \infty$. Given $M \geq 1$, set $f_M(t) = \min\{\log |t|^{-1}, \log M\}$. Write

$$\frac{1}{N^2} \sum_{j \neq k} \log |x_j^N - x_k^N|^{-1} = \frac{1}{N^2} \sum_{j \neq k} f_M(x_j^N - x_k^N)$$
$$+ \frac{1}{N^2} \sum_{\substack{j \neq k \\ |x_j^N - x_k^N| < 1/M}} (\log |x_j^N - x_k^N|^{-1} - f_M(x_j^N - x_k^N)). \tag{4.10}$$

The absolute value of the second sum in the right-hand side of (4.10) is

$$\leq \frac{1}{N^2} \sum_{\substack{1 \leq |j-k| \leq N/M \\ |j|,|k| \leq LN}} \log |\frac{N}{j-k}| \leq C \frac{\log M}{M}.$$

Thus, using the weak convergence (4.7) and then letting $M \to \infty$ we obtain

$$\lim_{N \to \infty} \frac{1}{N^2} \sum_{j \neq k} \log |x_j^N - x_k^N|^{-1} = \int_0^\infty \int_0^\infty \log |x - y|^{-1} \phi_V^s(x) \phi_V^s(y) dx dy,$$

which together with (4.8) and (4.9) proves the claim.

We turn now to the proof of Lemma 4.1. Let $\epsilon > 0$ be given. We want to estimate $Z_{N-1,N}^{\lambda,\sigma_N}$ from below. Choose N_0 so that $s_N \geq s - \epsilon$ if $N \geq N_0$. Then

$$Z_{N-1,N}^{\lambda,\sigma_N}(s_N) \geq Z_{N-1,N}^{\lambda,\sigma_N}(s - \epsilon),$$

if $N \geq N_0$. Choose $(x_k^N)_{k=1}^{N-1} \subseteq \mathbb{A}_N(s - \epsilon)$ as in the claim. Clearly,

$$\frac{1}{N^2} \log Z_{N-1,N}^{\lambda,\sigma_N}(s_N) \geq -\frac{\beta}{2} \left[\frac{1}{N^2} \sum_{j \neq k} \log |x_j^N - x_k^N|^{-1} \right.$$
$$\left. + \sum_{j=1}^{N-1} V_N(x_j^N) - \frac{1}{N^2} \sum_{j=1}^{N-1} \log |\sigma_N - x_j^N| \right],$$

and consequently, by Claim 4.2,

$$\liminf_{N \to \infty} \frac{1}{N^2} \log Z_{N-1,N}^{\lambda,\sigma_N}(s_N) \geq -\frac{\beta}{2} F_V^{s-\epsilon}.$$

Since $F_V^{s-\epsilon} \searrow F_V^s$ as $\epsilon \to 0+$,

$$\liminf_{N \to \infty} \frac{1}{N^2} \log Z_{N-1,N}^{\lambda,\sigma_N}(s_N) \geq -\frac{\beta}{2} F_V^s. \tag{4.11}$$

Thus, given $\delta > 0$, we can choose $N(\delta)$ so that if $N \geq N(\delta)$, then

$$\frac{1}{N^2} \log Z_{N-1,N}^{\lambda,\sigma_N}(s_N) \geq -\frac{\beta}{2}(F_V^s + \delta). \tag{4.12}$$

It follows from (4.2) with $M = N - 1$ and (4.3), that for any $0 < \rho < 1/2$,

$$
\begin{aligned}
&P^{\lambda,\sigma_N}_{N-1,N}(\Omega_{N,\eta+a}(S_N)^c; s_N) \\
&\le e^{\frac{\beta N^2}{2}(F_V^s+\delta)} \sum_{x\in\mathbb{A}(s_N)^{N-1}\setminus\Omega_{N,\eta+a}(s_N)} e^{-\frac{\beta}{2}K_{N-1,V_N}(x)-\frac{\beta}{2}\sum_j V_N(x_j)} \prod_{j=1}^{N-1} |\sigma_N - x_j|^{\lambda\beta} \\
&\le e^{\frac{\beta N^2}{2}(F_V^s+\delta)-\frac{\beta}{2}(1-\rho)(F_V^s+\eta+a)N^2} \left[\sum_{t\in\mathbb{A}_N} (t^2+1)^{-\frac{\beta}{2}\xi(N-1)}(1+\sigma_N^2)^{\lambda\beta/2}\right]^N \\
&\le e^{-\frac{\beta}{4}aN^2}
\end{aligned}
$$

if N is sufficiently large (independent of $a \ge 0$). Note that $\delta + \rho F_V^s - \eta < 0$ if we choose $\delta = \eta/2$ and ρ sufficiently small. This completes the proof. □

This lemma can be used to prove

Corollary 4.3. *For any $s \in (1, \infty]$,*

$$
\lim_{N\to\infty} \frac{1}{N^2} \log Z_N(s) = -\frac{\beta}{2} F_V^s. \tag{4.13}
$$

Furthermore $F_V^s - F_V > 0$ if $s < b_V$.

Proof. The lower limit follows by taking $\lambda = \sigma_N = 0$ in (4.11) (replacing $N - 1$ by N does not modify the argument above in any essential way). Given $0 < \rho < 1$, we can use (4.2) with $M = N$ and the continuity of $\exp K_{N,V_N}$ to see that

$$
\begin{aligned}
Z_N(s) &= \sum_{x\in\mathbb{A}_N(s)^N} e^{-\frac{\beta}{2}K_{N,V_N}(x)-\frac{\beta}{2}\sum_{j=1}^N V_N(x_j)} \\
&\le \sup_{x\in\mathbb{A}_N(s)^N} e^{-\frac{\beta}{2}(1-\rho)K_{N,V_N}(x)} \sum_{x\in\mathbb{A}_N(s)^N} e^{-\frac{\beta}{2}\rho\xi(N-1)\sum_j \log(1+x_j^2)} \qquad (4.14) \\
&\le e^{-\frac{\beta}{2}(1-\rho)K_{N,V_N}(y^N)+CN},
\end{aligned}
$$

if N is sufficiently large, where $y^N = (y_1^N, \dots, y_N^N) \in \mathbb{A}_N(s)^N$. Clearly, $y_j^N \ne y_k^N$ if $j \ne k$. Set $\lambda_N = N^{-1}\sum_j \delta_{y_j^N}$. It follows from (4.12), with $\lambda = \sigma = 0$ and $N - 1$ replaced by N, that $N^{-2}\log Z_N(s) \ge -\beta(F_V^s + \delta)/2$ for $N \ge N(\delta)$, so (4.2) and (4.14) yield

$$
\int_0^\infty \log(1+t^2)d\lambda_N(t) \le C.
$$

Thus $\{\lambda_N\}_{N=1}^\infty$ is tight. Pick a subsequence that gives the upper limit of $N^{-2}\log Z_N(s)$, and a further subsequence so that λ_{N_j} converges weakly to $\nu = \psi dx$. The measure ν has to be absolutely continuous with density satisfying $0 \le \psi \le 1$ because of the definition of λ_N. Using (4.1) and $|t - s| \le \sqrt{t^2+1}\sqrt{s^2+1}$ we see that $k_{V_N}(t, s) \ge 0$. Set, for given $M > 0$, $k_{V_N}^M(t, s) = \min(k_{V_N}(t, s), M)$ and choose $\phi_T(t)$ continuous so that $0 \le \phi_T \le 1$, $\phi_T(t) = 1$ if $|t| \le T$, $= 0$ if $|t| \ge T + 1$

and $\phi_T(t) \le \phi_{T'}(t)$ if $T \le T'$. Then, $k_{V_N}(t,s) \ge \phi_T(t)\phi_T(s)k^M_{V_N}(t,s)$ and using the estimate (4.14) we get

$$\frac{1}{N_j^2}\log Z_{N_j}(s)$$
$$\le \frac{C+\frac{\beta}{2}(1-\rho)M}{N_j} - \frac{\beta}{2}(1-\rho)\int_0^\infty\int_0^\infty \phi_T(t)\phi_T(s)K^M_{V_N}(t,s)d\lambda_{N_j}(t)d\lambda_{N_j}(s),$$

and thus, letting $j\to\infty$, $M\to\infty$, $T\to\infty$ and $\rho\to 0+$ in that order, we obtain

$$-\frac{\beta}{2}F^s_V \le \liminf_{N\to\infty}\frac{1}{N^2}\log Z_N(s) \le \limsup_{N\to\infty}\frac{1}{N^2}\log Z_N(s) \le -\frac{\beta}{2}I_V[\psi].$$

Thus $I_V[\psi] \le F^s_V$ and $\psi \in \mathcal{A}_s$, so we must have $\psi = \phi^s_V$.

Assume that $F^s_V \le F_V$ and $s < b_V$. Then $I_V[\phi^s_V] \le I_V[\phi_V]$ and consequently $\phi^s_V = \phi_V$ by the uniqueness of the minimizing measure. This contradicts the definition of b_V. The corollary is proved. □

Note that by (2.13) Corollary 4.3 implies (2.16) so we have proved the first part of Theorem 2.2. Before turning to the proof of the second part we need one more consequence of Lemma 4.1.

Corollary 4.4. *Let $\{s_N\}$ be as in Lemma 4.1 and assume that $f : [0,\sigma+\epsilon] \to \mathbb{R}$, $\epsilon > 0$, is continuous, or $f : [0,\infty) \to \mathbb{R}$ is continuous and bounded in case $s_N \equiv \infty$. Then*

$$\lim_{N\to\infty}\frac{1}{N}\log E^{y,\sigma_N}_{N-1,N}[e^{\sum_{j=1}^N f(x_j)}; s_N] = \int_0^\infty f(t)\phi^\sigma_V(t)dt. \tag{4.15}$$

Furthermore let

$$u^{y,\sigma_N}_{N-1,N}(t) = \frac{1}{N-1}E^{y,\sigma_N}_{N-1,N}[\sum_{i=1}^{N-1}\delta_{t,x_i}], \tag{4.16}$$

($\delta_{t,s}$ is Kronecker's delta), be the 1-dimensional marginal distribution of the probability measure (4.4) (with $s_N \equiv \infty$). Then for each $0 < y \le 1$:

(i) $0 \le u^{y,\sigma_N}_{N-1,N}(t) \le \frac{1}{N-1}$ *for all $t \in \mathbb{A}_N$,*
(ii) *if δ_t is the Dirac measure at t, then $\sum_{t\in\mathbb{A}_N} u^{y,\sigma_N}_{N-1,N}(t)\delta_t$ converges weakly to $\phi_V(t)dt$ as $N\to\infty$.*
(iii) $u^{y,\sigma_N}_{N-1,N}(\sigma_N) = 0$.

Proof. We can prove (4.15) using Lemma 4.1 in exactly the same way as in the proof of (2.5) on p. 194 in [Jo], see also [De]. The weak limit (ii) is a direct consequence of (4.15), see [De]. Note that the limit does not depend on y since the factor $\prod_{i=1}^{N-1}|\sigma_N - x_i|^{y\beta}$ does not affect the leading asymptotics.

In the expectation (4.16) all the x_i's have to be different, otherwise the probability is zero, and consequently the expectation is ≤ 1, which proves (i). That (iii) holds follows from the presence of the factor $\prod_{i=1}^{N-1}|\sigma_N - x_i|^{y\beta}$. The corollary is proved.

We turn now to the proof of the upper-tail limit. Note that

$$Q_{M,N}(x) = e^{-\frac{N\beta}{2}V_N(x_M)} \prod_{i=1}^{M-1} |x_M - x_i|^\beta Q_{M-1,N}(x'), \tag{4.17}$$

where $x' = (x_1, \dots, x_{M-1})$. Using this identity we see that

$$\begin{aligned} Z_{M,N}(t) &= M! \sum_{\substack{x\in\mathbb{A}_N^M \\ x_1\le\cdots\le x_M\le t}} Q_{M,N}(x) \\ &= M \sum_{s\in\mathbb{A}_N(t)} e^{-\frac{N\beta}{2}V_N(s)} \sum_{x\in\mathbb{A}_N(s)^{M-1}} \prod_{i=1}^{M-1} |s - x_i|^\beta Q_{M-1,N}(x). \end{aligned}$$

If we define

$$H_{M-1,N}(s) = \frac{1}{Z_{M-1,N}(s)} \sum_{x\in\mathbb{A}_N(s)^{M-1}} \prod_{i=1}^{M-1} |s - x_i|^\beta Q_{M-1,N}(x),$$

this can be written

$$Z_{M,N}(t) = M \sum_{s\in\mathbb{A}_N(t)} e^{-\frac{N\beta}{2}V_N(s)} Z_{M-1,N}(s) H_{M-1,N}(s), \tag{4.18}$$

or

$$F_{M,N}(t) = \frac{MZ_{M-1,N}}{Z_{M,N}} \sum_{s\in\mathbb{A}_N(t)} e^{-\frac{N\beta}{2}V_N(s)} F_{M-1,N}(s) H_{M-1,N}(s). \tag{4.19}$$

This is the main formula to be used in the proof of (2.17). We will need two choices of M, namely $M = N$ and $M = N - 1$. They are handled completely analogously and we will consider only the case $M = N$.

Write $\mathbb{A}_N(t, s) = \mathbb{A}_N \cap (t, s)$ for any $0 \le t < s \le \infty$ and $\mathbb{A}_N(t)^* = \mathbb{A}_N(t, \infty)$. If we let $t \to \infty$ in (4.19) and then subtract (4.19) from the limiting equality, we get

$$1 - F_N(t) = \frac{NZ_{N-1,N}}{Z_{N,N}} \sum_{s\in\mathbb{A}_N(t)^*} e^{-\frac{N\beta}{2}V_N(s)} F_{N-1,N}(s) H_{N-1,N}(s). \tag{4.20}$$

Set

$$\Phi_V = F_V - \frac{1}{2}\int_0^\infty V(s)\phi_V(s)ds.$$

From the variational relations for $\phi_V(t)$ it follows that

$$\int_0^\infty \log|b_V - s|^{-1}\phi_V(s)ds + \frac{1}{2}V(b_V) = \Phi_V. \tag{4.21}$$

Lemma 4.5. *We have*

$$\limsup_{N\to\infty} \frac{1}{N} \log \frac{Z_{N-1,N}}{Z_{N,N}} \le \beta\Phi_V. \tag{4.22}$$

Proof. By (4.17) we have

$$\begin{aligned}\frac{Z_{N,N}}{Z_{N-1,N}} &= \sum_{s\in\mathbb{A}_N} e^{-\frac{N\beta}{2}V_N(s)} E_{N-1,N}[\prod_{i=1}^{N-1} |s-x_i|^\beta] \\ &\geq e^{-\frac{N\beta}{2}V_N(r)} E_{N-1,N}[\prod_{i=1}^{N-1} |r-x_i|^\beta]\end{aligned} \tag{4.23}$$

for any $r \in \mathbb{A}_N$. One difficulty in estimating the right-hand side in (4.23) comes from the fact that, due to the discrete nature of the problem the integrand could, apriori, be zero for many r's with high probability. Note that we define $0^y = 0$ for any $y > 0$. Let $\psi_s(t) = 1$ if $t \neq s$ and $\psi_s(s) = 0$.

Consider

$$f_N(y;s) = \frac{1}{N}\log E_{N-1,N}[\prod_{i=1}^{N-1} |s-x_i|^{y\beta}\psi_s(x_i)].$$

Then,

$$\begin{aligned}f_N(0+;s) = \lim_{y\to 0+} f_N(y;s) &= \frac{1}{N}\log E_{N-1,N}[\prod_{i=1}^{N-1}\psi_s(x_i)] \\ &= \frac{1}{N}\log P_{N-1,N}[\text{all } x_i \neq s].\end{aligned} \tag{4.24}$$

Let $\epsilon > 0$ be given and write $\mathbb{B}_N(\epsilon) = \mathbb{A}_N(b_V+\epsilon, b_V+2\epsilon)$. Now,

$$\begin{aligned}\sum_{s\in\mathbb{B}_N(\epsilon)} P_{N-1,N}[\text{all } x_i \neq s] &\geq P_{N-1,N}[\bigcup_{s\in\mathbb{B}_N(\epsilon)} \{\text{all } x_i \neq s\}] \\ &= 1 - P_{N-1,N}[\bigcap_{s\in\mathbb{B}_N(\epsilon)} \{\text{one } x_i = s\}].\end{aligned} \tag{4.25}$$

Take $g : [0,\infty) \to [0,\infty)$ continuous such that $g(s) = 1$ if $b_V+\epsilon \leq s \leq b_V+2\epsilon$ and $g(s) = 0$ if $0 \leq s \leq b_V$ or $s \geq b_V+3\epsilon$. Then,

$$e^{\epsilon N} P_{N-1,N}[\bigcap_{s\in\mathbb{B}_N(\epsilon)} \{\text{one } x_i = s\}] \leq E_{N-1,N}[e^{\sum_{i=1}^N g(x_i)}] \leq e^{\epsilon N/2} \tag{4.26}$$

for all sufficiently large N. The first inequality follows from the definitions whereas the second follows from Corollary 4.4, (4.15). Combining (4.25) and (4.26) we see that

$$\max_{s\in\mathbb{B}_N(\epsilon)} P_{N-1,N}[\text{all } x_i \neq s] \geq \frac{1}{2N} \tag{4.27}$$

for all sufficiently large N. Hence, by (4.24) and (4.27) we can choose $\sigma_N = \sigma_N(\epsilon) \in \mathbb{B}_N(\epsilon)$ so that

$$\lim_{N\to\infty} f_N(0+;\sigma_N) = 0. \tag{4.28}$$

Take $r = \sigma_N$ in (4.23). Then

$$\begin{aligned}\frac{1}{N}\log\frac{Z_{N,N}}{Z_{N-1,N}} &\geq -\frac{\beta}{2}V_N(\sigma_N) + f_N(1;\sigma_N) \\ &= -\frac{\beta}{2}V_N(\sigma_N) + f_N(0+;\sigma_N) + \beta\int_0^1 f_N'(y;\sigma_N)dy.\end{aligned} \tag{4.29}$$

We can pick a subsequence $\{N_j\}$ which gives $\liminf_{N\to\infty} \frac{1}{N}\log\frac{Z_{N,N}}{Z_{N-1,N}}$ and such that $\sigma_{N_j}(\epsilon) \to \sigma(\epsilon) \in [b_V + \epsilon, b_V + 2\epsilon]$. Then, by (4.28) and (4.29),

$$\liminf_{N\to\infty} \frac{1}{N}\log\frac{Z_{N,N}}{Z_{N-1,N}} \geq -\frac{\beta}{2}V(\sigma(\epsilon)) + \beta \liminf_{j\to\infty}\int_0^1 f'_{N_j}(y;\sigma_{N_j})dy. \tag{4.30}$$

Now,

$$\begin{aligned} f'_N(y;\sigma_N) &= E^{y,\sigma_N}_{N-1,N}[\frac{1}{N}\sum_{i=1}^{N-1}\log|\sigma_N - x_i|] \\ &= \frac{N-1}{N}\sum_{t\in\mathbb{A}_N}\log|\sigma_N - t|u^{y,\sigma_N}_{N-1,N}(t). \end{aligned}$$

Hence, by Corollary 4.4 (i) and (iii),

$$f'_N(y;\sigma_N) \geq 2\frac{1}{N}\sum_{i=1}^{N}\log\frac{i}{N} \geq -2,$$

and consequently, by Fatou's lemma,

$$\liminf_{j\to\infty}\int_0^1 f'_{N_j}(y;\sigma_{N_j})dy \geq \int_0^1 \liminf_{j\to\infty} f'_{N_j}(y;\sigma_{N_j})dy. \tag{4.31}$$

Given $\delta > 0$, small, and $M > 0$ set

$$f_{M,\delta}(t) = \begin{cases} \log M, & \text{if } |t| \geq M \\ \log|t|, & \text{if } \delta \leq |t| < M \\ \log\delta, & \text{if } |t| \leq \delta. \end{cases}$$

By Corollary 4.4 (i) and (iii) we have

$$\begin{aligned} &\left|\sum_{t\in\mathbb{A}_N}(\min(\log M, \log|\sigma_N - t|) - f_{M,\delta}(\sigma_N - t))u^{y,\sigma_N}_{N-1,N}(t)\right| \\ &\leq \sum_{t\in\mathbb{A}_N\,;\,0<|t-\sigma_N|\leq\delta}\left|\log\left|\frac{\sigma_N - t}{\delta}\right|\right|\frac{1}{N-1} \leq \frac{2}{N-1}\sum_{k=1}^{[N\delta]}\log\frac{N\delta}{k} \\ &\leq \frac{2N}{N-1}\delta. \end{aligned}$$

Also, if $|\sigma_N - \sigma_\epsilon| \leq \delta$, which is true if N is large enough,

$$|f_{M,\delta}(|\sigma_N - t|) - f_{M,\delta}(|\sigma(\epsilon) - t|)| \leq \delta\log\frac{1}{\delta}.$$

Since $\log|\sigma_N - t| \geq \min(\log M, \log|\sigma_N - t|)$ and M,δ are arbitrary it follows from Corollary 4.4, (ii) that

$$\liminf_{j\to\infty} f'_{N_j}(y;\sigma_{N_j}) \geq \int_0^\infty \log|\sigma(\epsilon) - t|\phi_V(t)dt.$$

Together with (4.30) and (4.31) this gives

$$\liminf_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,N}}{Z_{N-1,N}} \geq -\frac{\beta}{2} V(\sigma(\epsilon)) + \beta \int_0^\infty \log|\sigma(\epsilon) - t|\phi_V(t)dt.$$

We can pick a sequence $\epsilon_j \to 0$ such that $\sigma(\epsilon_j) \to b_V$ and using (4.24) we obtain

$$\liminf_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,N}}{Z_{N-1,N}} \geq -\beta\Phi_V,$$

and the lemma is proved. □

Given $\delta > 0$ we can use Lemma 4.5 to find $N_0(\delta)$ so that

$$\frac{Z_{N-1,N}}{Z_{N,N}} \leq e^{N\beta(\Phi_V+\delta)} \tag{4.32}$$

if $N \geq N_0(\delta)$. Since $F_{N-1,N}(s) \leq 1$ we can combine (4.20) and (4.32) to get the estimate

$$1 - F_N(t) \leq Ne^{N\beta(\Phi_V+\delta)} \sum_{s\in\mathbb{A}_N(t)^*} e^{-\frac{N\beta}{2}V_N(s)} H_{N-1,N}(s). \tag{4.33}$$

We have

$$\begin{aligned} H_{N-1,N}(s) &= E_{N-1,N}[\prod_{i=1}^{N-1} |s - x_i|^\beta; s] \\ &\leq (1+s^2)^{\frac{\beta}{2}(N-1)} E^{0,0}_{N-1,N}[\prod_{i=1}^{N-1}(1+x_i^2)^{\beta/2}; s] \leq e^{CN}(1+s^2)^{\beta N/2}, \end{aligned}$$

where the last inequality is proved, using Lemma 4.1, just as (4.25) in [Jo]. Together with (4.1) this gives

$$e^{-\frac{N\beta}{2}V_N(s)} H_{N-1,N}(s) \leq e^{CN - \frac{N\beta\xi}{2}\log(1+s^2)}. \tag{4.34}$$

Hence, given a constant $D > 0$, there is a constant $d > 0$ such that

$$e^{N\beta(\Phi_V+\delta)} \sum_{s\in\mathbb{A}_N(d)^*} e^{-N\beta V_N(s)/2} H_{N-1,N}(s) \leq e^{-ND}. \tag{4.35}$$

For $t \geq s$ we define

$$H_{N-1,N}(t,s) = \frac{1}{Z_{N-1,N}(s)} \sum_{x\in\mathbb{A}_N(s)^{N-1}} \prod_{j=1}^{N-1} |t - x_i|^\beta Q_{N-1,N}(x).$$

Clearly,

$$H_{N-1,N}(s) = H_{N-1,N}(s,s) \leq H_{N-1,N}(t,s) \tag{4.36}$$

if $t \geq s$. Combining the estimates (4.33), (4.35) and (4.36) we obtain

$$1 - F_N(t) \leq Ne^{-ND} + Ne^{N\beta(\Phi_V+\delta)} \sum_{x\in\mathbb{A}_N(t,d)} e^{-\frac{N\beta}{2}V_N(s)} H_{N-1,N}(s+\epsilon, s) \tag{4.37}$$

for any $\epsilon > 0$. Let $s_N \in \mathbb{A}_N(t,d)$ be the s which gives the largest term in the sum in (4.37). Then

$$1 - F_N(t) \leq Ne^{-ND} + N^2(d-t)e^{N\beta(\Phi_V+\delta-\frac{1}{2}V_N(s_N))}H_{N-1,N}(s_N+\epsilon, s_N). \quad (4.38)$$

Choose a sequence which gives the upper limit of $N^{-1}\log(1 - F_N(t))$ and such that $s_{N_j} \to \sigma \in [t,d]$. We would like to prove that

$$\lim_{j\to\infty} \frac{1}{N_j} \log H_{N_j-1,N_j}(s_{N_j}+\epsilon, s_{N_j}) = -\beta \int \log|\sigma+\epsilon-t|\phi_V^\sigma(t)dt. \quad (4.39)$$

We will write N instead of N_j for simplicity. Looking at the definition of $H_{N-1,N}(t,s)$, we see that we are interested in the limit of

$$\frac{1}{N}\log E_{N-1,N}[e^{\beta\sum_{j=1}^{N-1}\log|s_N+\epsilon-x_i|}; s_N]$$

as $N \to \infty$, $s_N \to \sigma$. Since

$$|\log|s_N+\epsilon-x_i| - \log|\sigma+\epsilon-x_i|| = |\log|1+\frac{s_N-\sigma}{\sigma+\epsilon-x_i}|| \leq C\frac{|s_N-\sigma|}{\sigma+\epsilon-x_i}, \quad (4.40)$$

where C is a numerical constant, and $s_N \leq \sigma + \epsilon/2$ for N large enough, the limit (4.39) follows from Corollary 4.4.

If $t > b_V$, then $\phi_V^\sigma = \phi_V$, since $\sigma \geq t$, and combining (4.38) and (4.39) yields

$$\begin{aligned}&\limsup_{N\to\infty} \frac{1}{N}\log(1-F_N(t)) \\ &\leq \max\{-D, \beta\Phi_V + \delta - \frac{\beta}{2}V(\sigma) - \beta\int \log|\sigma+\epsilon-t|^{-1}\phi_V(t)dt\}.\end{aligned} \quad (4.41)$$

Note that σ could depend on ϵ and d. Pick a sequence $\epsilon = \epsilon_j \to 0+$ and then a subsequence so that $\sigma(\epsilon_{j_k}) \to \tau \in [t,d]$. Then, since D and δ are arbitrary, we get

$$\limsup_{N\to\infty} \frac{1}{N}\log(1-F_N(t)) \leq \beta(\Phi_V - \inf_{\tau\geq t}\int k_V(\tau,s)\phi_V(s)ds) \quad (4.42)$$

and we have proved one half of (2.17).

We now turn to the lower limit. If we start with $M = N-1$ instead of N then (4.42) holds with F_{N-1} replaced by $F_{N-1,N}(t)$. By assumption the right-hand side of (4.42) is negative for all $t > b_V$. Hence, if $t > b_V$, we see that

$$F_{N-1,N}(t) \geq 1/2 \quad (4.43)$$

for all sufficiently large N. Note that, if $t \geq s$, then

$$H_{N-1,N}(t) \geq \frac{Z_{N-1,N}(s)}{Z_{N-1,N}(t)}H_{N-1,N}(t,s) \geq F_{N-1,N}(s)H_{N-1,N}(t,s). \quad (4.44)$$

The function $f(\tau) = \int k_V(\tau, s)\phi_V(s)ds$ is continuous on $[t, \infty)$ and $f(\tau) \to \infty$ as $\tau \to \infty$, so it assumes its minimum in $[t, \infty)$ at some point $\tau_0 \geq t$. Let $\epsilon > 0$. Pick $s_N \in \mathbb{A}_N(t)^*$ such that $s_N \searrow \tau_0 + \epsilon$. Then, picking one term in the sum

$$\begin{aligned}&\sum_{s \in \mathbb{A}_N(t)^*} e^{-\frac{N\beta}{2}V_N(s)} F_{N-1,N}(s) H_{N-1,N}(s)\\ &\geq e^{-\frac{N\beta}{2}V_N(s_N)} F_{N-1,N}(\tau_0)^2 H_{N-1,N}(s_N, s_N - \epsilon).\end{aligned}$$

If we use the limit (4.39), the estimate (4.43) with $s = \tau_0$, and let $\epsilon \to 0+$, we see that

$$\begin{aligned}&\liminf_{N\to\infty} \frac{1}{N} \log \sum_{s \in \mathbb{A}_N(t)^*} e^{-\frac{N\beta}{2}V_N(s)} F_{N-1,N}(s) H_{N-1,N}(s)\\ &\geq -\frac{\beta}{2} V(\tau_0) - \beta \int \log |\tau_0 - t|^{-1} \phi_V(t) dt.\end{aligned} \tag{4.45}$$

To complete the proof we need

Lemma 4.6. *For any V_N satisfying the conditions (i)–(iii),*

$$\liminf_{N\to\infty} \frac{1}{N} \log \frac{Z_{N-1,N}}{Z_{N,N}} \geq \beta \Phi_V. \tag{4.46}$$

Proof. If we let $t \to \infty$ in (4.19), we see that, $\epsilon > 0$,

$$\begin{aligned}\frac{Z_{N,N}}{Z_{N-1,N}} &= N \sum_{s \in \mathbb{A}_N} e^{-\frac{N\beta}{2}V_N(s)} F_{N-1,N}(s) H_{N-1,N}(s)\\ &\leq N \sum_{s \in \mathbb{A}_N(b_V - \epsilon)} e^{-\frac{N\beta}{2}V_N(s)} F_{N-1,N}(s) H_{N-1,N}(s)\\ &\quad + N \sum_{s \in \mathbb{A}_N(b_V - \epsilon)^*} e^{-\frac{N\beta}{2}V_N(s)} H_{N-1,N}(s),\end{aligned} \tag{4.47}$$

since $F_{N-1,N}(s) \leq 1$. By adjusting the constant C we see that (4.34) holds for all $s \in \mathbb{A}_N$, so the first sum in the right-hand side of (4.47) is

$$\leq e^{CN} F_{N-1,N}(b_V - \epsilon) \sum_{s \in \mathbb{A}_N} e^{-\frac{\beta}{2}N\xi \log(1+s^2)} \leq e^{CN - \frac{\beta}{2}L(b_V - \epsilon)N^2}$$

for all sufficiently large N by the first part of Theorem 2.2. (Replacing $F_N(t)$ by $F_{N-1,N}(t)$ does not make any difference.) Since $L(b_V - \epsilon) > 0$ if $\epsilon > 0$, the first part of the right hand side of (4.47) is negligible.

The same argument that lead us from (4.33) to (4.42) allows us to treat the second term in the right-hand side of (4.47) and obtain

$$\begin{aligned}&\limsup_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,N}}{Z_{N-1,N}}\\ &\leq \max\{-D, -\frac{\beta}{2} V(\sigma) - \beta \int \log |\sigma + \eta - t|^{-1} \phi_V^\sigma(t) dt\},\end{aligned} \tag{4.48}$$

where $\sigma \in [b_V - \epsilon, d]$, $\eta > 2\epsilon$, $D > 0$ are given. Take $\epsilon = \epsilon_j \to 0+$ so that $\sigma(\epsilon_j) \to \tau \in [b_V, d]$. Note that $\phi_V^{\sigma(\epsilon_j)}(t)dt$ converges weakly to $\phi_V^{\tau}(t)dt = \phi_V(t)dt$. Using an inequality like (4.40) we get

$$\begin{aligned}&\limsup_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,N}}{Z_{N-1,N}} \\ &\le \max\{-D, -\frac{\beta}{2}V(\tau) - \beta \int \log|\tau+\eta-t|^{-1}\phi_V(t)dt\}.\end{aligned} \tag{4.49}$$

We can now repeat the argument that leads from (4.41) to (4.42) and obtain

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N} \log \frac{Z_{N,N}}{Z_{N-1,N}} &\le \frac{\beta}{2} \int V(s)\phi_V(s)ds \\ &- \beta \inf_{\tau \ge b_V} \int k_V(\tau, s)\phi_V(s)ds \le -\beta\Phi_V,\end{aligned}$$

since $\int k_V(\tau, s)\phi_V(s)ds \ge F_V$ if $\tau \ge b_V$. The lemma is proved. □

Combining (4.20), (4.45) and Lemma 4.6, we see that

$$\begin{aligned}&\liminf_{N\to\infty} \frac{1}{N} \log(1 - F_N(t)) \\ &\ge \beta(F_V - \int k_V(\tau_0, s)\phi_V(s)ds) = \beta(F_V - \inf_{\tau \ge t} \int k_V(\tau, s)\phi_V(s)ds),\end{aligned}$$

by the choice of τ_0. This completes the proof of Theorem 2.2.

5. Asymptotics for the Meixner Kernel

This section is devoted to the proof of Lemma 3.2, which is based on establishing the appropriate asymptotics of the Meixner polynomials. See [Go] and [JW] for some results on the asymptotics of Meixner polynomials.

From (3.3) we obtain, $x \in \mathbb{R}$,

$$\begin{aligned}m_n^{K,q}(x) &= (-1)^n \frac{(\sqrt{\gamma})^{n+K} n!}{(\sqrt{q})^n 2\pi i} \int_{\Gamma_r} \left(\frac{\sqrt{\gamma} + z/\sqrt{q}}{\sqrt{\gamma} + \sqrt{q}z}\right)^x \frac{dz}{(\sqrt{\gamma} + \sqrt{q}z)^K z^{n+1}} \\ &- \frac{\sin \pi x}{\pi} \frac{(\sqrt{\gamma})^{n+K} n!}{(\sqrt{q})^n} \int_{\sqrt{\gamma q}}^r \left|\frac{\sqrt{\gamma} - t/\sqrt{q}}{\sqrt{\gamma} - \sqrt{q}t}\right|^x \frac{dt}{(\sqrt{\gamma} - \sqrt{q}t)^K t^{n+1}},\end{aligned} \tag{5.1}$$

where Γ_r is the circle $|z| = r$, $0 < r < \sqrt{\gamma/q}$; if $0 < r \le \sqrt{\gamma q}$ the second integral should be omitted. Let $b = (1 + \sqrt{\gamma q})^2/(1 - q)$ as before, let σ be given by (1.11) and set

$$a = b + \gamma - 1 = \frac{(\sqrt{\gamma} + \sqrt{q})^2}{1 - q}.$$

Set

$$\begin{aligned}t(z) &= \left(\frac{\sqrt{\gamma q} + z}{\sqrt{\gamma q} + 1}\right)\left(\frac{\sqrt{\gamma} + \sqrt{q}}{\sqrt{\gamma} + \sqrt{q}z}\right), \\ s(z) &= \frac{\sqrt{\gamma} + \sqrt{q}}{\sqrt{\gamma} + \sqrt{q}z},\end{aligned}$$

and

$$A_N(x) = \frac{b^x}{x^{x+K}} \frac{(x+K-1)!N!}{x!(N+K-2)!} \frac{\gamma^{K+N}}{1-q} \sqrt{\frac{q}{\gamma}}.$$

For $0 < r < \sqrt{\gamma/q}$ we define

$$D_n^r(x; g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(re^{i\theta}) t(re^{i\theta})^x s(re^{i\theta})^K \frac{d\theta}{r^n e^{in\theta}}, \tag{5.3}$$

$F_n^r(x; g) = 0$ if $0 < r \le \sqrt{\gamma q}$, and if $\sqrt{\gamma q} < r < \sqrt{\gamma/q}$, then

$$F_n^r(x; g) = (-1)^{n+x+1} \int_{\sqrt{\gamma q}}^{r} |t(-\tau)|^x s(-\tau)^K g(-\tau) \frac{d\tau}{\tau^{n+1}}. \tag{5.4}$$

The powers are defined by taking the prinipal branch of the logarithm.

The Meixner kernel (3.4) can now be written, for x, y *integers* (which is the case we need),

$$K_N(x, y) = \sqrt{A_N(x)A_N(y)} \frac{D_N(x; g_1)D_N(y; g_2) - D_N(x; g_2)D_N(y; g_1)}{x - y} \tag{5.5}$$

if $x \ne y$, and

$$\begin{aligned} K_N(x, x) = A_N(x)[&D_N(x-1; g_3)D_N(x; g_2) - D_N(x; g_1)D_N(x-1; g_4) \\ &+ F_N(x; g_1)D_N(x; g_2) - F_N(x; g_2)D_N(x; g_1)], \end{aligned} \tag{5.6}$$

where $g_1(z) \equiv 1$, $g_2(z) = z - 1$, $g_3(z) = t(z) \log t(z)$ and $g_4(z) = g_2(z)g_3(z)$. The functions $g_i(z)$ are bounded for $|z| \le 1$.

Write $x = Nb + y$ and $K = [\gamma N] - N + 1 \doteq N(\gamma_N - 1) \doteq N(\gamma - 1) + \omega_N$, $0 < \omega_N \le 1$.

Lemma 5.1. *If $x = Nb + \xi\sigma N^{1/3}$ and $M_0 > 0$ is a given constant, there are constants $c_1(q, \gamma)$ and $c_2(q, \gamma)$, such that*

$$\frac{1}{N} A_N(x) \le c_1(q, \gamma) e^{c_2(q,\gamma)\xi N^{-2/3}} \tag{5.7}$$

for all $\xi \ge -M_0$. Furthermore,

$$\lim_{N\to\infty} \frac{1}{N} A_N(x) = \frac{\gamma\sqrt{q}}{(1-q)\sqrt{ab}} \tag{5.8}$$

uniformly for $|\xi| \le M_0$.

Proof. By Stirling's formula

$$\begin{aligned} A_N(x) = &\frac{(x+K)^{x+K} N^N b^x}{x^x (N+K)^{N+K} a^{x+K}} \gamma^{K+N} \frac{(N+K)(N+K-1)}{x+K} \\ &\times \sqrt{\frac{(x+K)N}{x(N+K)}} \frac{1}{1-q} \sqrt{\frac{q}{\gamma}} e^{o(1)}. \end{aligned} \tag{5.9}$$

Write $a_N = b + \gamma_N - 1$. Then,

$$\begin{aligned}&\frac{(x+K)^{x+K}N^N b^x}{x^x(N+K)^{N+K}a^{x+K}}\gamma^{K+N}\\&=\left(\frac{Nb}{x}\right)^x\left(\frac{x+K}{Na_N}\right)^{x+K}\left(\frac{a_N}{a}\right)^{x+K}\left(\frac{\gamma}{\gamma_N}\right)^{N+K}.\end{aligned}\tag{5.10}$$

If we write $u = Na_N$ and $v = Nb < u$. Then

$$\left(\frac{Nb}{x}\right)^x\left(\frac{x+K}{Na_N}\right)^{x+K} = \left(1+\frac{y}{u}\right)^{u+y}\left(1+\frac{y}{v}\right)^{-v-y} \doteq e^{g(y)}.$$

Since $g(0) = g'(0) = 0$ and $g''(t) = (v-u)(u+t)^{-1}(v+t)^{-1} < 0$, we have $\exp g(t) \le 1$ if $\xi \ge 0$. If $-M_0 \le \xi \le M_0$, then

$$|g(t)| = |\int_0^t (t-s)g''(s)ds| \le CN^{-1/3}.$$

Furthermore

$$\left(\frac{a_N}{a}\right)^{x+K} = e^{\omega_N + O(\xi N^{-2/3}) + o(1)}$$

and

$$\left(\frac{\gamma}{\gamma_N}\right)^{K+N} = e^{-\omega_N + o(1)}.$$

Inserting these estimates into (5.10) we obtain

$$\frac{(x+K)^{x+K}N^N b^x}{x^x(N+K)^{N+K}a^{x+K}}\gamma^{K+N} \le Ce^{C\xi N^{-2/3}}$$

for $\xi \ge -M_0$ and

$$\lim_{N\to\infty}\frac{(x+K)^{x+K}N^N b^x}{x^x(N+K)^{N+K}a^{x+K}}\gamma^{K+N} = 1$$

uniformly for $|\xi| \le M_0$. By (5.9) this proves (5.7) and (5.8). The lemma is proved. □

Set

$$u(z) = b\log(\sqrt{\gamma q} + z) - a\log(\sqrt{\gamma} + \sqrt{q}z) - \log z$$

so that

$$D_N^r(x;g) = \frac{1}{2\pi}\int_{\Gamma_r} e^{N(u(z)-u(1)) + y\log t(z) + \omega_N \log s(z)} g(z)\frac{dz}{iz}.\tag{5.11}$$

Now,

$$\begin{aligned}u'(z) &= -\rho(1-z)^2\\&+\rho(1-z)^3\frac{\sqrt{q}z^2 + (\sqrt{q}+\sqrt{\gamma}+q\sqrt{\gamma})z + \sqrt{q}+\sqrt{\gamma}+q\sqrt{\gamma}+\gamma\sqrt{q}}{z(z+\sqrt{\gamma q})(\sqrt{\gamma}+\sqrt{q}z)},\end{aligned}$$

where

$$\rho = \frac{\gamma\sqrt{q}}{(1+\sqrt{\gamma q})(\sqrt{\gamma}+\sqrt{q})}.$$

Hence we can write

$$u(z) - u(1) = \frac{1}{3}\rho(1-z)^3 + \rho(1-z)^4 v(z), \tag{5.12}$$

where one verifies that $|v(z)| \le 28/27$ if $|z-1| \le 1/4$.

By taking absolute values in (5.3) we obtain

$$|D_N^r(x; g)| \le \frac{C}{2\pi}\left(\frac{a}{b}\right)^{x/2} \frac{a^K(1-q)^K}{r^N} \int_{-\pi}^{\pi} e^{f(\cos\theta)} d\theta, \tag{5.13}$$

where

$$f(\tau) = \frac{x}{2}\log(\gamma q + r^2 + 2\sqrt{\gamma q} r\tau) + \frac{x-K}{2}\log(\gamma + qr^2 + 2\sqrt{\gamma q} r\tau).$$

Write $r = 1-\delta$, $0 \le \delta < 1$. A computation shows that $f'(\tau) \ge 0$ if (say)

$$y \ge -\delta \frac{1 + q + 2\sqrt{\gamma q}}{1-q} N, \tag{5.14}$$

which covers all the y's we are interested in. Thus, if (5.14) is fullfilled, then

$$|D_N^r(x; g)| \le C \exp(N(u(1-\delta) - u(1)) + y \log t(1-\delta)). \tag{5.15}$$

By (5.12),

$$u(1-\delta) - u(1) \le \rho\delta^3(\frac{1}{3}\delta\frac{28}{27}) \le \frac{2}{3}\rho\delta^3 \tag{5.16}$$

if $0 \le \delta \le 1/4$. Now,

$$\begin{aligned}
\log t(1-\delta) &= \log\Big(1 - \frac{1}{1 - \frac{\sqrt{q}}{\sqrt{\gamma}+\sqrt{q}}\delta} \frac{(1-q)\sqrt{\gamma}}{(1+\sqrt{\gamma q})(\sqrt{\gamma}+\sqrt{q})}\delta\Big) \\
&\le -\rho(1-q)\frac{1}{\sqrt{\gamma q}}\delta,
\end{aligned}$$

and consequently it follows from (5.15) and (5.16) that, if $y \ge 0$, then

$$|D_N^r(x; g)| \le C[\exp\Big[\frac{2N}{3}\rho\delta^3 - \rho(1-q)\frac{1}{\sqrt{\gamma q}}\delta y\Big]. \tag{5.17}$$

Recall that $y = \sigma N^{1/3}\xi$ with σ given by (1.11). Note that $\sigma = (1-q)^{-1}\sqrt{\gamma q}\rho^{-2/3}$. Choose $\delta = (\rho N)^{-1/3}\sqrt{\xi}$ if $\xi \le (N\rho)^{2/3}/16$ and $\delta = 1/4$, if $\xi \ge (N\rho)^{2/3}/16$. Inserting this into (5.17) gives

$$|D_N^r(x; g)| \le C \exp\Big[-\frac{1}{3}\min(\sqrt{\xi}, \frac{1}{4}(N\rho)^{1/3})\xi\Big], \tag{5.18}$$

for $\xi \ge 0$.

Let $\epsilon \in [0, \pi]$ and set

$$\begin{aligned}
I_1' &= \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} g(re^{i\theta}) t(re^{i\theta})^x s(re^{i\theta})^K \frac{d\theta}{r^N e^{iN\theta}}, \\
I_1'' &= D_N^r(x; g) - I_1'.
\end{aligned}$$

By the same argument that was used for (5.13) above, we see that if y satisfies (5.14), then

$$\begin{aligned} |I_1''| &\le C|t(re^{i\epsilon})|^x|s(re^{i\epsilon})|^K\frac{1}{r^N} \\ &\le C\exp\big[N\mathrm{Re}\,(u(re^{i\epsilon})-u(1))+y\log|t(re^{i\epsilon})|\big]. \end{aligned} \tag{5.19}$$

Next, we consider $F_N^r(x;g)$, $\sqrt{\gamma q}<r\le 1$. Taking absolute values in (5.4) yields

$$|F_N^r(x;g)|\le C\int_{\sqrt{\gamma q}}\left|\frac{\sqrt{\gamma q}-\tau}{\sqrt{\gamma q}+1}\right|^x\left|\frac{\sqrt{\gamma}+\sqrt{q}}{\sqrt{\gamma}-\sqrt{q}\tau}\right|^{x+K}\frac{d\tau}{\tau^{N+1}}. \tag{5.20}$$

The integrand in (5.20) is a increasing function of τ for all x that we are considering. The monotonicity argument used for (5.13) now shows that, if (5.14) is fulfilled, then

$$\begin{aligned} |F_N^r(x;g)| &\le C|t(-r)|^x|s(-r)|^K\frac{1}{r^N} \\ &\le C|t(re^{i\epsilon})|^x|s(re^{i\epsilon})|^K\frac{1}{r^N} \\ &\le C\exp\big[N\mathrm{Re}\,(u(re^{i\epsilon})-u(1))+y\log|t(re^{i\epsilon})|\big], \end{aligned} \tag{5.21}$$

where the last inequality is the same as in (5.19). If we take $\epsilon=0$, we get the same right-hand side as in (5.15) and hence we obtain the same estimates, i. e.

$$|F_N^r(x;g)|\le C\exp\big[-\frac{1}{3}\min(\sqrt{\xi},\frac{1}{4}(N\rho)^{1/3})\xi\big].$$

Combining this with (5.6), (5.7) and (5.18) yields

$$|K_N(x,x)|\le CN\exp\big[-\frac{1}{4}\min(\sqrt{\xi},\frac{1}{4}(N\rho)^{1/3})\xi\big] \tag{5.22}$$

for any $\xi\ge 0$; x an integer.

Consider now $\xi\in[-M_0,(\rho N)^{1/6}]$. Take $\epsilon=(\rho N)^{-1/4}$, $\delta=\eta(\rho N)^{-1/3}\le(\rho N)^{-1/4}$, where $\eta>0$ will be chosen below. By (5.12), we have

$$\begin{aligned} I_1' = \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} & g((1-\delta)e^{i\theta})\exp\big\{N\big[\frac{1}{3}\rho(1-(1-\delta)e^{i\theta})^3 \\ &+\rho(1-(1-\delta)e^{i\theta})^4v((1-\delta)e^{i\theta})\big]+y\log t((1-\delta)e^{i\theta}) \\ &+\omega_N\log s((1-\delta)e^{i\theta})\big\}d\theta. \end{aligned} \tag{5.23}$$

We make the change of variables $\theta=\omega(\rho N)^{-1/3}$. For $0<\eta\le(\rho N)^{1/12}$, $|\theta|\le\epsilon$, we have

$$\begin{aligned} &\frac{1}{3}\rho(1-(1-\delta)e^{i\theta})^3+\rho(1-(1-\delta)e^{i\theta})^4v((1-\delta)e^{i\theta}) \\ &=\frac{1}{3}(\eta-i\omega)^3+R_1, \end{aligned} \tag{5.24}$$

where $R_1\to 0$ uniformly as $N\to\infty$. Furthermore, if $\xi\in[-M_0,(\rho N)^{1/6}]$, then

$$y\log t((1-\delta)e^{i\theta})=(-\eta+i\omega)\xi+R_2, \tag{5.25}$$

where $R_2\to 0$ uniformly as $N\to\infty$.

Suppose $g^{(j)}(1) = 0$, $j = 0, \dots, \ell - 1$ but $g^{(\ell)}(1) \neq 0$, so that

$$g((1-\delta)e^{i\theta}) = \frac{1}{\ell!} g^{(\ell)}(1)\rho^{-\ell/3}(-\eta + i\omega)^{\ell} + \dots . \tag{5.26}$$

We now have all the estimates we need. Let $\eta = \sqrt{\xi}$ if $\xi \geq M_0$ and $\eta = 1$ if $|\xi| \leq M_0$.
By (5.12) and (5.24) we obtain

$$\operatorname{Re} Nu((1-\delta)e^{i\theta}) = \frac{1}{3}\eta^3 - \eta\omega^2 + R_1$$

and hence, if $\xi \in [-M_0, (\rho N)^{1/6}]$, $\epsilon = \omega(\rho N)^{-1/3}$ with $\omega = (\rho N)^{1/12}$,(5.19) yields,

$$\begin{aligned} |I_1''| &\leq C \exp\big[\frac{1}{3}\eta^3 - \eta(\rho N)^{1/6} - \eta\xi + R_3\big] \\ &\leq \frac{C}{N^{(\ell+1)/3}} \exp\big[-\frac{2}{3}|\xi|^{3/2}\big]. \end{aligned} \tag{5.27}$$

Similarly, by (5.21), for $\xi \in [-M_0, (\rho N)^{1/6}]$,

$$|I_1'| \leq \frac{C}{N^{(\ell+1)/3}} \exp\big[-\frac{2}{3}|\xi|^{3/2}\big]. \tag{5.29}$$

The dominated convergence theorem gives

$$\begin{aligned} &\lim_{N\to\infty} N^{(\ell+1)/3} I_1' \\ &= \frac{\rho^{-(\ell+1)/3}}{\ell!} g^{(\ell)}(1)\frac{1}{2\pi}\int_{-\infty}^{\infty} (-\eta + i\omega)^{\ell} \exp\big[\frac{i}{3}(\omega + i\eta)^3 + i\xi(\omega + i\eta)\big] d\omega \\ &= \frac{\rho^{-(\ell+1)/3}}{\ell!} g^{(\ell)}(1)\operatorname{Ai}^{(\ell)}(\xi), \end{aligned} \tag{5.30}$$

uniformly for $|\xi| \leq M_0$. Observe that $g_1(1) = 1$,$g_2(1) = 0$ but $g_2'(1) = 1$, $g_3(1) = 0$ but $g_3'(1) = \rho(1-q)(\gamma q)^{-1/2}$ and $g_4(1) = g_4'(1) = 0$ but $g_4''(1) = 2\rho(1-q)(\gamma q)^{-1/2}$.
Combining (5.27) and (5.29) we obtain

$$|D_N^r(x; g)| \leq \frac{C}{N^{(\ell+1)/3}} \exp\big[-\frac{2}{3}|\xi|^{3/2}\big], \tag{5.31}$$

for $\xi \in [-M_0, (\rho N)^{1/6}]$. The estimate (5.27) and the limit (5.30) give

$$\lim_{N\to\infty} N^{1/3} D_N^r(x; g_1) = \rho^{-1/3}\operatorname{Ai}(\xi), \tag{5.32a}$$

$$\lim_{N\to\infty} N^{2/3} D_N^r(x; g_2) = \rho^{-2/3}\operatorname{Ai}'(\xi), \tag{5.32b}$$

$$\lim_{N\to\infty} N^{2/3} D_N^r(x; g_3) = \frac{\rho^{1/3}(1-q)}{\sqrt{\gamma q}}\operatorname{Ai}'(\xi), \tag{5.32c}$$

and

$$\lim_{N\to\infty} N D_N^r(x; g_4) = \frac{(1-q)}{\sqrt{\gamma q}}\operatorname{Ai}''(\xi). \tag{5.32d}$$

We can now use (5.22), (5.28), (5.31) and (5.32) in (5.5) and (5.6) to prove (3.6), (3.7) and (3.8) for the Meixner kernel. The lemma is proved.

6. The Equilibrium Measure

The equilibrium measure $\phi_V(t)dt$ satisfies certain variational conditions.

Proposition 6.1. *Assume that* $\phi \in \mathcal{A}_s$ *satisfies*

(i) $\int_0^s k_V(t,\tau)\phi(\tau)d\tau \geq \lambda$ *if* $\phi(t) = 0$,
(ii) $\int_0^s k_V(t,\tau)\phi(\tau)d\tau \leq \lambda$ *if* $\phi(t) = 1$,
(iii) $\int_0^s k_V(t,\tau)\phi(\tau)d\tau = \lambda$ *if* $0 < \phi(t) < 1$, *for some* λ *(which* $= F_V$*). Then* $\phi = \phi_V$.

We will not prove this here, see [LL] for a very similar result. The way to compute ϕ_V is to seek a candidate solution ϕ and then verify that ϕ satisfies the variational conditions. In a region where $0 < \phi(t) < 1$ we can differentiate (iii) and obtain

$$\int_0^s \frac{\phi(\tau)}{\tau - t}d\tau = -\frac{1}{2}V'(t). \tag{6.1}$$

Since $V^{\gamma,q}$ is convex the support of ϕ_V is a single interval. If we consider the variational problem without the constraint $\phi \leq 1$, and this problem has a solution ψ_0 such that $0 \leq \psi_0 \leq 1$, then this ψ_0 is the solution we are seeking. This is the case when $\gamma \geq 1/q$, and then $[a_V, b_V] = [a, b]$ and

$$\int_a^b \frac{\phi(\tau)}{\tau - t}d\tau = -\frac{1}{2}V'(t), \quad a \leq t \leq b. \tag{6.2}$$

We must have $\phi(b) = 0$ and $\phi(a)$ bounded ($\phi(a) = 0$ if $\gamma > 1/q$).

If the solution $\psi_0(t) > 1$ in some interval, e.g. $\psi_0(t) > 1$ in $[0, a_0)$ but $0 < \psi_0(t) < 1$ in (a_0, b_0), we make an ansatz that $\phi(t) = 1$ in $[0, a]$ and $0 < \phi(t) < 1$ in (a, b) for some a, b, $[a_V, b_V] = [0, b]$. This is the situation when $\gamma < 1/q$. By (6.1),

$$\int_a^b \frac{\phi(\tau)}{\tau - t}d\tau = -\frac{1}{2}V'(t) - \int_0^a \frac{d\tau}{\tau - t}, \tag{6.3}$$

and $\phi(a) = 1$, $\phi(b) = 0$. By making the substitution $x = 2(t-a)/c - 1$, $y = 2(\tau - a)/c - 1$, $c = b - a$, in (6.2) and (6.3) we get an equation of the form

$$\frac{1}{\pi}\int_{-1}^1 \frac{v(x)}{x - y}dx = f(y), \quad -1 \leq x \leq 1, \tag{6.4}$$

with some f. This equation has the general solution, [Tr],

$$v(x) = -\frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^1 \frac{f(y)\sqrt{1-y^2}}{y - x}dy + \frac{C}{\pi\sqrt{1-x^2}},$$

where C is an arbitrary constant. In this way we obtain (2.19) and (2.20).

Equation (2.21) is obtained by substituting (2.19) or (2.20) into (2.15) (the infimum is assumed for $\tau = t$). Consider the case $\gamma > 1/q$, the other case is similar. Then, with $t = a + c(x+1)/2$,

$$J(t) = \int_b^t J'(s)ds = \frac{c}{2}\int_1^x J'(a + c(y+1)/2)dy$$

and

$$g(y) \doteq J'(a+c(y+1)/2) = \frac{c}{2}\int_{-1}^{1}\frac{v(x)}{x-y}dx + \frac{1}{2}V'(a+c(y+1)/2)$$
$$= \frac{c}{2}\int_{-1}^{1}\log|y-x|v'(x)dx + \frac{1}{2}[\log\frac{1}{q} - \log(y+B) + \log(y+D)].$$

Now,

$$v'(x) = \frac{1}{2\pi}\left[\frac{\sqrt{D^2-1}}{x+D} - \frac{\sqrt{B^2-1}}{x+B}\right]\frac{1}{\sqrt{1-x^2}}$$

and

$$\int_{-1}^{1}\log|y-x|v'(x)dx = \frac{1}{2}F(y,D) - \frac{1}{2}F(y,B),$$

where

$$F(y,R) = \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{R^2-1}}{(x+R)\sqrt{1-x^2}}\log|y-x|dx.$$

Note that

$$\frac{d}{dy}F(y,R) = \frac{\sqrt{R^2-1}}{y+R}[\frac{1}{\sqrt{y^2-1}} + \frac{1}{\sqrt{R^2-1}}].$$

Using these formulas we see that $g(-1)=0$ and hence

$$J(t) = \frac{c}{4}\int_1^x g(y)dy = \frac{c}{4}\int_1^x (x-y)g'(y)dy$$
$$= \frac{c}{4}\int_1^x (x-y)(\frac{\sqrt{B^2-1}}{x+B} - \frac{\sqrt{D^2-1}}{x+D})\frac{dy}{\sqrt{y^2-1}},$$

which gives (2.21).

If $f(y) = (\gamma - q)(y+B)^{-1} + (1-q\gamma)(y+D)^{-1}$, then $f(y) > 0$ for all $y \geq 1$ and $a_0 = \inf_{1\leq y\leq 1/c} f(y) > 0$. Thus for $0 \leq \delta \leq 1$, by (2.21),

$$J(b+\delta) \geq \frac{a_0 c}{8\sqrt{q\gamma}}\int_1^{1+\delta/c}(1 - \frac{2\delta}{c} - y)\frac{dy}{\sqrt{y+1}\sqrt{y-1}} \geq c_1\delta^{3/2},$$

for some constant $c_1 > 0$. If $\delta \geq 1$, then

$$J(b+\delta) \geq \frac{a_0 c}{8\sqrt{q\gamma}}\int_1^{1+1/c}(1 - \frac{2\delta}{c} - y)\frac{dy}{\sqrt{y+1}\sqrt{y-1}},$$

which proves (2.22). A more careful computation for small δ yields (2.23).

Acknowledgements. I thank C. Newman for drawing my attention to the fact that the exponent $\chi = 1/3$ occurs in many problems. I want to express my gratitude to A. Dembo and P. Diaconis for telling me about the interpretation of $G(M,N)$ in terms of randomly growing Young diagrams and the exclusion process. Remark 2.6 was motivated by discussions with J. Baik. Finally I thank the referee for pointing out some mistakes in a previous version. This work was supported by the Swedish Natural Science Research Council (NFR).

References

[BR] Baik, J., Rains, E.: The asymptotics of monotone subsequences of involutions. math. CO/9905084

[BDJ] Baik, J., Deift, P.A. and Johansson, K.: On the distribution of the longest increasing subsequence in a random permutation. J. Amer. Math. soc. **12**, 1119–1178 (1999)

[BDG] Ben Arous, G., Dembo, A., Guionnet, A.: Ageing of Spherical Spin Glasses. Preprint

[BG] Ben Arous, G., Guionnet, A.: Large Deviations for Wigner's Law and Voiculescu's Non-commutative Entropy. Probab. Theory Relat. Fields, **108**, 517–542 (1997)

[BO] Borodin, A., Olshanski, G.: Statistics on Partitions, Point Processes and the Hypergeometric Kernel. math.CO/9904010

[Ch] Chihara, T.S.: *An Introduction to Orthogonal Polynomials*. New York: Gordon and Breach, 1978

[De] Deift, P.A.: *Orthogonal polynomials and random matrices: A Riemann–Hilbert approach*. New York: Courant Lecture Notes in Mathematics, **3**, 1999

[DS] Dragnev, P.D., Saff, E.B.: Constrained energy problems with applications to orthogonal polynomials of a discrete variable. J. Anal. Math. **72**, 223–259 (1997)

[Fu] Fulton, W.: *Young Tableaux*. London Mathematical Society, Student Texts **35**, Cambridge: Cambridge Univ. Press, 1997

[Go] Goh, W.M.Y.: Plancherel–Rotach Type Asymptotics of the Meixner Polynomials. Preprint

[HP] Hiai, F., Petz, D.: A Large Deviation Theorem for the Empirical Eigenvalue Distribution of Random Unitary Matrices. Math. Inst. of the Hungarian Academy of Sciences, Preprint No. 17/1997

[HJ] Horn, R.A., Johnson, C.R.: *Matrix Analysis*. Cambridge: Cambridge Univ. Press, 1985

[Hö] Hörmander, L.: *The Analysis of Linear Partial Differential Operators I*. Berlin–Heidelberg: Springer-Verlag, 1983

[Ja] James, A.T.: Distributions of Matrix Variates and Latent Roots Derived from Normal Samples. Ann. Math. Statist. **35**, 475–501 (1964)

[JW] Jin, X.-S., Wang, R.: Uniform Asymptotic Expansions for Meixner Polynomials. Constr. Approx. **14**, 113–150 (1998)

[JPS] Jockush, W., Propp, J., Shor, P.: Random domino tilings and the arctic circle theorem. Preprint 1995, math.CO/9801068

[Jo] Johansson, K.: On Fluctuations of Eigenvalues of Random Hermitian Matrices. Duke Math. J. **91**, 151–204 (1998)

[Kn] Knuth, D.E.: Permutations, Matrices and Generalized Young Tableaux. Pacific J. Math. **34**, 709–727 (1970)

[KS] Krug, J., Spohn, H.: Kinetic Roughening of Growing Interfaces. In: *Solids far from Equilibrium: Growth, Morphology and Defects,* Ed. C. Godrèche, Cambridge: Cambridge University Press, 1992, pp. 479–582

[LL] Lax, P.D., Levermore, C.D.: The Small Dispersion Limit of the Korteweg–deVries Equation. I. Commun. Pure and Appl. Math **36**, 253–290 (1983)

[Li] Ligget, T.M.: *Interacting particle Systems*. New York: Springer-Verlag, 1985

[NP] Newman, C.M., Piza, M.S.T.: Divergence of Shape Fluctuations in Two Dimensions. Ann. Prob. **23**, 977–1005 (1995)

[NSU] Nikiforov, A.F., Suslov, S.K., Uvarov, V.B.: *Classical Orthogonal Polynomials of a Discrete Variable*. Springer Series in Computational Physics, Berlin–Heidelberg: Springer-Verlag, 1991

[Pi] Piza, M.S.T.: Directed Polymers in a Random Environment: Some Results on Fluctuations. J. Stat. Phys. **89**, 581–603 (1997)

[Ro] Rost, H.: Non–Equilibrium Behaviour of a Many Particle Process: Density Profile and Local Equilibria. Zeitschrift f. Warsch. Verw. Gebiete, **58**, 41–53 (1981)

[ST] Saff, E.B., Totik,V.: *Logarithmic Potentials with External Fields*. Grundlehren der Matematischen Wissenschaften, **316**, Berlin: Springer-Verlag, 1997

[Sa] Sagan, B.: *The Symmetric Group*. Brooks/Cole Publ. Comp., 1991

[Se1] Seppäläinen, T.: Large Deviations for Increasing Subsequences on the Plane. Probab. Theory Relat. Fields **112**, 221–244 (1998)

[Se2] Seppäläinen, T.: Coupling the totally asymmetric simple exclusion process with a moving interface. Markov Process. Related Fields **4**, 593– 628 (1998)

[Sz1] Szegö, G.: *Orthogonal Polynomials*. American Mathematical Society Colloquium Publications, Vol. **XXII**, New York, 1939

[TW1] Tracy, C.A., Widom, H.: Level Spacing Distributions and the Airy Kernel. Commun. Math. Phys. **159**, 151–174 (1994)

[TW2] Tracy, C.A., Widom, H.: On Orthogonal and Symplectic Matrix Ensembles. Commun. Math. Phys. **177**, 727–754 (1996)

[TW3] Tracy, C.A., Widom, H.: Correlation Functions, Cluster Functions, and Spacing Distributions for Random Matrices. J. Statist. Phys. **92**, 809–835 (1998)

[Tr] Tricomi, F.G.: *Integral Equations*. Pure Appl. Math **V**, London: Interscience, 1957

Communicated by A. Kupiainen