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## SHAPE OPTIMISATION OF TWO-DIMENSIONAL ANISOTROPIC STRUCTURES USING THE BOUNDARY ELEMENT METHOD

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### ABSTRACT

A shape optimisation procedure is developed, using the boundary element method, for two-dimensional anisotropic structures to minimise weight while satisfying certain constraints upon stresses and geometry. A directly differentiated form of boundary integral equation with respect to geometric design variables is used to calculate shape design sensitivities of anisotropic materials. The boundary element method is very suitable for shape optimisation and in comparison with the finite element method needs fewer data, related only to the boundary of the structure being considered. Because a directly differentiated form of the boundary integral equation can be used to determine the derivatives of the objective and constraint functions, the accuracy of computation is very high. Because of the non-linear nature of weight and stresses, the numerical optimisation method used in the program is the feasible direction approach, together with the Golden Section method for the one-dimensional search. Three example problems with anisotropic material properties are presented to demonstrate the applications of this general purpose program.

**Keywords:** Shape optimisation, boundary element method, design sensitivity analysis, anisotropic materials

### NOTATION

$A$	matrix containing the integrals of the traction kernels
$A_{jk}$	complex constants
$B$	matrix containing the integrals of the displacement kernels
$C_{jk}(P)$	limiting value of the surface integral of $T_{jk}(p, Q)$
$D_s$	operator (s=1,4)
$E_k$	Young's modulus in the $x_k$ direction
$F$	Objective function
$G_{12}$	shear modulus
$J(\zeta)$	Jacobian of transformation from global Cartesian coordinates to intrinsic coordinates of the element

$m_{1k}, m_{2k}$	unit vectors tangent and normal to the surface
$n_1, n_2$	direction cosines of the unit outward normal vector to the surface of the elastic body
$N^c(\zeta)$	Quadratic shape function corresponding to the $c$ th node of the element
P	load point at the surface of the elastic domain
Q	field point at the surface of the elastic domain
$(R_i, \theta_i)$	polar coordinates
$r_{ij}$	complex constants
S	area of a component in 2D
$S_{mn}$	elastic compliance matrix
$\bar{S}_{jk}$	transformed lamina compliance matrix
$t_j$	traction vector
$T_{jk}(P, Q)$	$j$ th component of the traction vector at point Q due to a unit point load in the $k$ th direction at P
$u_j$	displacement vector
$U_{jk}$	$j$ th component of the displacement vector at point Q due to a unit point load in the $k$ th direction at P
$V_1, V_2, V_3, V_4$	Invariants
W	weight
$W_0$	initial weight
$x_i$	rectangular Cartesian coordinates
$z_j$	Complex coordinates
$\alpha_j, \beta_j$	real constants
$\gamma$	lamina orientation angle with respect to the x and y axes
$\delta_{jk}$	Kronecker delta
$\varepsilon_{jk}$	strain tensor
$\zeta$	intrinsic coordinates of isoparametric quadratic element
$\zeta_i$	coordinates of load point
$\Lambda_1, \Lambda_2$	real functions of the Cartesian and intrinsic coordinates respectively at each integration point
$\mu_s$	roots of the characteristic equation
$\nu_{jk}$	Poisson's ratio

$\sigma_{\max}$	maximum equivalent stress
$\sigma_{jk}$	stress tensor
$\phi$	Airy stress function
$\Omega_1, \Omega_2$	real functions of the Cartesian and intrinsic coordinates at each integration point

## 1. INTRODUCTION

Shape optimisation is an important area of current development in mechanical and structural design. Computerised procedures using optimisation algorithms can iteratively determine the optimum shape of a component while satisfying some objectives, without at the same time violating the design constraints. However, in this field of research the analysis has been mostly concentrated on isotropic materials.

The utilisation of composites in aerospace applications is well established today due to the known benefits such as high specific stiffness or strength and the materials tailoring facilities for creating high performance structures. It would be beneficial to apply the numerical shape optimisation algorithms for the design of anisotropic structures.

The boundary element method is an attractive alternative to the finite element technique for a wide range of applications in stress analysis, is particularly well established for linear elastic problems, and has a number of advantages over the finite element technique. It greatly simplifies mesh data preparation, because only the surface of the component or structure to be analysed needs to be discretised. It needs less computing time and storage for the same level of accuracy because within the solution domain the governing differential equations are satisfied exactly, rather than approximately as in the case of the finite element method. In using the boundary element method, less unwanted information about internal points is obtained. Since it is a surface-oriented technique, it is particularly well suited for shape optimisation problems.

This work presents the weight minimisation of two-dimensional anisotropic structures using the boundary element method. The steps that are required are as follows: shape representation, boundary element analysis to calculate stresses and displacements, design sensitivity analysis for calculating derivatives, numerical optimisation to find the optimum solution iteratively, and boundary element mesh re-generation as the optimisation proceeds. It should be noted that to the

author's knowledge no other publications are available on the shape optimisation of anisotropic structures.

In a recent study by the author [1], a directly differentiated form of boundary integral equation with respect to geometric design variables was used to calculate stress and displacement derivatives for 2D anisotropic structures. The accuracy was compared against the results of the finite difference applied to the boundary element analysis. Not surprisingly, results obtained by analytical differentiation are much more accurate. For the finite difference method to give the same level of accuracy, double precision arithmetic is necessary and the computational costs are substantially higher.

Also an optimum shape design algorithm [1] in two dimensions was developed by the coupling of an optimising technique and a boundary element stress analyser for stress minimisation of anisotropic structures. The numerical optimisation method used in the program is the extended penalty function approach, using the BFGS variable metric for unconstrained minimisation, together with the Golden Section method for the one-dimensional search. Hermitian cubic spline functions were used to represent boundary shapes. Hermitian cubic splines are well suited for the boundary shape representation, and complex geometries can be described in a very compact way by a small number of design variables. Applications of this general purpose computer program to the optimum shape design of bars and holes in plates with anisotropic materials were presented[1]. It has to be mentioned that in an earlier study by the author [2-5] shape design optimisation of isotropic structures using the boundary element method was carried out and applications of the developed programmes to the optimum design of a series of loaded structures were presented.

This paper describes the implementation of numerical techniques in a general-purpose computer program to perform shape optimisation to minimise weight of two-dimensional anisotropic structures while satisfying certain constraints upon stresses and geometry. To solve the overall optimisation problem, the feasible direction method together with the Golden Section method [6] for the one-dimensional search are employed. Since both weight and stresses are nonlinear functions of design variables, then the feasible direction method is more reliable because it deals directly with the nonlinearity of the problem. Assuming a uniform mass density and structural thickness, minimising the structural weight is equivalent to minimising the area and therefore the objective function is simply the area of the structure. Since during the optimisation procedure the

shape of the boundary is continuously changing and is not regular, area is calculated by the boundary element method.

In order to show the applications of this general-purpose program, three examples; a cantilever beam under lateral load, a circular plate with a cavity subject to internal pressure and a link plate under tensile and bending moment loads with anisotropic material properties are selected for the analysis.

## 2. CONSTITUTIVE EQUATIONS FOR PLANE ANISOTROPIC ELASTICITY

The stress-strain relations for a two-dimensional homogeneous, anisotropic elastic body in plane stress is

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} \quad (1)$$

where  $\sigma_{jk}$  and  $\varepsilon_{jk}(j,k=1,2)$ , are the stresses and strains, respectively, and the coefficients  $S_{mn}$  are the elastic compliances of the material. These compliances can be written in terms of engineering constants as

$$\begin{aligned} S_{11} &= \frac{1}{E_1} & S_{12} &= -\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2} & S_{16} &= \frac{\eta_{12,1}}{E_1} = \frac{\eta_{1,12}}{G_{12}} \\ S_{22} &= \frac{1}{E_2} & S_{26} &= \frac{\eta_{12,2}}{E_2} = \frac{\eta_{2,12}}{G_{12}} & S_{66} &= \frac{1}{G_{12}} \end{aligned} \quad (2)$$

where  $E_k$  is the Young's modulus in the  $x_k$  direction,  $G_{12}$  is the shear modulus in the  $x_1$ - $x_2$  plane and  $\nu_{jk}$  is the Poisson's ratio. For specially orthotropic materials,  $S_{16}=S_{26}=0$ . [7].

The compatibility equation of strains is

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 2 \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} \quad (3)$$

and equilibrium is satisfied by taking stresses in terms of derivatives of the Airy stress function  $\phi(x_1, x_2)$  as

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} \quad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} \quad \sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2} \quad (4)$$

Combining equations 1,3 and 4, the governing equation for the two dimensional problem of anisotropic elasticity can be obtained

$$S_{22} \frac{\partial^4 \phi}{\partial x_1^4} - 2S_{26} \frac{\partial^4 \phi}{\partial x_1^3 \partial x_2} + (2S_{12} + S_{66}) \frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} - 2S_{16} \frac{\partial^4 \phi}{\partial x_1 \partial x_2^3} + S_{11} \frac{\partial^4 \phi}{\partial x_2^4} = 0. \quad (5)$$

By introducing the operator  $D_s$  ( $s=1,4$ ) as

$$D_s = \frac{\partial}{\partial x_2} - \mu_s \frac{\partial}{\partial x_1} \quad (6)$$

Equation (5) becomes

$$D_1 D_2 D_3 D_4 (\phi) = 0 \quad (7)$$

and  $\mu_s$  are the four roots of the characteristic equation

$$[S_{22} - 2\mu S_{26} + (2S_{12} + S_{66})\mu^2 - 2S_{16}\mu^3 + S_{11}\mu^4] \frac{d^4 \phi}{dz^4} = 0. \quad (8)$$

In order to have a solution for the stress function, the term in square brackets must be zero. Leknitskii [7] has shown that, for an anisotropic material, these roots are distinct and must be either purely imaginary or complex and they may be denoted by

$$\mu_1 = \alpha_1 + i\beta_1, \quad \mu_2 = \alpha_2 + i\beta_2, \quad \mu_3 = \bar{\mu}_1, \quad \mu_4 = \bar{\mu}_2 \quad (9)$$

where  $\alpha_j$  and  $\beta_j$ , ( $j=1,2$ ), are real constants,  $i = \sqrt{-1}$  and the overbar represents the complex conjugate. The characteristic directions may thus be denoted by

$$z_j = x_1 + \mu_j x_2 \quad j = 1,2 \quad (10)$$

and their complex conjugates.

For a generally orthotropic lamina, the strains can be expressed in terms of the stresses in nonprincipal coordinates of the laminae as[8]

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad (11)$$

where the  $\bar{S}_{jk}$  are the components of the transformed lamina compliance matrix which are defined as follows:

$$\begin{aligned} \bar{S}_{11} &= V_1 + V_2 \cos 2\gamma + V_3 \cos 4\gamma \\ \bar{S}_{12} &= V_4 - V_3 \cos 4\gamma \\ \bar{S}_{22} &= V_1 - V_2 \cos 2\gamma + V_3 \cos 4\gamma \\ \bar{S}_{16} &= V_2 \sin 2\gamma + 2V_3 \sin 4\gamma \\ \bar{S}_{26} &= V_2 \sin 2\gamma - 2V_3 \sin 4\gamma \\ \bar{S}_{66} &= 2(V_1 - V_4) - 4V_3 \cos 4\gamma \end{aligned} \quad (12)$$

where the invariants ( $V_1, V_2, V_3, V_4$ ) are



$$\begin{aligned}
 V_1 &= \frac{1}{8}(3S_{11} + 3S_{22} + 2S_{12} + S_{66}) \\
 V_2 &= \frac{1}{2}(S_{11} - S_{22}) \\
 V_3 &= \frac{1}{8}(S_{11} + S_{22} - 2S_{12} - S_{66}) \\
 V_4 &= \frac{1}{8}(S_{11} + S_{22} + 6S_{12} - S_{66})
 \end{aligned} \tag{13}$$

The subscripts 1 and 2 are the principal coordinates of the lamina and  $\gamma$  is the lamina orientation angle with respect to the x and y axes.

### 3. REVIEW OF THE BOUNDARY ELEMENT METHOD FOR ANISOTROPIC MATERIALS

The boundary integral equation in the direct formulation of the BEM for anisotropic materials is an integral constraint equation relating boundary tractions( $t_j$ ) and boundary displacements( $u_j$ ) and it may be written as

$$C_{jk}u_j(P) + \int_s T_{jk}(P,Q)u_j(Q)ds(Q) = \int_s U_{jk}(P,Q)t_j(Q)ds(Q) \quad j,k = 1,2 \tag{14}$$

$P(\zeta_1, \zeta_2)$  and  $Q(x_1, x_2)$  are the load and field points, respectively. The constant  $C_{jk}$  depends on the local geometry of the boundary at P, whether it lies on a smooth surface or a sharp corner. In terms of generalised complex variables

$$\begin{aligned}
 z_1 &= (x_1 - \zeta_1) + \mu_1(x_2 - \zeta_2) \\
 z_2 &= (x_1 - \zeta_1) + \mu_2(x_2 - \zeta_2)
 \end{aligned} \tag{15}$$

the fundamental solution for displacements and tractions, respectively, are as follows:

$$\begin{aligned}
 U_{jk} &= 2 \operatorname{Re} [r_{k1}A_{j1} \ln(z_1) + r_{k2}A_{j2} \ln(z_2)] \\
 T_{j1} &= 2n_1 \operatorname{Re} [\mu_1^2 A_{j1} / z_1 + \mu_2^2 A_{j2} / z_2] - 2n_2 \operatorname{Re} [\mu_1 A_{j1} / z_1 + \mu_2 A_{j2} / z_2] \\
 T_{j2} &= -2n_1 \operatorname{Re} [\mu_1 A_{j1} / z_1 + \mu_2 A_{j2} / z_2] + 2n_2 \operatorname{Re} [A_{j1} / z_1 + A_{j2} / z_2]
 \end{aligned} \tag{16}$$

$n_j$  are the unit outward normal components at Q with respect to the  $x_1$ - $x_2$  coordinate system.

The constants  $r_{kj}$  are

$$\begin{aligned}
 r_{1j} &= S_{11}\mu_j^2 + S_{12} - S_{16}\mu_j \\
 r_{2j} &= S_{12}\mu_j + S_{22} / \mu_j - S_{26}
 \end{aligned} \tag{17}$$

and  $A_{jk}$  are complex constants which may be obtained from the following set of equations

$$\begin{aligned}
 A_{j1} - \bar{A}_{j1} + A_{j2} - \bar{A}_{j2} &= \delta_{j2} / 2\pi i \\
 \mu_1 A_{j1} - \bar{\mu}_1 \bar{A}_{j1} + \mu_2 A_{j2} - \bar{\mu}_2 \bar{A}_{j2} &= -\delta_{j1} / 2\pi i \\
 r_{11} A_{j1} - \bar{r}_{11} \bar{A}_{j1} + r_{21} A_{j2} - \bar{r}_{21} \bar{A}_{j2} &= 0. \\
 r_{12} A_{j1} - \bar{r}_{12} \bar{A}_{j1} + r_{22} A_{j2} - \bar{r}_{22} \bar{A}_{j2} &= 0.
 \end{aligned} \tag{18}$$

$\delta_{jk}$  is the Kronecker delta. For the details of these, the reader is referred to references [9-11].

The boundary element implementation of equation 14 entails boundary discretization. Quadratic isoparametric elements are chosen for the analyses. Substitution of these isoparametric representations into equation 14 will result in a set of linear algebraic equations for the unknown displacements and tractions at the nodes on the boundary of the solution domain as follows

$$AU=B \quad (19)$$

A and B are the matrices which contain the integrals of the traction and displacement kernels, respectively. These linear algebraic equations may then be solved by standard matrix solution techniques.

To calculate surface stresses from the already calculated surface tractions and displacements, it is necessary to consider a local system of coordinates  $(\zeta_1, \zeta_2)$ . Let  $m_{1k}$  be the unit vector in direction tangential to the surface, and  $m_{2k}$  the one in the direction normal to the surface. Let  $u_j$ ,  $t_j$ ,  $\varepsilon_{jk}$ , and  $\sigma_{jk}$  be the displacements, tractions, strains, and stresses, respectively, in the local coordinates. The displacement in the tangential direction is

$$\bar{u}_j(\zeta) = N^c(\zeta)u_j^c m_{1k} \quad (20)$$

and strain in the tangential direction is obtained by this expression to give

$$\bar{\varepsilon}_{11}(\zeta) = \frac{dN^c(\zeta)}{d\zeta} u_j^c m_{1k} \frac{1}{J(\zeta)} \quad (21)$$

Then, using the constitutive equation (Eq. 1), the components of the stress tensor in the local coordinate system can be calculated, and by a simple transformation the stress components in the global system can be obtained.

#### **4. SHAPE DESIGN SENSITIVITY ANALYSIS OF 2D ANISOTROPIC MATERIALS**

Implicit differentiation of the BIE equation with respect to a design variable,  $x_l(l=1,2)$ (which is most likely to be the coordinate of a node on the movable part of the boundary) results in the following equation

$$\begin{aligned}
 & C_{jk}(P) \frac{\partial u_j(P)}{\partial x_l} + \frac{\partial c_{jk}}{\partial x_l} u_j(P) + \int_s \left( \frac{\partial T_{jk}(P,Q)}{\partial x_l} u_j(Q) + T_{jk}(P,Q) \frac{\partial u_j(Q)}{\partial x_l} \right) ds(Q) + \\
 & \int_s T_{jk}(P,Q) u_j(Q) \frac{\partial (dS(Q))}{\partial x_l} = \int_s \left( \frac{\partial U_{jk}(P,Q)}{\partial x_l} t_j(Q) + U_{jk}(P,Q) \frac{\partial t_j(Q)}{\partial x_l} \right) ds(Q) + \\
 & \int_s U_{jk}(P,Q) t_j(Q) \frac{\partial (dS(Q))}{\partial x_l}
 \end{aligned} \tag{22}$$

The derivatives of the terms which only depend on the geometry will be carried out similar to the isotropic materials [2-5]. The derivatives of the remaining terms such as  $U_{jk}$  and  $T_{jk}$  for anisotropic materials will be as follows:

$$\begin{aligned}
 \frac{\partial U_{jk}}{\partial x_l} &= 2 \frac{\partial}{\partial x_l} \left[ \text{Re} \left( r_{k1} A_{j1} \ln(z_1) + r_{k2} A_{j2} \ln(z_2) \right) \right] \\
 \frac{\partial T_{j1}}{\partial x_l} &= 2n_1 \frac{\partial}{\partial x_l} \left[ \text{Re} \left( \frac{\mu_1^2 A_{j1}}{z_1} + \frac{\mu_2^2 A_{j2}}{z_2} \right) \right] + 2 \text{Re} \left( \frac{\mu_1^2 A_{j1}}{z_1} + \frac{\mu_2^2 A_{j2}}{z_2} \right) \frac{\partial (n_1)}{\partial x_l} \\
 &\quad - 2n_2 \frac{\partial}{\partial x_l} \left[ \text{Re} \left( \frac{\mu_1 A_{j1}}{z_1} + \frac{\mu_2 A_{j2}}{z_2} \right) \right] - 2 \text{Re} \left( \frac{\mu_1 A_{j1}}{z_1} + \frac{\mu_2 A_{j2}}{z_2} \right) \frac{\partial (n_2)}{\partial x_l} \\
 \frac{\partial T_{j2}}{\partial x_l} &= -2n_1 \frac{\partial}{\partial x_l} \left[ \text{Re} \left( \frac{\mu_1 A_{j1}}{z_1} + \frac{\mu_2 A_{j2}}{z_2} \right) \right] - 2 \text{Re} \left( \frac{\mu_1 A_{j1}}{z_1} + \frac{\mu_2 A_{j2}}{z_2} \right) \frac{\partial (n_1)}{\partial x_l} \\
 &\quad + 2n_2 \frac{\partial}{\partial x_l} \left[ \text{Re} \left( \frac{A_{j1}}{z_1} + \frac{A_{j2}}{z_2} \right) \right] + 2 \text{Re} \left( \frac{A_{j1}}{z_1} + \frac{A_{j2}}{z_2} \right) \frac{\partial (n_2)}{\partial x_l}
 \end{aligned} \tag{23}$$

where the coefficients  $\mu_j$  and  $A_{jk}$  depend on the material properties and are independent of the design variables. To calculate the above derivatives the complex values,  $\ln(z_j)$  and  $\frac{1}{z_j}$ , can be

written as

$$\begin{aligned}
 \ln(z_j) &= \ln|z_j| + i \arg(z_j) \\
 \frac{1}{z_j} &= \frac{\bar{z}_j}{|z_j|^2}
 \end{aligned} \tag{24}$$

Defining the real functions  $\Lambda_j$  and  $\Omega_j$  as

$$\begin{aligned}
 \Lambda_1 &= (x_1 - \zeta_1) + \alpha_1(x_2 - \zeta_2) \\
 \Lambda_2 &= (x_1 - \zeta_1) + \alpha_2(x_2 - \zeta_2) \\
 \Omega_1 &= -\beta_1\zeta_2 + \beta_1x_2 \\
 \Omega_2 &= -\beta_2\zeta_2 + \beta_2x_2
 \end{aligned} \tag{25}$$

The complex coordinates and their conjugates can be written as

$$z_j = \Lambda_j + i\Omega_j, \bar{z}_j = \Lambda_j - i\Omega_j \tag{26}$$

By substituting equations 24-26 into equations 23 the derivatives of the kernel products with respect to the design variable  $x_l$  can be obtained.

$$\begin{aligned}
 \frac{\partial U_{jk}}{\partial x_l} &= 2 \operatorname{Re}(r_{k1}A_{j1}) \frac{\partial}{\partial x_l} [\ln|z_1|] + 2 \operatorname{Re}(r_{k1}A_{j1}i) \frac{\partial}{\partial x_l} [\arg(z_1)] \\
 &\quad + 2 \operatorname{Re}(r_{k2}A_{j2}) \frac{\partial}{\partial x_l} [\ln|z_2|] + 2 \operatorname{Re}(r_{k2}A_{j2}i) \frac{\partial}{\partial x_l} [\arg(z_2)]
 \end{aligned} \tag{27}$$

$$\begin{aligned}
 \frac{\partial T_{j1}}{\partial x_l} &= 2 \operatorname{Re} \left[ \frac{\mu_1^2 A_{j1}}{z_1} + \frac{\mu_2^2 A_{j2}}{z_2} \right] \frac{\partial n_1}{\partial x_l} - 2 \operatorname{Re} \left[ \frac{\mu_1 A_{j1}}{z_1} + \frac{\mu_2 A_{j2}}{z_2} \right] \frac{\partial n_2}{\partial x_l} \\
 &\quad + 2n_1 \operatorname{Re}(\mu_1^2 A_{j1}) \frac{\partial}{\partial x_l} \left( \frac{\Lambda_1}{|z_1|^2} \right) + 2n_1 \operatorname{Re}(-\mu_1^2 A_{j1}i) \frac{\partial}{\partial x_l} \left( \frac{\Omega_1}{|z_1|^2} \right) + 2n_1 \operatorname{Re}(\mu_2^2 A_{j2}) \frac{\partial}{\partial x_l} \left( \frac{\Lambda_2}{|z_2|^2} \right) \\
 &\quad + 2n_1 \operatorname{Re}(-\mu_2^2 A_{j2}i) \frac{\partial}{\partial x_l} \left( \frac{\Omega_2}{|z_2|^2} \right) - 2n_2 \operatorname{Re}(\mu_1 A_{j1}) \frac{\partial}{\partial x_l} \left( \frac{\Lambda_1}{|z_1|^2} \right) - 2n_2 \operatorname{Re}(-\mu_1 A_{j1} i) \frac{\partial}{\partial x_l} \left( \frac{\Omega_1}{|z_1|^2} \right) \\
 &\quad - 2n_2 \operatorname{Re}(\mu_2 A_{j2}) \frac{\partial}{\partial x_l} \left( \frac{\Lambda_2}{|z_2|^2} \right) - 2n_2 \operatorname{Re}(-\mu_2 A_{j2}i) \frac{\partial}{\partial x_l} \left( \frac{\Omega_2}{|z_2|^2} \right)
 \end{aligned} \tag{28}$$

$$\begin{aligned}
 \frac{\partial T_{j2}}{\partial x_l} &= -2 \operatorname{Re} \left[ \frac{\mu_1 A_{j1}}{z_1} + \frac{\mu_2 A_{j2}}{z_2} \right] \frac{\partial n_1}{\partial x_l} + 2 \operatorname{Re} \left[ \frac{A_{j1}}{z_1} + \frac{A_{j2}}{z_2} \right] \frac{\partial n_2}{\partial x_l} \\
 &\quad - 2n_1 \operatorname{Re}(\mu_1 A_{j1}) \frac{\partial}{\partial x_l} \left( \frac{\Lambda_1}{|z_1|^2} \right) - 2n_1 \operatorname{Re}(-i\mu_1 A_{j1}) \frac{\partial}{\partial x_l} \left( \frac{\Omega_1}{|z_1|^2} \right) - 2n_1 \operatorname{Re}(\mu_2 A_{j2}) \frac{\partial}{\partial x_l} \left( \frac{\Lambda_2}{|z_2|^2} \right) \\
 &\quad - 2n_1 \operatorname{Re}(-i\mu_2 A_{j2}) \frac{\partial}{\partial x_l} \left( \frac{\Omega_2}{|z_2|^2} \right) + 2n_2 \operatorname{Re}[A_{j1}] \frac{\partial}{\partial x_l} \left( \frac{\Lambda_1}{|z_1|^2} \right) + 2n_2 \operatorname{Re}[-iA_{j1}] \frac{\partial}{\partial x_l} \left( \frac{\Omega_1}{|z_1|^2} \right) \\
 &\quad + 2n_2 \operatorname{Re}[A_{j2}] \frac{\partial}{\partial x_l} \left( \frac{\Lambda_2}{|z_2|^2} \right) + 2n_2 \operatorname{Re}(-iA_{j2}) \frac{\partial}{\partial x_l} \left( \frac{\Omega_2}{|z_2|^2} \right)
 \end{aligned} \tag{29}$$

Therefore, the design sensitivity analysis is carried out by implicit differentiation of the structural response (equation 19) with respect to design variables  $x_l$ , which are the coordinates of some nodes of the movable part of the boundary,

$$\frac{\partial A}{\partial x_l} U + A \frac{\partial U}{\partial x_l} = \frac{\partial B}{\partial x_l} \quad \Rightarrow \quad A \frac{\partial U}{\partial x_l} = \left( \frac{\partial B}{\partial x_l} - U \frac{\partial A}{\partial x_l} \right) \quad (30)$$

This is a set of linear algebraic equations for the unknown gradients,  $\frac{\partial U}{\partial x_l}$ , and equivalent to solving the same equation as 19. Thus, if the quantity in brackets in equation 30 is separately assembled, then the displacement derivative vector  $\frac{\partial U}{\partial x_l}$  can be computed in one pass by re-entering the equation solver.

The gradients of stresses usually require the intermediate calculation of the gradients of displacements with respect to the design variables. For the derivatives of the stresses, both sides of equation (21) are differentiated and the gradients of strain in the tangential direction is obtained from

$$\frac{\partial \bar{\varepsilon}_{11}}{\partial x_l} = \frac{dN^c(\zeta)}{d\zeta} \frac{\partial u_j}{\partial x_l} m_{1k} \frac{1}{J(\zeta)} + \frac{dN^c(\zeta)}{d\zeta} u_j^c \frac{\partial m_{1k}}{\partial x_l} \frac{1}{J(\zeta)} + \frac{dN^c(\zeta)}{d\zeta} u_j^c m_{1k} \frac{-1}{[J(\zeta)]^2} \frac{\partial [J(\zeta)]}{\partial x_l} \quad (31)$$

which is a function of gradients of displacements previously calculated. The gradients of the stresses can then be determined by differentiating the constitutive equations (11) for the anisotropic materials.

In reference [1], the derivatives of displacements and stresses with respect to design variables for anisotropic materials are calculated both by this direct analytical differentiation method and also by the finite difference method. The former is shown to be both more accurate and less time consuming.

## **5. ANALYTIC CALCULATION OF WEIGHT AND ITS DERIVATIVES BY THE BOUNDARY ELEMENT METHOD**

For a component in a two-dimensional case with a uniform mass density and thickness, minimising the structural weight is equivalent to minimising the area. Here, uniform mass density and thickness for anisotropic materials have been assumed.

Let R be a closed bounded region in the  $x_1$ - $x_2$  plane whose boundary c consists of a finite number of smooth curves. Let  $f(x_1, x_2)$  and  $g(x_1, x_2)$  be functions which are continuous and have continuous partial derivatives within the domain R. According to Green's theorem [12]

$$\iint_R \left( \frac{\partial g}{\partial x_1} - \frac{\partial f}{\partial x_2} \right) dx_1 dx_2 = \int_c (f dx_1 + g dx_2) \quad (32)$$

Therefore, the area of the domain R ( $S = \iint_R dx_1 dx_2$ ), using Green's theorem, can be written as a

line integral over the boundary

$$S = \frac{1}{2} \int_c (x_1 dx_2 - x_2 dx_1) \quad (33)$$

Now assume the boundary of the domain is discretized to M quadratic isoparametric elements, and the coordinates at nodal points are expressed as

$$x_j(\zeta) = N^c(\zeta) x_j^c \quad (34)$$

where  $N^c(\zeta)$  is the quadratic shape function for local node number c, and  $\zeta$  is the intrinsic coordinate local to each element. Therefore, the area of the domain can be obtained.

$$S = \frac{1}{2} \sum_{b=1}^M \int_{-1}^1 [x_1(\zeta) n_1 + x_2(\zeta) n_2] J(\zeta) d\zeta \quad (35)$$

$J(\zeta)$  is the Jacobian of transformation and  $n_1$  and  $n_2$  are direction cosines of the unit outward normal vector to the surface of the elastic body.

$$n_1 = \frac{dx_2}{ds} = \frac{dx_2/d\zeta}{ds/d\zeta} = \frac{dx_2/d\zeta}{J(\zeta)}, \quad n_2 = -\frac{dx_1}{ds} = -\frac{dx_1/d\zeta}{ds/d\zeta} = -\frac{dx_1/d\zeta}{J(\zeta)} \quad (36)$$

By substituting equations 36 in equation 35 the area of the domain can be obtained,

$$S = \frac{1}{2} \sum_{b=1}^M \int_{-1}^1 \left[ x_1(\zeta) \frac{dx_2}{d\zeta} - x_2(\zeta) \frac{dx_1}{d\zeta} \right] d\zeta \quad (37)$$

By differentiating equation(37) with respect to the design variable  $x_l$ , the weight derivatives can be calculated considering that if  $x_l$  is the  $x_1$  coordinate of movable node then

$$\frac{\partial}{\partial x_l} \left( \frac{dx_2(\zeta)}{d\zeta} \right) = 0. \text{ and } \frac{\partial}{\partial x_l} (x_2(\zeta)) = 0., \text{ therefore,}$$

$$\frac{\partial S}{\partial x_l} = \frac{1}{2} \sum_{b=1}^M \int_{-1}^1 \left[ \frac{\partial x_1(\zeta)}{\partial x_l} \frac{dx_2}{d\zeta} - x_2(\zeta) \frac{\partial}{\partial x_l} \left( \frac{dx_1}{d\zeta} \right) \right] d\zeta \quad (38)$$

If  $x_1$  is the  $x_2$  coordinate of movable node then  $\frac{\partial}{\partial x_1} \left( \frac{dx_1(\zeta)}{d\zeta} \right) = 0$ . and  $\frac{\partial}{\partial x_1} (x_1(\zeta)) = 0$ ., therefore,

$$\frac{\partial \mathcal{S}}{\partial x_1} = \frac{1}{2} \sum_{b=1}^M \int_{-1}^1 \left[ x_1(\zeta) \frac{\partial}{\partial x_1} \left( \frac{dx_2}{d\zeta} \right) - \frac{\partial x_2(\zeta)}{\partial x_1} \left( \frac{dx_1}{d\zeta} \right) \right] d\zeta \quad (39)$$

## 6. NUMERICAL OPTIMISATION METHOD

Minimum weight design of structures or mechanical components, as long as the stresses and displacements are within some allowable range, always has been a major desire for engineers. Structures must be strong enough while a minimum quantity of materials for their manufacture is used. The minimisation of weight to meet imposed constraints such as maximum allowable stresses, displacements, frequencies, etc. is a mathematical programming exercise.

The general problem to be dealt with here is the minimisation of structural weight while satisfying certain constraints upon stresses and geometry. Since both weight and stress constraints are nonlinear functions of the design variables, then the feasible direction method has been selected as the numerical optimisation technique. This method determines a usable-feasible direction in which the design point may be moved in the design space. This direction is 'feasible' because it does not violate, at least over an infinitely small step, any of the constraints; and 'usable' because it results in a reduction of the objective function. Using gradient information, the direction is found and along this direction, a one-dimensional search is next performed. This technique is very suitable for highly non-linear shape optimisation problems because the design point is always feasible [13]. For more details, the reader is referred to references [2-6].

## 7. NUMERICAL RESULTS

In order to apply the boundary element method for the weight minimisation of two-dimensional anisotropic linear elastic problems, a general purpose computer program has been developed. This program uses an iterative technique and involves three major steps within each iteration:

- a) an analysis of the stresses for a given design
- b) Sensitivity analysis corresponding to possible changes in design
- c) Improvements to the design, the regeneration of the boundary element mesh.

The flowchart of the programme is shown in Fig.1.

Three examples are selected to illustrate the use of the program. Three different materials are used [14-16] to investigate the effect of engineering constants on the optimum shape design of the components. It should be noted that no specific material is being studied. Every component is being treated as a lamina that has four engineering constants ( $E_1$ ,  $E_2$ ,  $G_{12}$  and  $\nu_{12}$ ) with a lamina orientation angle of zero. Material No. 1 is isotropic, therefore, [ $E_1=E_2$ ,  $G_{12}=E_1/2(1+\nu_{12})$ ]. The properties of materials Nos. 2 and 3 are [ $E_1/E_2=1.0$ ,  $G_{12}/E_2=2.94$ ,  $\nu_{12}=0.845$ ] and [ $E_1/E_2=13.36$ ,  $G_{12}/E_2=0.58$ ,  $\nu_{12}=0.295$ ], respectively. See Table 1.

### 7.1 Cantilever beam under lateral load

Firstly, an isotropic cantilever beam subjected to a uniformly distributed load is analysed (material No. 1). See Fig. 2. This example is being used to solve isotropic behaviour as a special case of anisotropy. The objective is to find the optimum shape of edge AB, with CD unchanged. The coordinates of seven points P1-P7, which are equally spaced along AB, are selected as design variables. The model contains 54 quadratic elements, 24 elements on each side AB and CD together with 3 elements on each edge AD and BC. The initial shape chosen is a rectangle, which is shown together with the optimum shape in Fig. 3. Fig. 4 shows the variation of weight as a proportion of the initial weight during the optimisation procedure. The analysis is completed in ten iterations with a total weight reduction of 44 percent. The maximum equivalent stress initially is 9.07 and for the optimum shape is 9.1. The results are identical with those of references [2,4].



## 7.2 Circular plate with a cavity under internal pressure

Fig. 5a shows a circular plate with a central cavity subject to internal pressure. The objective is to minimise the weight of the plate. Materials Nos 2 and 3 (Table 1) are chosen for the analysis. In each case the optimisation procedure is carried out with the maximum allowable limit on the equivalent stress not to exceed 1.3 times the maximum equivalent stress of the original geometry. The geometry is symmetric about both coordinate axes, so only the quadrant region ABCD needs to be modelled. See Fig. 5b. The boundary conditions are  $v=0$  along AB and  $u=0$  along CD, where  $u$  and  $v$  are the horizontal and vertical displacements, respectively. The model contains 36 quadratic boundary elements. 12 elements are located on the edge AD and ten elements on each side AB and CD.

The radii of points  $P_1$  to  $P_5$  located on the inner boundary at the fixed angles, shown in Fig. Q5c, are selected as the design variables. Fig. 6 shows the initial geometry and optimum shapes of the cavity in the plate with the selected materials. Fig. 7 shows the variation of weight as a proportion of the initial weight during convergence to the optimum. The analyses for the materials Nos 2 and 3 are completed in 17 and 13 iterations with the weight reductions of 13 and 16 percent, respectively. For material No. 2 the maximum stress for the optimum shape is almost the same as the maximum stress of the initial geometry. For the material No. 3, the maximum stress of the optimum geometry is 26% higher than the maximum stress of the original geometry.

## 7.3 Link plate

This example concerns a link plate loaded through pin joints at its two ends. Therefore, the plate experiences both direct tensile and bending moment loads. See Fig. 8a. Taking advantage of the symmetry, only the right half of the plate is considered. The loading applied is a uniform internal pressure of 1 unit over the semi-circular region of each hole assumed to be in contact with the pin. The model contains 59 quadratic elements. The coordinates of thirteen points, P1-P13, are selected as design variables. The objective is to find the optimum shape of the outer boundary of the link plate. Materials Nos 2 and 3 (Table 1) are chosen for the analysis. For each selected material the optimisation procedure is carried out with the maximum allowable limit on the equivalent stress not to exceed 1.3 times the maximum equivalent stress of the original geometry.

Fig. 9 shows the original geometry together with the optimum shapes of the two selected materials. The analysis of the material No. 2 is completed in 10 iterations with 47 per cent reduction of weight. For material No. 3 the analysis is completed in 7 iterations with 24 per cent

weight reduction. For both cases the maximum stresses for optimum shapes are 25-28% higher than the maximum stresses of the original geometries.

## **8. SUMMARY**

Following a brief review of the mathematical basis of the boundary integral equation method (for two-dimensional stress analysis of anisotropic structures), analytical differentiation of the boundary integral equation was carried out to compute the derivatives of displacements and stresses with respect to changes of the shape design variables. The design sensitivity analysis using the boundary element method was combined with the feasible direction method to form an optimum shape design program in two-dimensions for anisotropic structures. The objective has been to minimise weight subject to stress and geometrical constraints. Three examples have been analysed and the results are presented.

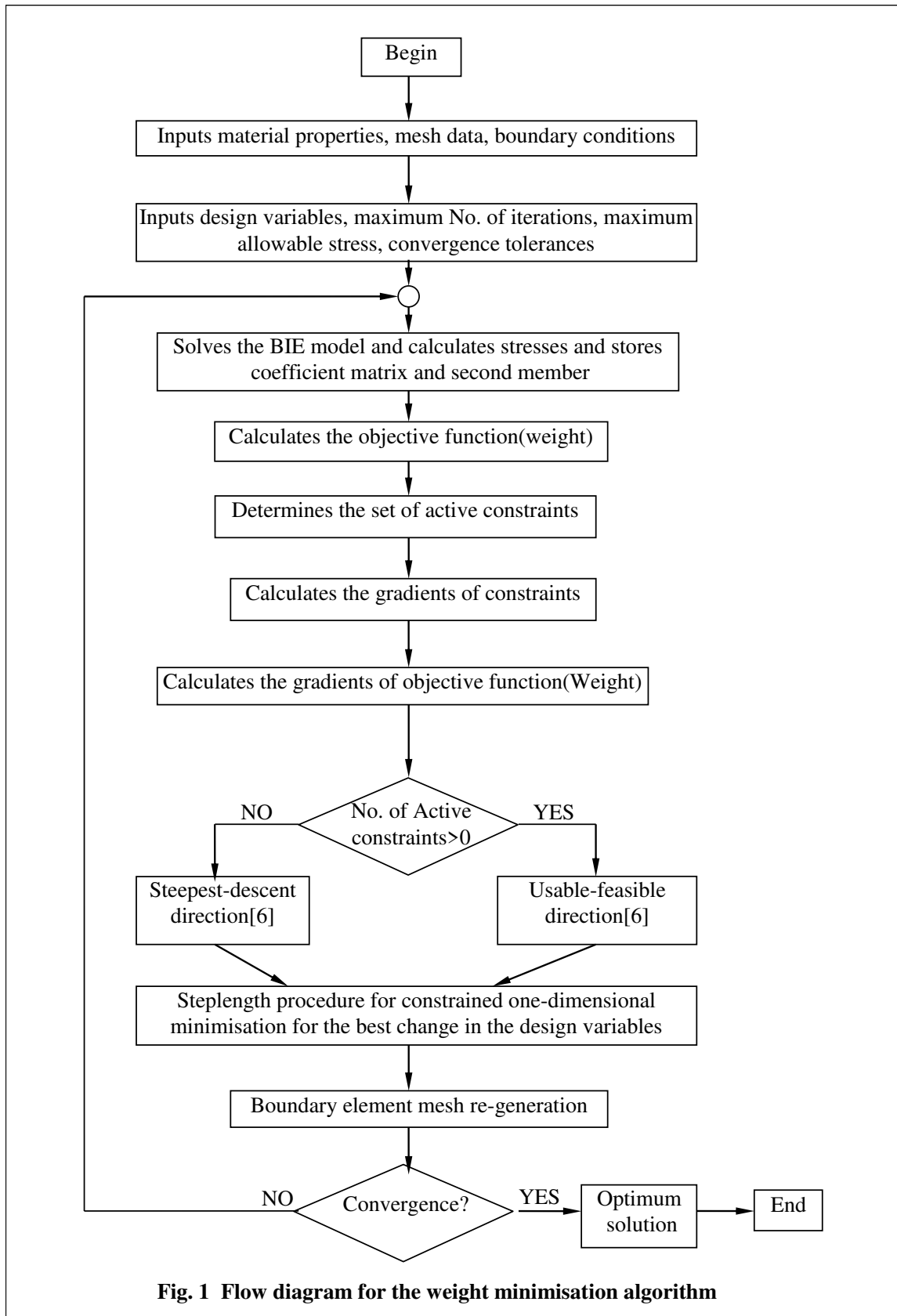
## REFERENCES

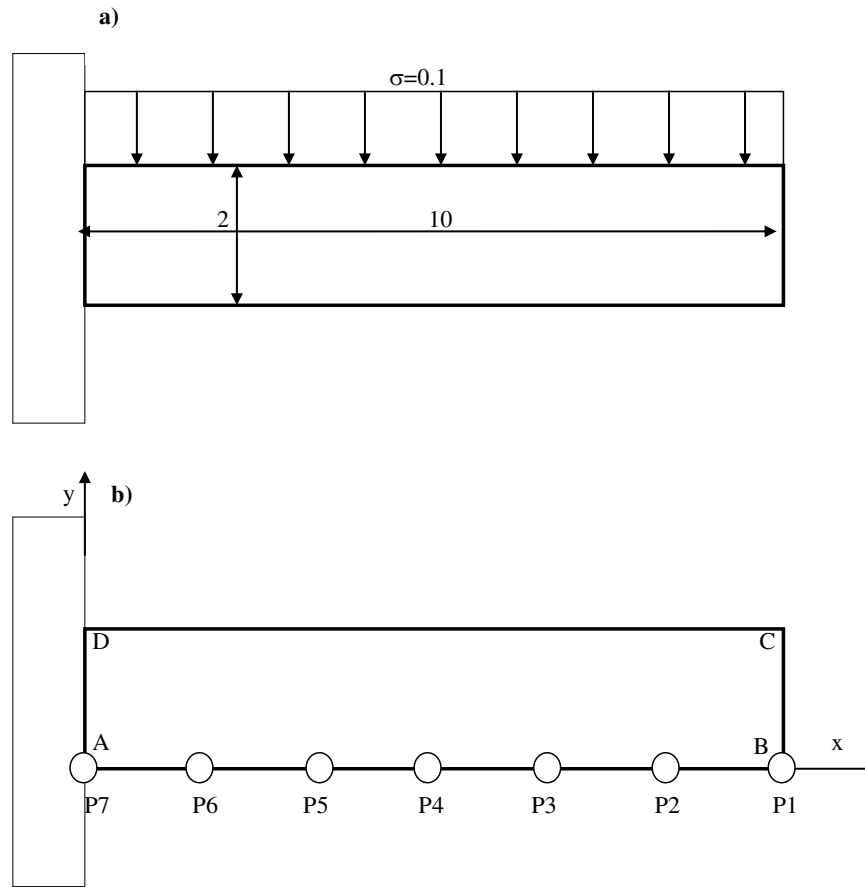
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**Table 1 Elastic properties of the selected materials**

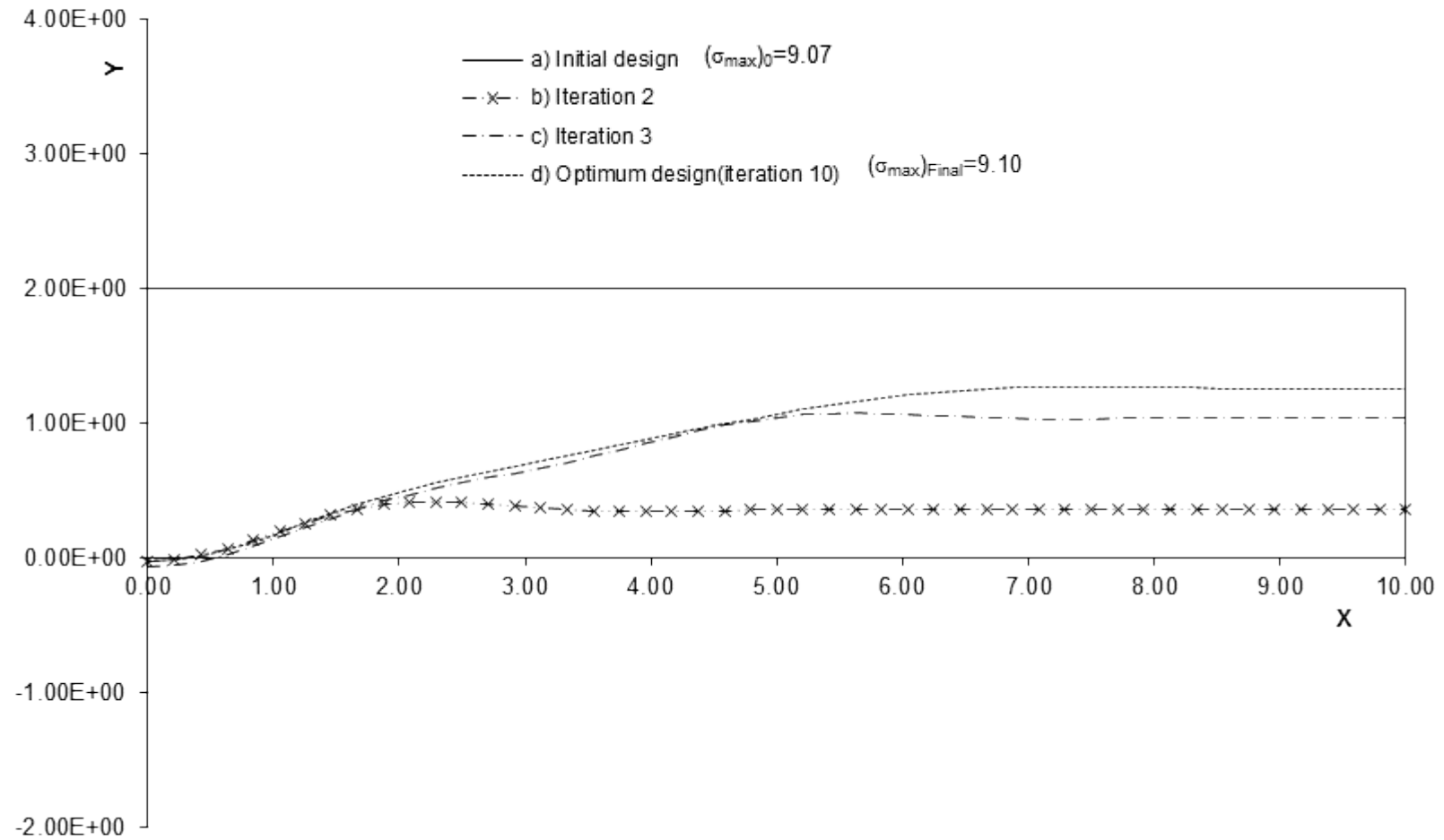
Elastic properties Material No.	$E_{11}$ (GPa)	$E_{22}$ (GPa)	$\nu_{12}$	$G_{12}$ (GPa)	Complex parameters	
					$\mu_1$	$\mu_2$
1)	210.9	210.9	0.29	81.8	i	i
2)	18.9	18.9	0.845	55.6	-0.915+0.403i	0.915+0.403i
3)	148.07	11.08	0.295	6.40	4.683i	0.780i





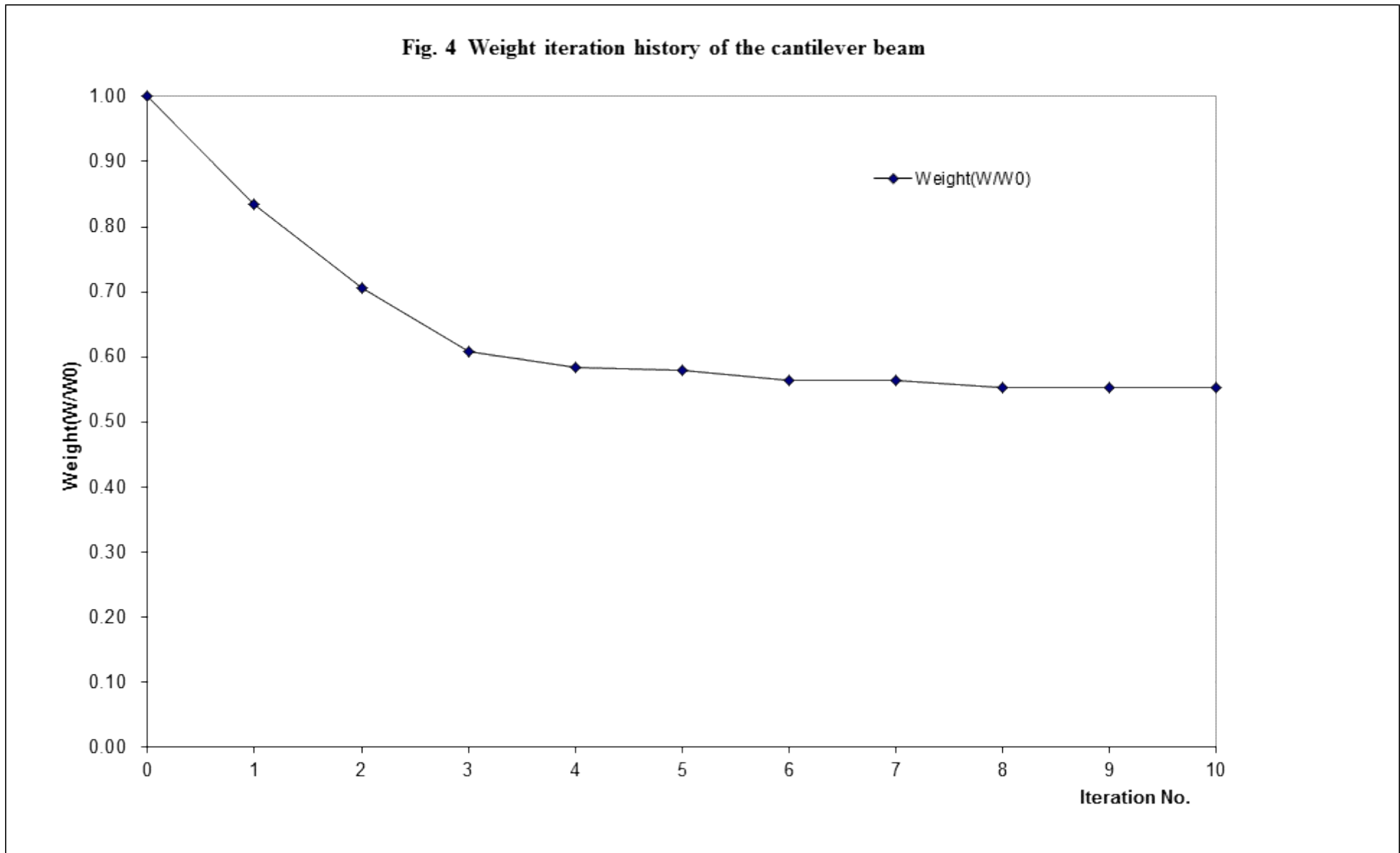
**Fig. 2** a) Cantilever beam under lateral load  
b) Position of the design variable points on the edge AB

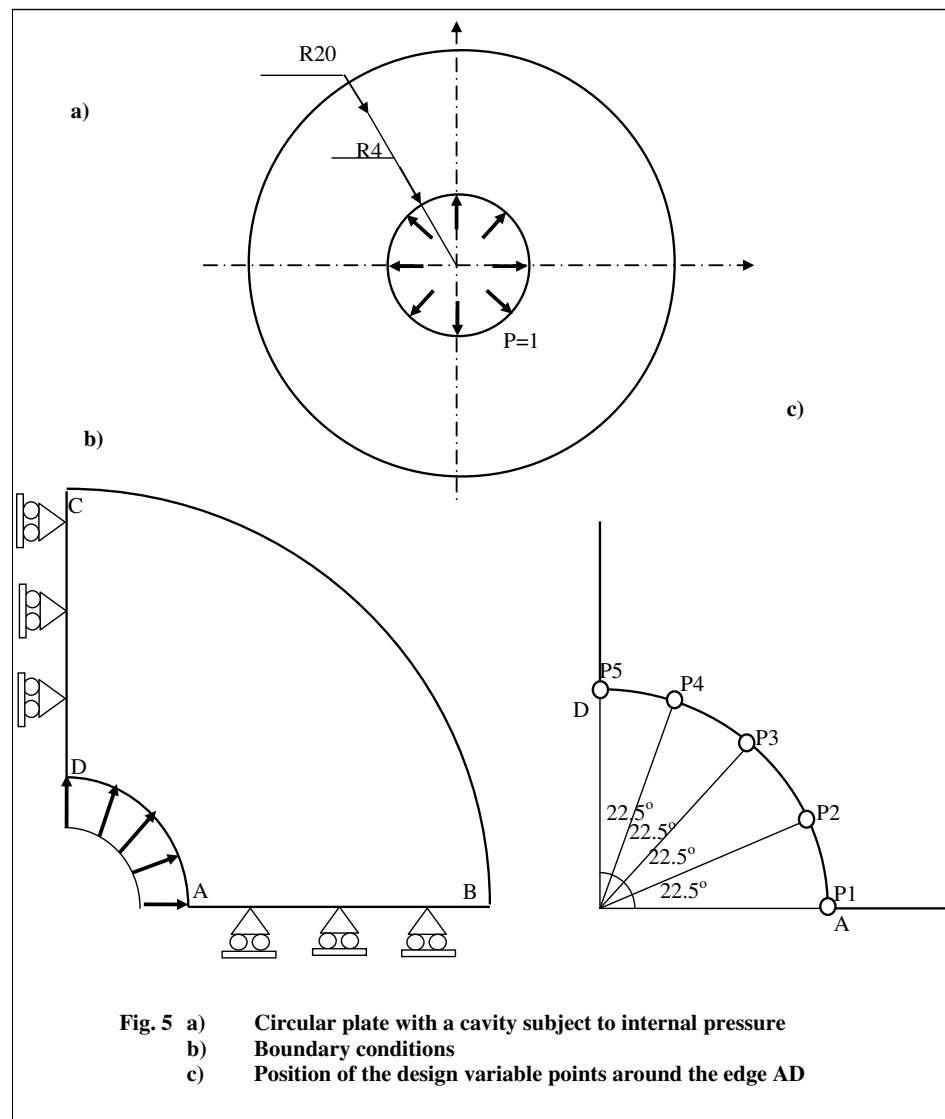
**Fig. 3 Optimum weight design of the cantilever beam(Material No. 1)**  
**a) Initial geometry, d) Optimum design**





**Fig. 4 Weight iteration history of the cantilever beam**





**Fig. 6 Initial and optimum designs of a circular plate with a central cavity subject to internal pressure**

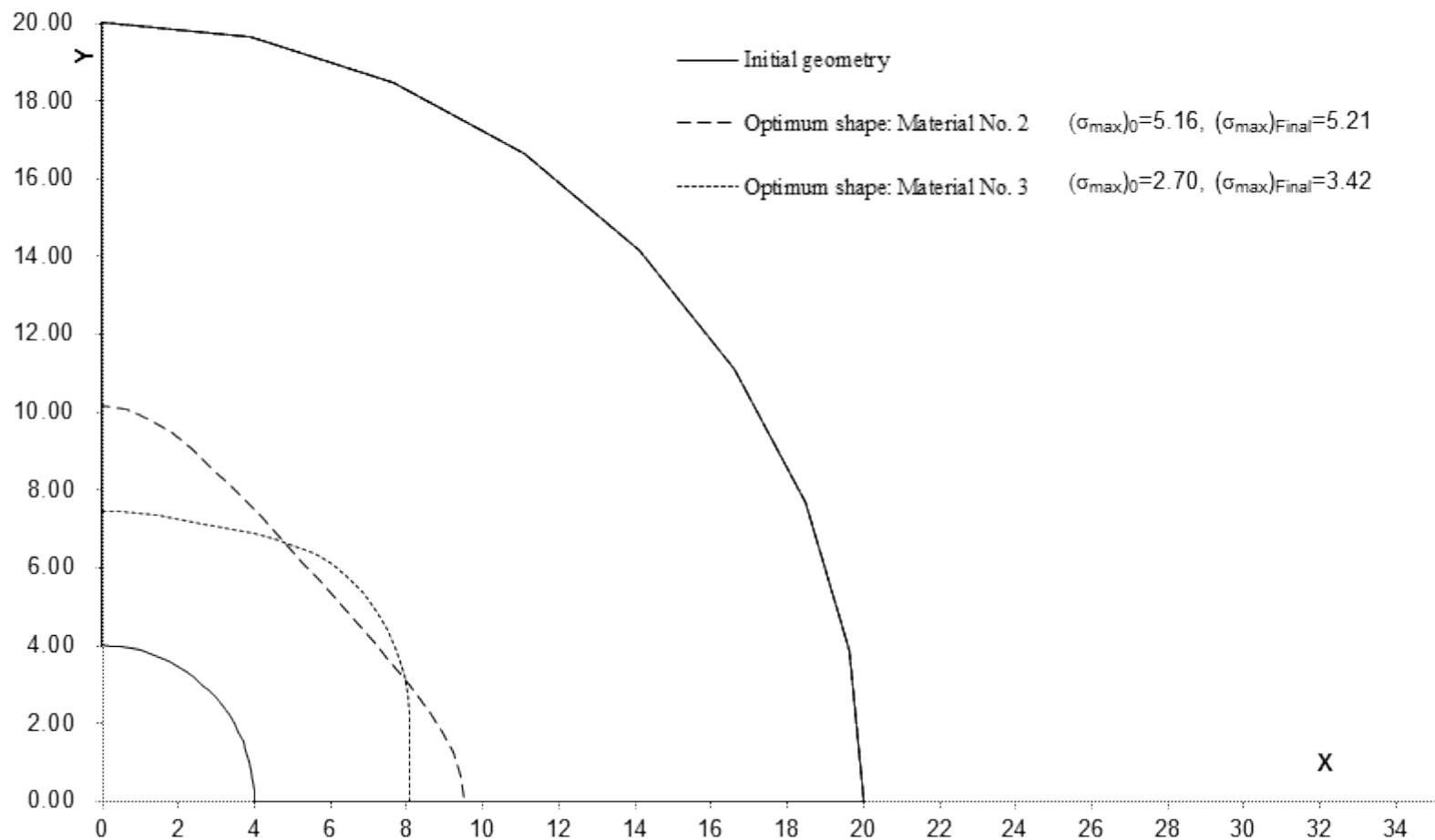
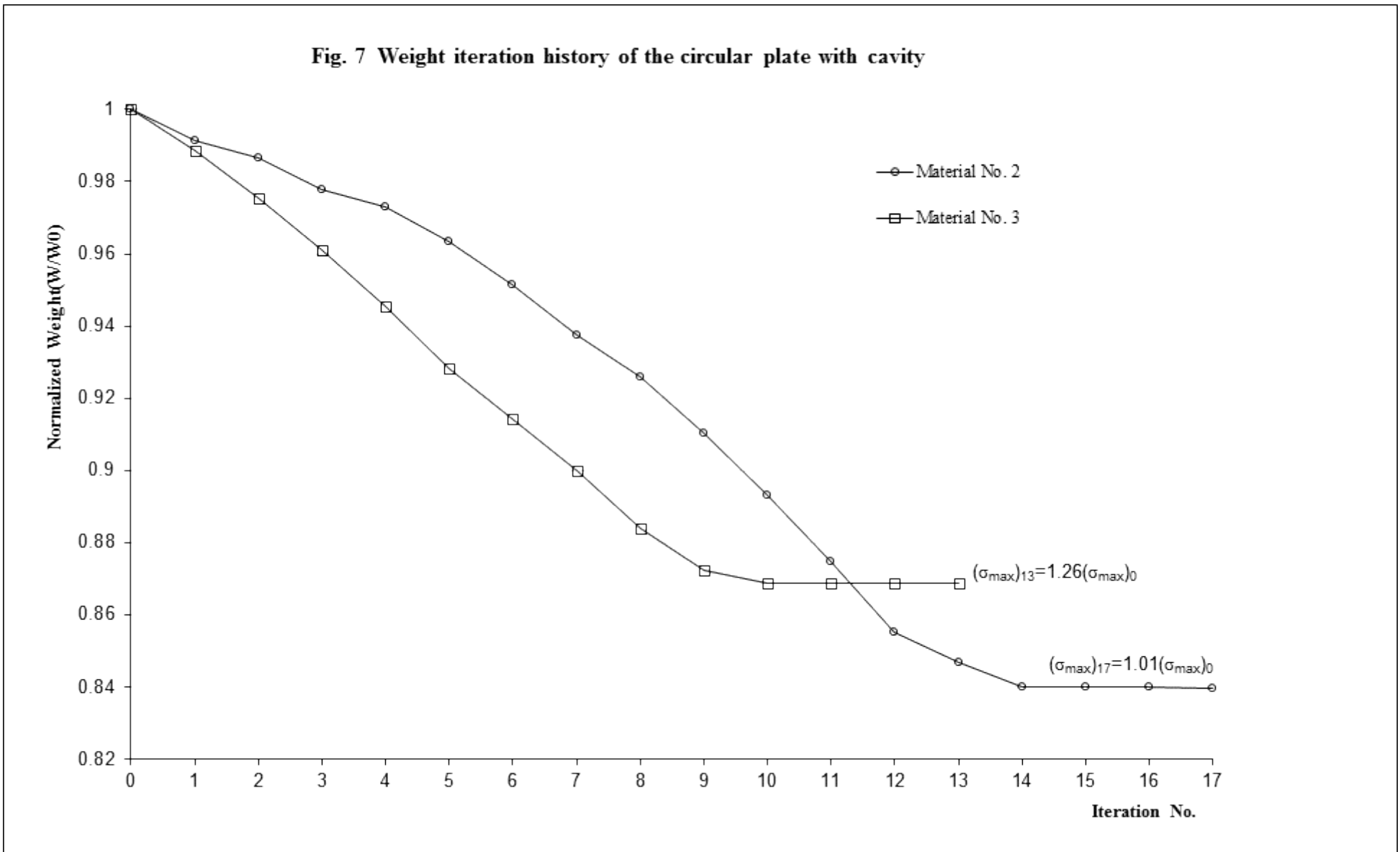
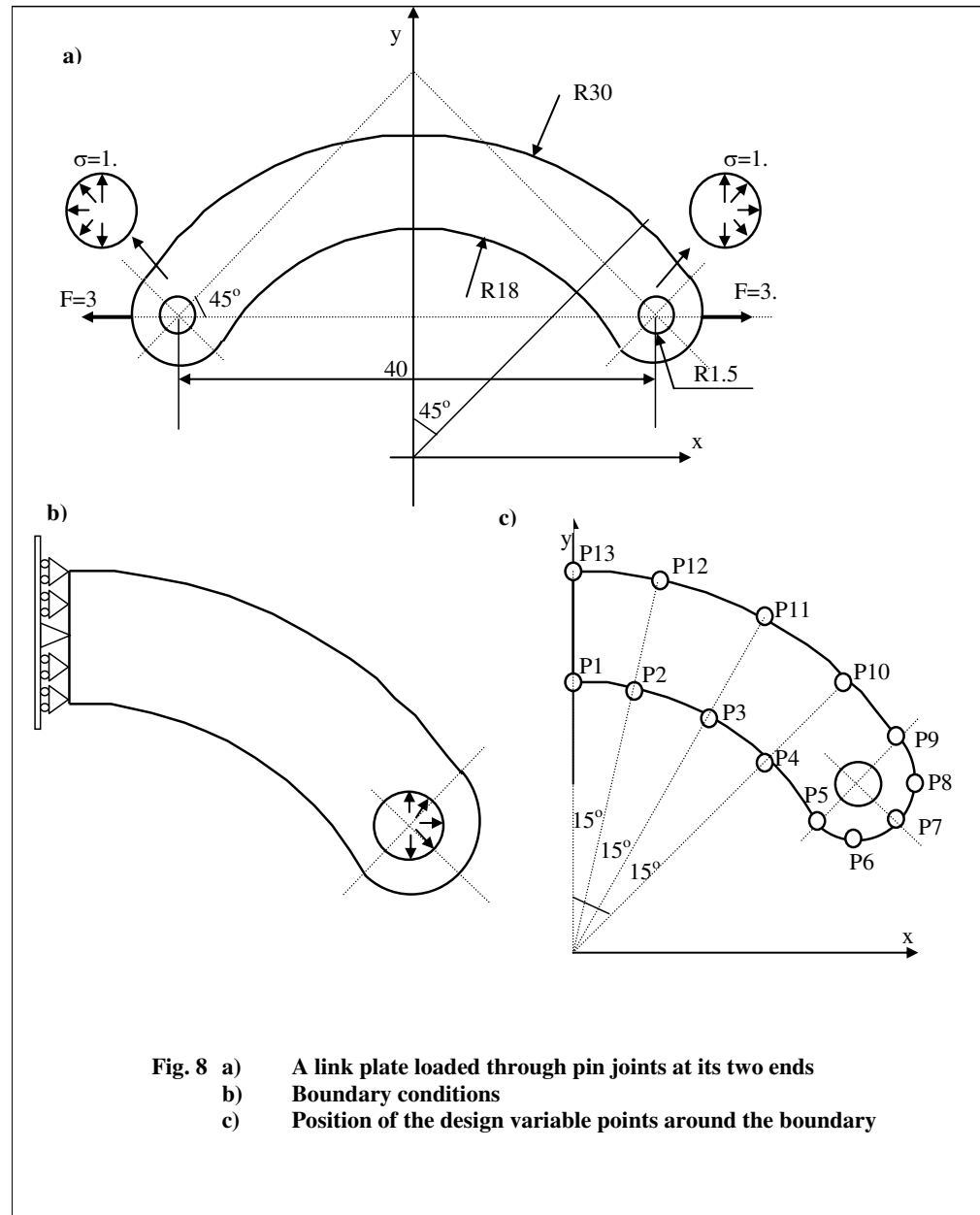


Fig. 7 Weight iteration history of the circular plate with cavity





**Fig. 9 Initial and optimum designs of the link plate a) Initial geometry  
b) Optimum design: Material No. 2  
c) Optimum design: Material No. 3**

