SHAPE OPTIMIZATION PROBLEMS WITH ROBIN CONDITIONS ON THE FREE BOUNDARY

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Abstract. We provide a free discontinuity approach to a class of shape optimization problems involving Robin conditions on the free boundary. More precisely, we identify a large family of domains on which such problems are well posed in a way that the extended problem can be considered a relaxed version of the corresponding one on regular domains, we prove existence of a solution and obtain some qualitative information on the optimal sets.

Keywords: Free boundary problems, shape optimization problems, Robin boundary conditions, free discontinuity problems, functions of bounded variation

Contents

1. Introduction

Let $D \subseteq \mathbb{R}^d$ be a *design region* which we assume to be open, bounded and with a Lipschitz boundary. Consider $B \subset D$ open and with a C^1-boundary and $g \in C^1(\mathbb{R}^d)$ such that

$$0 < c_1 \leq g \leq c_2 \qquad \text{on } B.$$

The main concern of the paper is the following *shape optimization problem*.

> (P) Find Ω with Lipschitz boundary such that $B \subseteq \Omega \subseteq D$ and which minimizes the *shape functional*
>
> $$J(\Omega) := \min_{\substack{u \in W^{1,p}(\Omega) \\ u = g \text{ on } B}} \left[\int_\Omega f(x, \nabla u)\, dx + \int_{\partial\Omega} \beta(x) |u|^p \, d\mathcal{H}^{d-1} + \gamma |\Omega| \right],$$
>
> where $p > 1$, $\gamma \geq 0$, $f : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty[$ is continuous, with $\xi \mapsto f(x, \xi)$ convex and positively p-homogeneous, and $\beta : \mathbb{R}^d \to [0, +\infty[$ is continuous ($\mathcal{H}^{d-1}$ stands for the $(d-1)$-dimensional Hausdorff measure).

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The problem amounts to the determination of the "*free boundary*" $\partial\Omega$ of the optimal domain Ω on which the associated state function u (which realizes $J(\Omega)$) satisfies a boundary condition of *Robin type*. In the case $p = 2$, $f(x,\xi) = |\xi|^2$ and $\beta(x) = \beta$, the condition reduces precisely to the classical Robin condition

$$\frac{\partial u}{\partial n} + \beta u = 0 \qquad \text{on } \partial\Omega,$$

where n denotes the unit external normal.

The function u satisfies also extra conditions on $\partial\Omega$ coming from optimality. Those new conditions, are referred to be *overdetermined*, but they do not play a fundamental role in our approach to the minimization problem.

In the two dimensional case with $f(x,\xi) = A(x)\xi\cdot\xi$, $\gamma = 0$ and $p = 2$, the problem can be interpreted as that of finding the shape of the membrane with minimal total energy among those with elastic properties given by the elastic moduli $A(x)$, prescribed transversal displacement g on the part B, which are elastically supported at the boundary (with elastic forces with constant $\beta(x)$).

The existence of optimal domains for problem (P) is unclear. In general, there are very few results in shape optimization where the existence of an optimal domain can be proved in a "natural" way, i.e. without imposing extra restrictive conditions, and the most of them hold for Dirichlet boundary conditions. For Robin b.c. the only analysis carried to understand existence concerns the first eigenvalue of the Robin Laplacian. Contrary to Dirichlet b.c., the general relaxed form of a Robin problem (i.e. a precise description of the limit of a sequence of Robin problems on a sequence of *arbitrary, non-smooth, non-uniform* domains) is not known. In this paper, we analyze a class of energy type functionals generalizing the Bernoulli free boundary problem in a nonlinear framework complemented by elastic boundary conditions.

The main result of our paper consists in the identification of a class of admissible domains $\mathcal{A}_B(D)$ containing the Lipschitz ones on which the minimization of J can be carried out and can be considered as a *relaxation* of the original problem.

The class $\mathcal{A}_B(D)$ and the extension of J to such a class is suggested by the study of the following free discontinuity functional

$$F(u) := \int_{\mathbb{R}^d} f(x,\nabla u)\,dx + \int_{J_u} \beta(x)[(u^+)^p + (u^-)^p]\,d\mathcal{H}^{d-1} + \gamma|\{u>0\}| \tag{1.1}$$

on the set of functions

$$\mathcal{F}_{B,g}(D) := \{u \in SBV(\mathbb{R}^d) \,:\, supp(u) \subseteq \bar{D}, u \geq 0, u = g \text{ on } B\}.$$

Here SBV denotes the space of *special functions of bounded variation* introduced by De Giorgi and Ambrosio [9] to deal with image segmentation problems. The link between J and F is obtained easily noticing that if u is the state function of the regular domain Ω (which we can assume positive), then the extension of u to $\mathbb{R}^d$ by zero outside Ω yields an element $\tilde{u}$ of $\mathcal{F}_{B,g}(D)$ such that

$$F(\tilde{u}) = \int_\Omega f(x,\nabla u)\,dx + \int_{\partial\Omega} \beta(x)|u|^p\,d\mathcal{H}^{d-1} + \gamma|\Omega| = J(\Omega). \tag{1.2}$$

The surface energy in (1.1) is rather unusual, involving the *sum* of the p-power of the traces of u. Its form, among the many yielding equality (1.2), is suggested by lower semicontinuity issues for the functional F.

We expect that the minimization problem

$$\min_{u\in\mathcal{F}_{B,g}(D)} F(u) \tag{1.3}$$

should provide information for the shape optimization problem (P), in particular we expect optimal domains being given by the *supports* of optimal functions. Notice that this approach through a problem on functions is similar to that employed by Alt and Caffarelli in their pioneering paper [1], where a free boundary problem under Dirichlet conditions was studied. So the free discontinuity problem (1.3) is a sort of SBV-counterpart in the context of Robin boundary conditions of the problem considered in [1].

Clearly the connection between the free discontinuity problem (1.3) and problem (P) is subordinated to the regularity properties of the optimal functions. We proceed as follows.

First of all, we show that the minimum problem (1.3) is well posed. This is not trivial because the compactness properties available for minimizing sequences are not compatible with Ambrosio's theorem in SBV (see Theorem 2.1). Using standard results in SBV, one shows the existence of a candidate minimizer u such that $u \geq 0$ and $u^p \in SBV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. The SBV-regularity is a consequence of optimality and relies on a bound from below on the support of its positivity set (see Theorem 3.5 and [6])

$$u \geq \alpha > 0 \qquad \text{a.e. on } \{u > 0\}. \tag{1.4}$$

Such a property is immediately available in the classical context thanks to the Hopf Lemma and the Robin boundary condition (see Remark 3.8). In the context of the free discontinuity problems it is a sort of non-degeneracy property, coming from optimality.

In view of the bound from below (1.4) and that from above (given by $\|g\|_\infty$), the first two terms in the free discontinuity functional F turn out to be estimated from above and below on the minimizer u by the Mumford-Shah type functional

$$MS(u) := \int_{\mathbb{R}^d} f(x, \nabla u)\, dx + \mathcal{H}^{d-1}(J_u). \tag{1.5}$$

In particular the minimality property of u for F entails that (a suitable multiple of) u is an *almost-quasi minimizer* of the functional MS (see Proposition 3.11). Such local minimizers are defined as follows: there exist $\Lambda \geq 1$, $\alpha > 0$, $c_\alpha > 0$ such that for every $B_\rho(x) \subset\subset D$, $v \in SBV_{loc}(D)$ with $v = g$ on B and $\{v \neq u\} \subseteq B_\rho(x)$

$$\int_{B_\rho(x)} f(x, \nabla u)\, dx + \mathcal{H}^{d-1}(J_u \cap \bar{B}_\rho(x)) \leq \int_{B_\rho(x)} f(x, \nabla v)\, dx + \Lambda \mathcal{H}^{d-1}(J_v \cap \bar{B}_\rho(x)) + c_\alpha \rho^{d-1+\alpha}.$$

This minimality property is weaker than that considered by De Giorgi, Carriero and Leaci in [10] since it can be $\Lambda > 1$. Under suitable assumptions on f, we show (Theorem 2.3) that this weaker minimality still yields the essential closedness of J_u in D, i.e.,

$$\mathcal{H}^{d-1}\left((\overline{J_u} \setminus J_u) \cap D\right) = 0.$$

In the case $f(x, \xi) = |\xi|^2$, under certain supplementary hypotheses and without boundary condition (associated to g), such a result was proved by Siaudeau [13] following the lines of [10]. More recently, the same result has been proved in [6] on the basis of a monotonicity formula for the Mumford-Shah functional. Here we extend the analysis [10] to cover more general energy densities f (for which monotonicity is unclear) and to treat boundary data (as in the case of [7]). A monotonicity formula similar to [6] can not hold for arbitrary f, but for some particular cases as $f(x, \xi) = |\xi|^p$, with $p \in (1, 2)$ this can not be trivially excluded. Such a formula relies on a precise estimate from below of the ratio

$$\inf_{u \in W^{1,p}(\partial B_R)} \frac{R \int_{\partial B_R} |\nabla u|^p d\mathcal{H}^{d-1}}{\int_{B_R} |\nabla u|^p dx},$$

for $p \in (1, 2)$, which, up to our knowledge, is not known.

The analysis of problem (1.3) thus shows that the support of minimizers of F is given by the connected component of $D \setminus \overline{J_u}$ on which u does not vanish. This set belongs to the family

$$\mathcal{A}_B(D) := \{\Omega \text{ open } : B \subseteq \Omega \subseteq D, \partial\Omega \text{ is } \mathcal{H}^{d-1}\text{-countably rectifiable with } \mathcal{H}^{d-1}(\partial\Omega) < +\infty\},$$

which turns out to be the class we are looking for. Indeed we prove in Theorem 3.2 that

$$\min_{\Omega \in \mathcal{A}_B(D)} J(\Omega),$$

where the extension of J to irregular domains is given in (3.3) below and is suggested by the free discontinuity functional F, is well posed and satisfies

$$\min_{\Omega \in \mathcal{A}_B(D)} J(\Omega) = \inf_{\Omega \text{ Lipschitz}} J(\Omega),$$

i.e., the extended problem is a *relaxed* version of the original one. This last property is a consequence of density results in the sense of Cortesani and Toader [8] for the functional F in SBV (see Proposition 3.12).

The regularity of domains in $\mathcal{A}_B(D)$ is in the weak sense of geometric measure theory: in particular they are open sets with finite perimeter and they can admit in principle *inner cracks*. It is our opinion that the class

$$\{\Omega \text{ open and bounded in } \mathbb{R}^d : \partial\Omega \text{ is } \mathcal{H}^{d-1}\text{-countably rectifiable with } \mathcal{H}^{d-1}(\partial\Omega) < +\infty\},$$

provides a natural framework for shape optimization problems under Robin conditions. In [5] we employed it to deal with the Faber-Krahn inequality of the first eigenvalue and associated semilinear variants (including the case of the torsional rigidity) of the Robin-Laplacian: the existence of an optimal domain in the class above permits to use geometrical arguments typical of shape optimization problems to show that minimizers are balls.

The paper is organized as follows. In Section 2 we fix the notation employed throughout the paper, and recall some basic facts concerning SBV-functions and free discontinuity problems. The main problem and the associated analysis outlined above is detailed in Section 3, while Section 4 contains the proof of the essential closedness of the jump sets of almost-quasi minimizers of (1.5).

2. Notation and preliminaries

Throughout the paper, $B_r(x)$ will denote the open ball of center $x \in \mathbb{R}^d$ and radius $r > 0$. We will write B_r in place of $B_r(0)$. We say that $A \subset\subset B$ if $\bar{A}$ is compact and contained in B. If $E \subset \mathbb{R}^d$, we will denote its volume by $|E|$, and 1_E will stand for its characteristic function, i.e., $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$. We set $\omega_d := |B_1|$.

For $A \subseteq \mathbb{R}^d$ open set and $p \geq 1$, $L^p(A)$ will denote the usual Lebesgue space of p-summable functions, while $W^{1,p}(A)$ will denote the Sobolev space of functions in $L^p(A)$ whose derivative in the sense of distributions is p-summable. Moreover $\|u\|_\infty$ will stand for the sup-norm of u, while $supp(u)$ will denote the set $\{u \neq 0\}$, well defined up to zero Lebesgue measure. We will say that $\{u \neq v\} \subseteq B_\rho(x)$ if $u = v$ a.e. outside $B_\rho(x)$.

Finally we will use the following notation: for $a, b \in \mathbb{R}$

$$a \wedge b := \min\{a, b\} \qquad \text{and} \qquad a \vee b := \max\{a, b\}.$$

2.1. **Functions of bounded variation.** Let $A \subseteq \mathbb{R}^d$ be an open set. We say that $u \in BV(A)$ if $u \in L^1(A)$ and its derivative in the sense of distributions is a finite Radon measure on A, i.e., $Du \in \mathcal{M}_b(A; \mathbb{R}^d)$. $BV(A)$ is called the space of *functions of bounded variation* on A. $BV(A)$ is a Banach space under the norm $\|u\|_{BV(A)} := \|u\|_{L^1(A)} + \|Du\|_{\mathcal{M}_b(A;\mathbb{R}^d)}$. We refer the reader to [2] for an exhaustive treatment of the space BV.

Concerning the fine properties, a function $u \in BV(A)$ (or better every representative of u) is a.e. approximately differentiable on A, with approximate gradient $\nabla u \in L^1(A; \mathbb{R}^d)$. Moreover, the jump set J_u is a $\mathcal{H}^{d-1}$-countably rectifiable set, i.e., $J_u \subseteq \cup_{i\in\mathbb{N}} M_i$ up to a $\mathcal{H}^{d-1}$-negligible set, with M_i a C^1-hypersurface in $\mathbb{R}^d$. The measure Du admits the following representation for every Borel set $B \subseteq A$:

$$Du(B) = \int_B \nabla u \, dx + \int_{J_u \cap B} (u^+ - u^-)\nu_u \, d\mathcal{H}^{d-1} + D^c u(B),$$

where $\nu_u(x)$ is the normal to J_u at x, and $D^c u$ is singular with respect to the Lebesgue measure and concentrated outside J_u. $D^c u$ is usually referred to as the *Cantor part* of Du. $u^\pm$ are the upper and lower approximate limits of u at x. The normal ν_u coincides $\mathcal{H}^{d-1}$-a.e. on J_u with the normal to the hypersurfaces M_i. The direction of $\nu_u(x)$ is chosen in such a way that $u^\pm(x)$ is the approximate limit of u at x on the sets $\{y \in \mathbb{R}^d : \nu_u(x) \cdot (y - x) \gtrless 0\}$. Moreover, $u^\pm$ coincide $\mathcal{H}^{d-1}$-almost everywhere on J_u with the traces $\gamma^\pm(u)$ of u on J_u which are defined by the following Lebesgue-type limit quotient relation

$$\lim_{r\to 0} \frac{1}{r^d} \int_{B_r^\pm(x)} |u(x) - \gamma^\pm(u)(x)| \, dx = 0$$

where $B_r^{\pm}(x) := \{y \in B_r(x) : \nu_u(x) \cdot (y-x) \gtrless 0\}$ (see [2, Remark 3.79]).

If A is bounded and with a Lipschitz boundary, then $BV(A) \hookrightarrow L^{d/d-1}(A)$. Moreover, the following compactness result holds: if $(u_n)_{n\in\mathbb{N}}$ is bounded in $BV(A)$, there exist $u \in BV(A)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that

$$u_{n_k} \to u \qquad \text{strongly in } L^1(A)$$

and

$$Du_{n_k} \to Du \qquad \text{weakly* in the sense of measures.}$$

We say in this case that $u_{n_k} \overset{*}{\rightharpoonup} u$ weakly* in $BV(A)$.

We say that $u \in SBV(A)$ if $u \in BV(A)$ and $D^c u = 0$. $SBV(A)$ is called the space of *special functions of bounded variation* on A. This space is very useful in free discontinuity problems in view of the following compactness and lower-semicontinuity result due to L. Ambrosio (see [2, Theorems 4.7-4.8]).

Theorem 2.1. *Let $A \subset \mathbb{R}^d$ be open and bounded, $p \in]1, +\infty[$, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV(A)$ such that*

$$\sup_n \int_A |\nabla u_n|^p \, dx + \mathcal{H}^{d-1}(J_{u_n}) + \|u_n\|_\infty < +\infty.$$

Then there exist $u \in SBV(A)$ with $\nabla u \in L^p(A;\mathbb{R}^d)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that

$$u_{n_k} \to u \qquad \textit{strongly in } L^1(A),$$

$$\nabla u_{n_k} \rightharpoonup \nabla u \qquad \textit{weakly in } L^p(A;\mathbb{R}^d)$$

and

$$\mathcal{H}^{d-1}(J_u) \le \liminf_{n\to+\infty} \mathcal{H}^{d-1}(J_{u_n}).$$

In the following we will use the notation

$$SBV^p(A) := \{u \in SBV(A) \,:\, \nabla u \in L^p(A;\mathbb{R}^d) \text{ and } \mathcal{H}^{d-1}(J_u) < +\infty\}.$$

2.2. **Regularity results for almost-quasi minimizers of free discontinuity problems.** Let $\Omega \subseteq \mathbb{R}^d$ be an open set, and let $\Omega' \subseteq \mathbb{R}^d$ be open with $\Omega \subseteq \Omega'$ and such that $\Omega' \setminus \overline{\Omega}$ has a C^1-boundary. Let $g \in C^1(\mathbb{R}^d)$. For $u \in SBV_{loc}(\Omega')$ with $u = g$ on $\Omega' \setminus \overline{\Omega}$, let us consider the free discontinuity functional

$$F(u) := \int_{\Omega'} f(x, \nabla u) \, dx + \mathcal{H}^{d-1}(J_u). \tag{2.1}$$

We assume that $f : \Omega' \times \mathbb{R}^d \to [0, +\infty[$ satisfies the following assumptions.

- (H_1) f is continuous and there exist $L > 0$ and $p > 1$ such that for every $x \in \Omega'$, $\xi \in \mathbb{R}^d$ and $t > 0$
$$f(x, t\xi) = t^p f(x, \xi), \qquad L^{-1}|\xi|^p \le f(x, \xi) \le L|\xi|^p.$$
- (H_2) For every $x \in \Omega'$, the map $\xi \mapsto f(x, \xi)$ is convex.
- (H_3) There exists $\mu > 0$ such that for every $x_0 \in \Omega'$, $\varphi \in C_c^1(B_1)$ and $\xi \in \mathbb{R}^d$,
$$\int_{B_1} [f(x_0, \xi + \nabla\varphi) - f(x_0, \xi)] \, dx \ge \mu \int_{B_1} (|\xi|^2 + |\nabla\varphi|^2)^{\frac{p-2}{2}} |\nabla\varphi|^2 \, dx.$$

We are interested in the following notion of local minimality.

Definition 2.2 (Almost-quasi minimality). *Let $u \in SBV^p_{loc}(\Omega')$ be such that $u = g$ on $\Omega' \setminus \overline{\Omega}$. We say that u is an almost-quasi minimizer for the functional F in (2.1) with boundary condition g if there exist $\Lambda \ge 1$, $\alpha > 0$ and $c_\alpha > 0$ such that for every ball $B_\rho(x_0) \subset\subset \Omega'$*

$$\int_{B_\rho(x_0)} f(x, \nabla u) \, dx + \mathcal{H}^{d-1}(J_u \cap \bar{B}_\rho(x_0)) \le \int_{B_\rho(x_0)} f(x, \nabla v) \, dx + \Lambda \mathcal{H}^{d-1}(J_v \cap \bar{B}_\rho(x_0)) + c_\alpha \rho^{d-1+\alpha}$$

for every $v \in SBV^p_{loc}(\Omega')$, $v = g$ on $\Omega' \setminus \overline{\Omega}$, and such that $\{v \ne u\} \subseteq B_\rho(x_0)$.

The following result will be pivotal in our analysis. It states that the jump set of almost-quasi minimizers enjoys the Ahlfors regularity à la De Giorgi-Carriero-Leaci [10], i.e., it is essentially closed.

Theorem 2.3 (**Essential closedness of the jump set**). *Let $\Omega \subseteq \Omega' \subseteq \mathbb{R}^d$ be open and such that $\Omega' \setminus \overline{\Omega}$ has a C^1-boundary. Let $g \in C^1(\mathbb{R}^d)$ and let $f : \Omega' \times \mathbb{R}^d \to [0, +\infty[$ satisfy $(H_1)-(H_3)$. Let $u \in SBV^p_{loc}(\Omega')$ be an almost-quasi minimizer for the functional (2.1) with boundary condition g according to Definition 2.2.*

Given $\tilde{\Omega} \subset\subset \Omega'$, there exist $\varepsilon_0 > 0$ and $\rho_0 > 0$ depending on $\tilde{\Omega}$ such that for every $x \in J_u$, $x \in \tilde{\Omega}$ and $\rho < \rho_0$

$$\frac{1}{\varepsilon_0}\rho^{d-1} \geq \mathcal{H}^{d-1}(J_u \cap B_\rho(x)) \geq \varepsilon_0 \rho^{d-1}. \tag{2.2}$$

In particular

$$\mathcal{H}^{d-1}\left((\overline{J_u} \setminus J_u) \cap \Omega'\right) = 0,$$

i.e., the jump set of u is essentially closed in Ω'.

For $\Lambda = 1$, $f = |\xi|^2$ and $\Omega' = \Omega$, i.e., for quasi minimizers for the original Mumford-Shah functional, the previous property reduces to the celebrated of De Giorgi, Carriero and Leaci [10]. Their approach has then be adapted to cover more general energy densities: the case of p-homogeneous functions $f(\xi)$ can be found e.g. in [2, Chapter 7]. The case with boundary conditions has been dealt by Carriero and Leaci in [7].

Ahlfors regularity for $\Lambda > 1$ without boundary conditions has been proved by Siaudeau in [13] along the lines of [10], but in the presence of some supplementary a priori hypotheses, and also in [6] on the basis of a general monotonicity formula for the Mumford-Shah functional.

We will give the proof of Theorem 2.3 in Section 4: the main difference with respect to [13] is that we consider a weaker setting with more general energy densities f and we take into account boundary conditions, and so our contribution is essentially technical. The reader already acquainted with the ideas of [10] and of [7] could skip the proof of Theorem 2.3 without prejudice.

2.3. **A density result for free discontinuity functionals.** We will make use of a density result in SBV due to Cortesani and Toader [8]. In order to formulate the statement, we will say that $u \in SBV(\Omega)$ with Ω open set in $\mathbb{R}^d$ has polyhedral jumps if $\overline{J_u} \cap \Omega$ is the intersection with Ω of the union of a finite number of $(d-1)$-dimensional simplexes. The density result is the following (see [8, Theorem 3.1]).

Theorem 2.4. *Let $\Omega \subseteq \mathbb{R}^d$ be open and with Lipschitz boundary, and let $p > 1$. Let $u \in SBV(\Omega) \cap L^\infty(\Omega)$ be such that $\nabla u \in L^p(\Omega; \mathbb{R}^d)$ and $\mathcal{H}^{d-1}(J_u) < +\infty$.*

There exists $(u_n)_{n\in\mathbb{N}}$ such that the following items hold true for every $n \in \mathbb{N}$.

(a) $\mathcal{H}^{d-1}\left((\overline{J_{u_n}} \setminus J_{u_n}) \cap \Omega\right) = 0$.
(b) *J_{u_n} is polyhedral in Ω.*
(c) *$u_n \in W^{k,\infty}(\Omega \setminus \overline{J_{u_n}})$ for every $k \geq 1$.*

Moreover

$$u_n \to u \qquad \text{strongly in } L^1(\Omega),$$

$$\nabla u_n \to \nabla u \qquad \text{strongly in } L^p(\Omega; \mathbb{R}^d).$$

and

$$\limsup_{n\to+\infty} \int_{J_{u_n}\cap A} \varphi(x, u_n^+, u_n^-, \nu_{u_n})\, d\mathcal{H}^{d-1} \leq \int_{J_u \cap A} \varphi(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{d-1}$$

for every open set $A \subset\subset \Omega$ and every upper semicontinuous function $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \times S^{d-1} \to [0, +\infty[$ such that $\varphi(x, a, b, \nu) = \varphi(x, b, a, -\nu)$ for every $x \in \Omega$, $a, b \in \mathbb{R}$ and $\nu \in S^{d-1}$.

3. A free boundary problem with Robin conditions

With the notations of Section 2, let $\Omega' = \mathbb{R}^d$ and $f : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty[$ satisfy the assumptions $(H_1), (H_2), (H_3)$. Let $\beta : \mathbb{R}^d \to [0, +\infty[$ be a continuous function such that

$$\forall x \in \mathbb{R}^d \ : \ 0 < \beta_1 \leq \beta(x) \leq \beta_2 < +\infty \tag{3.1}$$

for some positive constants β_1, β_2. Let $D \subseteq \mathbb{R}^d$ be a *design region* which we assume to be an open bounded set with a Lipschitz boundary. Consider $B \subset D$ open and with C^1-boundary, and $g \in C^1(\mathbb{R}^d)$ such that

$$0 < c_1 \leq g \leq c_2 \qquad \text{on } B. \tag{3.2}$$

For every domain Ω with Lipschitz boundary such that $B \subseteq \Omega \subseteq D$, let us consider the *shape functional*

$$J(\Omega) := \min_{\substack{u \in W^{1,p}(\Omega) \\ u = g \text{ on } B}} \left[\int_\Omega f(x, \nabla u)\, dx + \int_{\partial\Omega} \beta(x) |u|^p \, d\mathcal{H}^{d-1} \right] + \gamma |\Omega|,$$

where $\gamma \in [0, +\infty[$.

We are interested in the minimization of J among all admissible domains. Unfortunately, the existence of domains which minimize J is unclear, due to the lack of compactness properties for minimizing sequences of Lipschitz sets.

In order to achieve the existence of optimal domains, following [5] we relax the previous problem to the family of sets

$$\mathcal{A}_B(D) := \{\Omega \text{ open } \ : \ B \subseteq \Omega \subseteq D, \partial\Omega \text{ is } \mathcal{H}^{d-1}\text{-countably rectifiable with } \mathcal{H}^{d-1}(\partial\Omega) < +\infty\}$$

by setting for $\Omega \in \mathcal{A}_B(D)$

$$J(\Omega) := \min_{\substack{u \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \\ u = g \text{ on } B}} \left[\int_\Omega f(x, \nabla u)\, dx + \int_{\partial\Omega} \beta(x) [|u^+|^p + |u^-|^p] \, d\mathcal{H}^{d-1} \right] + \gamma |\Omega|. \tag{3.3}$$

To make sense of the previous expression, notice that if we extend u to zero outside Ω, we obtain a function of bounded variation on $\mathbb{R}^d$ such that $J_u \subseteq \partial\Omega$: $u^\pm$ are thus the traces of u on $\partial\Omega$ in the sense of BV-theory, which turn out to be well defined up to a $\mathcal{H}^{d-1}$-negligible set. Notice that we admit *two traces* since the geometry of Ω is compatible with the presence of *inner cracks*. If Ω is regular, then the previous expression for J coincides clearly with the classical one.

Remark 3.1. The existence of a minimizer u in the definition (3.3) of J such that

$$0 \leq u \leq \|g\|_\infty$$

is easily proved. Indeed, if $(u_n)_{n\in\mathbb{N}}$ is a minimizing sequence, by truncation we may assume $0 \leq u_n \leq \|g\|_\infty$. If $u \in W^{1,p}(\Omega)$ is a limit point of the sequence in the weak topology, then it is a minimizer since

$$\int_\Omega f(x, \nabla u)\, dx \leq \liminf_{n\to+\infty} \int_\Omega f(x, \nabla u_n)\, dx$$

and, using a standard slicing argument (see e.g. [5, Lemma 6.16])

$$\int_{\partial\Omega} \beta(x) [|u^+|^p + |u^-|^p] \, d\mathcal{H}^{d-1} \leq \liminf_{n\to+\infty} \int_{\partial\Omega} \beta(x) [|u_n^+|^p + |u_n^-|^p] \, d\mathcal{H}^{d-1}.$$

The main result of the paper is the following.

Theorem 3.2. *Let $D \subseteq \mathbb{R}^d$ be an open, bounded set with Lipschitz boundary, let $f : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty[$ satisfy $(H_1) - (H_3)$, and let $\beta : \mathbb{R}^d \to [0, +\infty[$ satisfy (3.1). Given $B \subset D$ open and with a C^1-boundary, and $g \in C^1(\mathbb{R}^d)$ satisfying (3.2), the shape optimization problem*

$$\min_{\Omega \in \mathcal{A}_B(D)} J(\Omega) \tag{3.4}$$

admits a solution $\tilde{\Omega}$, which moreover satisfies

$$J(\tilde{\Omega}) = \inf_{\substack{\Omega \in \mathcal{A}_B(D) \\ \Omega \text{ is Lipschitz}}} J(\Omega). \tag{3.5}$$

Theorem 3.2 thus proves that the new shape optimization problem (3.4) is well posed and it is a *relaxed version* of the original one on regular domains.

In order to address problem (3.4) we consider the free discontinuity functional

$$F(u) := \int_{\mathbb{R}^d} f(x,\nabla u)\,dx + \int_{J_u} \beta(x)[(u^+)^p + (u^-)^p|]\,d\mathcal{H}^{d-1} + \gamma|\{u>0\}| \tag{3.6}$$

on the set

$$\mathcal{F}_{B,g}(D) := \{u \in SBV(\mathbb{R}^d) \,:\, supp(u) \subseteq \bar{D}, u \geq 0, u = g \text{ on } B\}.$$

The basic idea is that the minimization of F defined on functions should be easier than that of J defined on sets. Moreover we expect to recover an optimal domain by considering the support $\{u>0\}$ of an optimal function u. The key point for this to hold true is to show that minimizers of (3.6) are regular enough to guarantee that their support belongs to $\mathcal{A}_B(D)$.

Existence of minimizers for (3.6) on $\mathcal{F}_{B,g}(D)$ is by no means obvious, since coercivity properties of minimizing sequences are not compatible with the usual compactness in SBV given by Ambrosio's theorem. Let indeed $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence. By truncation, is not restrictive to assume

$$0 \leq u_n \leq \|g\|_\infty.$$

Then we get by comparing with $g1_D$

$$\int_D |\nabla u_n|^p\,dx + \int_{J_{u_n}} |u_n^+|^p + |u_n^-|^p\,d\mathcal{H}^{d-1} \leq C, \tag{3.7}$$

with C independent of n. We thus see that no bounds on the $\mathcal{H}^{d-1}$-measure of the jump sets are available, so that a direct application of Ambrosio's theorem is forbidden.

We have however the following compactness result.

Proposition 3.3. *There exists $u \in L^\infty(\mathbb{R}^d)$ with $u \geq 0$, $supp(u) \subseteq \bar{D}$ and $u^p \in SBV(\mathbb{R}^d)$ such that, up to a subsequence,*

$$u_n \to u \qquad a.e.\ in\ \mathbb{R}^d.$$

Moreover for every $v \in \mathcal{F}_{B,g}(D)$

$$\int_{\mathbb{R}^d} f(x,\nabla u)\,dx + \int_{J_u} \beta(x)[(u^+)^p + (u^-)^p]\,d\mathcal{H}^{d-1} + \gamma|\{u>0\}| \leq F(v)$$

(notice that u is approximately differentiable a.e. since $u^p \in SBV(\mathbb{R}^d)$).

Proof. By the chain rule in BV we get that $w_n := u_n^p \in SBV(\mathbb{R}^d)$ with

$$\nabla w_n = p u_n^{p-1} \nabla u_n.$$

In view of (3.7) and since $u_n \leq \|g\|_\infty$ we obtain

$$\sup_n |Dw_n|(\mathbb{R}^d) < +\infty.$$

Up to a subsequence we get

$$w_n \to w \qquad \text{strongly in } L^1(\mathbb{R}^d) \tag{3.8}$$

for some $w \in BV(\mathbb{R}^d)$ with $w \geq 0$ and $supp(w) \subseteq \bar{D}$. Notice that Ambrosio's theorem can be applied locally to $w_n \vee \varepsilon$ for every $\varepsilon > 0$. We thus get $w \vee \varepsilon \in SBV_{loc}(\mathbb{R}^d)$ for every $\varepsilon > 0$, so that $w \in SBV(\mathbb{R}^d)$.

We set

$$u := w^{1/p},$$

so that the first part of the statement is proved.

Let us come to the minimality property of u. Notice that the sequence $((u_n-\varepsilon)_+)_{n\in\mathbb{N}}$satisfies the assumptions of Ambrosio's compactness theorem for every $\varepsilon>0$. By the lower semicontinuity for surface energies [4, Theorem 2.12] we get in view of the arbitrariness of ε

$$\int_{J_u}\beta(x)[(u^+)^p+(u^-)^p]\,d\mathcal{H}^{d-1}\le\liminf_{n\to+\infty}\int_{J_{u_n}}\beta(x)[(u_n^+)^p+(u_n^-)^p]\,d\mathcal{H}^{d-1}. \tag{3.9}$$

If up to a subsequence

$$\nabla u_n\rightharpoonup\Phi\qquad\text{weakly in } L^p(\mathbb{R}^d;\mathbb{R}^d)$$

we deduce from (3.8) and the equality $\nabla(u_n-\varepsilon)_+=\nabla u_n 1_{\{u_n\ge\varepsilon\}}$ that

$$\Phi=\nabla u\qquad\text{a.e. on }\{u>0\}.$$

By lower semicontinuity we have

$$\int_{\mathbb{R}^d}f(x,\nabla u)\,dx\le\int_{\mathbb{R}^d}f(x,\Phi)\,dx\le\liminf_{n\to+\infty}\int_{\mathbb{R}^d}f(x,\nabla u_n)\,dx. \tag{3.10}$$

Since we also have

$$|\{u>0\}|\le\liminf_{n\to\infty}|\{u_n>0\}|,$$

the minimality follows collecting (3.9) and (3.10) □

The function provided by the previous proposition does not a priori belong to $\mathcal{F}_{B,g}(D)$ since the SBV-regularity is unclear. In order to address such an issue, we introduce the following notion of *subsolution*.

Definition 3.4 (Subsolution). *We say that $u\in\mathcal{F}_{B,g}(D)$ is a subsolution for the functional*

$$u\mapsto\int_{\mathbb{R}^d}f(x,\nabla u)\,dx+\int_{J_u}\beta(x)[(u^+)^p+(u^-)^p]\,d\mathcal{H}^{d-1} \tag{3.11}$$

if for every $v\in\mathcal{F}_{B,g}(D)$ with $0\le v\le u$ we have

$$F(u)\le F(v).$$

Clearly, for every $\gamma\ge0$ any minimizer given by Proposition 3.3 is also a subsolution in the sense defined above. The following non degeneracy result for subsolutions, which is related to the Hopf principle (see Remark 3.8), holds true.

Theorem 3.5 (Bound from below). *Let u be a subsolution for* (3.11)*. Then there exists $\alpha>0$ such that*

$$u\ge\alpha\qquad\text{a.e. on }\{u>0\}. \tag{3.12}$$

Proof. Let $\varepsilon<c_1$ (defined in (3.2)) be such that $u1_{\{u>\varepsilon\}}\in SBV(\mathbb{R}^d)$. Then such a function belongs to $\mathcal{F}_{B,g}(D)$, so that, by comparison with u

$$\begin{aligned}\int_{\mathbb{R}^d}f(x,\nabla u)\,dx+\int_{J_u}\beta(x)[(u^+)^p+(u^-)^p]\,d\mathcal{H}^{d-1}&\\ \le\int_{\{u>\varepsilon\}}f(x,\nabla u)\,dx+\int_{\{u^-<\varepsilon\le u^+\}\cap J_u}\beta(x)(u^+)^p\,d\mathcal{H}^{d-1}+&\\ +\int_{\{\varepsilon\le u^-<u^+\}\cap J_u}\beta(x)[(u^+)^p+(u^-)^p]\,d\mathcal{H}^{d-1}+\int_{\partial^e\{u>\varepsilon\}\setminus J_u}\beta(x)\varepsilon^p\,d\mathcal{H}^{d-1}.&\end{aligned}$$

By the assumptions on f and β we deduce

$$L^{-1}\int_{\{u\le\varepsilon\}}|\nabla u|^p\,dx+\beta_1\int_{\{u^-<u^+\le\varepsilon\}\cap J_u}[(u^-)^p+(u^+)^p]\,d\mathcal{H}^{d-1}\le\beta_2\varepsilon^p\mathcal{H}^{d-1}(\partial^e\{u>\varepsilon\}\setminus J_u)$$

which entails for a.e. $0<\delta<\varepsilon$

$$\int_{\{u\le\varepsilon\}}|\nabla u|^p\,dx+L\beta_1\delta^p\mathcal{H}^{d-1}(\partial^e\{\delta<u<\varepsilon\}\cap J_u)\le L\beta_2\varepsilon^p\mathcal{H}^{d-1}(\partial^e\{u>\varepsilon\}\setminus J_u).$$

Setting

$$E(\varepsilon) := \int_{\{u<\varepsilon\}} |\nabla u|^p \, dx, \qquad \gamma(\delta,\varepsilon) := \mathcal{H}^{d-1}(\partial^e\{\delta < u < \varepsilon\} \cap J_u), \tag{3.13}$$

$$\text{and} \qquad h(\varepsilon) := \mathcal{H}^{d-1}(\partial^e\{u \geq \varepsilon\} \setminus J_u),$$

we deduce thus that for a.e. $0 < \delta < \varepsilon < c_1$

$$E(\varepsilon) + L\beta_1\delta^p\gamma(\delta,\varepsilon) \leq L\beta_2\varepsilon^p h(\varepsilon). \tag{3.14}$$

Let $\eta > 0$ and set

$$\varepsilon_i := \frac{5}{6}\eta + \frac{2^{-i}}{6}\eta \qquad \text{and} \qquad \delta_i := \frac{2}{3}\eta - \frac{2^{-i}}{6}\eta. \tag{3.15}$$

We have for $i \to +\infty$

$$\varepsilon_i \to e_\infty := \frac{5}{6}\eta \qquad \text{and} \qquad \delta_i \to \delta_\infty := \frac{2}{3}\eta.$$

Setting

$$\Omega(\delta,\varepsilon) := \{\delta < u < \varepsilon\}, \tag{3.16}$$

we will see that there exists $\eta_0 > 0$ small enough such that for $\eta < \eta_0$

$$\left|\Omega\left(\frac{2}{3}\eta, \frac{5}{6}\eta\right)\right| \int_{\frac{2}{3}\eta}^{\frac{5}{6}\eta} h(s)\, ds = 0. \tag{3.17}$$

Since in view of the isoperimetric inequality we have

$$|\Omega(\delta,\varepsilon)|^{\frac{d-1}{d}} \leq C_d(h(\varepsilon) + h(\delta) + \gamma(\delta,\varepsilon)), \tag{3.18}$$

relation (3.17) together with (3.14) entails that

$$\left|\Omega\left(\frac{2}{3}\eta, \frac{5}{6}\eta\right)\right| = 0.$$

Since $\eta < \eta_0$ is arbitrary, we get that

$$u \geq \frac{5}{6}\eta_0 \qquad \text{a.e. on } supp(u).$$

The proof of (3.17) will be the outcome of an iteration scheme which resembles under certain aspects the iteration à la De Giorgi in the study of elliptic regularity. We divide the proof in several steps.

Step 1: The main inequalities. Let us set for $i \in \mathbb{N}$

$$a_i := \int_{\delta_i}^{\varepsilon_i} h(s)\, ds \qquad \text{and} \qquad b_i := |\Omega(\delta_i, \varepsilon_i)|,$$

where h is defined in (3.13), while $\Omega(\delta,\varepsilon)$ is given in (3.16).

We claim that there exist $c_1, c_2 > 0$ depending only on d, p and β such that for every $i \geq 1$

$$\begin{cases} a_i \leq c_1 2^i b_{i-1}^{\frac{1}{dp'}} a_{i-1} \\ \\ b_i \leq \dfrac{c_2\left(2^{\frac{d}{d-1}}\right)^i}{\eta^{\frac{d}{d-1}}} a_{i-1}^{\frac{d}{d-1}}, \end{cases} \tag{3.19}$$

where $p' := p/(p-1)$.

Let us start with the second inequality. For every $\eta/2 < \delta < \varepsilon < \eta$, the isoperimetric inequality (3.18) together with inequality (3.14) entails

$$|\Omega(\delta,\varepsilon)|^{\frac{d-1}{d}} \leq C_{d,\beta}(1 + 2^p)[h(\varepsilon) + h(\delta)], \tag{3.20}$$

where $C_{d,\beta} := C_d\beta_2/\beta_1$. Integrating in ε between $[\varepsilon_i, \varepsilon_{i-1}]$ we get

$$\int_{\varepsilon_i}^{\varepsilon_{i-1}} |\Omega(\delta,\varepsilon)|^{\frac{d-1}{d}}\, d\varepsilon \le C_{d,\beta}(1+2^p)\left[\int_{\varepsilon_i}^{\varepsilon_{i-1}} h(\varepsilon)\, d\varepsilon + h(\delta)(\varepsilon_{i-1}-\varepsilon_i)\right],$$

so that, recalling the definition of ε_i in (3.15)

$$|\Omega(\delta,\varepsilon_i)|^{\frac{d-1}{d}}\left[\frac{1}{6}2^{-i+1}\eta - \frac{1}{6}2^{-i}\eta\right] \le C_{d,\beta}(1+2^p)\left[\int_{\varepsilon_i}^{\varepsilon_{i-1}} h(\varepsilon)\, d\varepsilon + h(\delta)\left(\frac{1}{6}2^{-i+1}\eta - \frac{1}{6}2^{-i}\eta\right)\right].$$

Integrating in δ on $[\delta_{i-1}, \delta_i]$ we deduce

$$|\Omega(\delta_i,\varepsilon_i)|^{\frac{d-1}{d}}\left[\frac{1}{6}2^{-i+1}\eta - \frac{1}{6}2^{-i}\eta\right]^2 \le C_{d,\beta}(1+2^p)\left[\frac{1}{6}2^{-i+1}\eta - \frac{1}{6}2^{-i}\eta\right]\left[\int_{\varepsilon_i}^{\varepsilon_{i-1}} h(\varepsilon)\, d\varepsilon + \int_{\delta_{i-1}}^{\delta_i} h(\delta)\, d\delta\right].$$

We get

$$|\Omega(\delta_i,\varepsilon_i)|^{\frac{d-1}{d}} \le C_{d,\beta}(1+2^p)\frac{6\cdot 2^i}{\eta}\left[\int_{\varepsilon_i}^{\varepsilon_{i-1}} h(\varepsilon)\, d\varepsilon + \int_{\delta_{i-1}}^{\delta_i} h(\delta)\, d\delta\right],$$

so that the second inequality in (3.19) follows.

Let us come to the first inequality. Notice that for every $\eta/2 < \delta < \varepsilon < \eta$, the coarea formula, inequality (3.20), and the main inequality (3.14) entail

$$\begin{aligned}\int_\delta^\varepsilon h(s)\, ds &= \int_{\Omega(\delta,\varepsilon)} |\nabla u|\, dx \le |\Omega(\delta,\varepsilon)|^{\frac{1}{p'}}\left(\int_{\Omega(\delta,\varepsilon)} |\nabla u|^p\, dx\right)^{\frac{1}{p}} \\ &\le |\Omega(\delta,\varepsilon)|^{\frac{1}{p'}}(L\beta_2)^{\frac{1}{p}}\varepsilon h(\varepsilon)^{\frac{1}{p}} = |\Omega(\delta,\varepsilon)|^{\frac{1}{p'd}}|\Omega(\delta,\varepsilon)|^{\frac{d-1}{p'd}}(L\beta_2)^{\frac{1}{p}}\varepsilon h(\varepsilon)^{\frac{1}{p}} \\ &\le |\Omega(\delta,\varepsilon)|^{\frac{1}{p'd}}[C_{d,\beta}(1+2^p)]^{\frac{1}{p'}}[h(\varepsilon)+h(\delta)]^{\frac{1}{p'}}(L\beta_2)^{\frac{1}{p}}\varepsilon h(\varepsilon)^{\frac{1}{p}} \\ &\le |\Omega(\delta,\varepsilon)|^{\frac{1}{p'd}}[C_{d,\beta}(1+2^p)(L\beta_2)^{\frac{p'}{p}}]^{\frac{1}{p'}}\varepsilon[h(\varepsilon)+h(\delta)].\end{aligned}$$

Integrating in ε on $[\varepsilon_i, \varepsilon_{i-1}]$ we get

$$\int_{\varepsilon_i}^{\varepsilon_{i-1}}\left[\int_\delta^\varepsilon h(s)\, ds\right] d\varepsilon \le [C_{d,\beta}(1+2^p)(L\beta_2)^{\frac{p'}{p}}]^{\frac{1}{p'}}|\Omega(\delta,\varepsilon_{i-1}|^{\frac{1}{p'd}}\varepsilon_{i-1}\left[\int_{\varepsilon_i}^{\varepsilon_{i-1}} h(\varepsilon)\, d\varepsilon + h(\delta)[\varepsilon_{i-1}-\varepsilon_i]\right]$$

so that

$$\int_\delta^{\varepsilon_i} h(s)\, ds\left[\frac{1}{6}2^{-i}\eta\right] \le [C_{d,\beta}(1+2^p)(L\beta_2)^{\frac{p'}{p}}]^{\frac{1}{p'}}|\Omega(\delta,\varepsilon_{i-1}|^{\frac{1}{p'd}}\varepsilon_{i-1}\left[\int_{\varepsilon_i}^{\varepsilon_{i-1}} h(\varepsilon)\, d\varepsilon + h(\delta)\frac{1}{6}2^{-i}\eta\right].$$

Integrating now in δ on $[\delta_{i-1}, \delta_i]$ we obtain

$$\int_{\delta_i}^{\varepsilon_i} h(s)\, ds\left[\frac{1}{6}2^{-i}\eta\right]^2 \le [C_{d,\beta}(1+2^p)(L\beta_2)^{\frac{p'}{p}}]^{\frac{1}{p'}}|\Omega(\delta_{i-1},\varepsilon_{i-1}|^{\frac{1}{p'd}}\varepsilon_{i-1}\left[\frac{1}{6}2^{-i}\eta\right]\left[\int_{\varepsilon_i}^{\varepsilon_{i-1}} h(\varepsilon)\, d\varepsilon + \int_{\delta_{i-1}}^{\delta_i} h(\delta)d\delta\right]$$

which imply the first inequality in (3.19) since $\varepsilon_{i-1} \le \eta$.

Step 2: Combining the main inequalities. We claim that we can find $\alpha > 0$ such that setting

$$U_i := a_i^\alpha b_i$$

we have for $i \ge 1$

$$U_i \le \frac{\tilde{c}}{\eta^{\frac{d}{d-1}}}A^i U_{i-1}^\vartheta \tag{3.21}$$

where $\tilde{c}, A > 0$, and $\vartheta > 1$.

Indeed using (3.19) we have for $\alpha>0$ and $i\geq 1$

$$a_i^\alpha b_i \leq \frac{c_1^\alpha c_2}{\eta^{\frac{d}{d-1}}}[2^{\alpha+\frac{d}{d-1}}]^i b_{i-1}^{\frac{\alpha}{dp'}} a_{i-1}^{\alpha+\frac{d}{d-1}}.$$

Writing

$$\begin{cases}\alpha+\frac{d}{d-1}=\vartheta\alpha\\ \frac{\alpha}{p'd}=\vartheta\end{cases}$$

we obtain

$$\alpha=\frac{1+\sqrt{1+\frac{4}{p'(d-1)}}}{\frac{2}{p'd}} \tag{3.22}$$

and

$$\vartheta=\frac{1+\sqrt{1+\frac{4}{p'(d-1)}}}{2}>1,$$

so that inequality (3.21) follows

Step 3: Decay for $E(\varepsilon)$. We claim that there exists $\varepsilon_0>0$ and $c_0>0$ such that for $\varepsilon\leq\varepsilon_0$

$$E(\varepsilon)\leq c_0\varepsilon^p. \tag{3.23}$$

Indeed from the inequality $E(\varepsilon)\leq L\beta_2\varepsilon^p h(\varepsilon)$ we infer using the coarea formula

$$\begin{aligned}\varepsilon E(\varepsilon)\leq\int_\varepsilon^{2\varepsilon}E(s)\,ds&\leq\int_\varepsilon^{2\varepsilon}L\beta_2 s^p h(s)\,ds\leq L\beta_2 2^p\varepsilon^p\int_\varepsilon^{2\varepsilon}h(s)\,ds\\&=L\beta_2 2^p\varepsilon^p\int_{\Omega(\varepsilon,2\varepsilon)}|\nabla u|\,dx\leq L\beta_2 2^p\varepsilon^p\left(\int_{\Omega(\varepsilon,2\varepsilon)}|\nabla u|^p\,dx\right)^{\frac{1}{p}}|\Omega(\varepsilon,2\varepsilon)|^{\frac{1}{p'}}\end{aligned}$$

so that

$$E(\varepsilon)\leq L\beta_2 2^p\varepsilon^{p-1}|\Omega(\varepsilon,2\varepsilon)|^{\frac{1}{p'}}E(2\varepsilon)^{\frac{1}{p}}.$$

Let $\varepsilon_0>0$ be such that

$$L\beta_2 2^p|\Omega(0,2\varepsilon_0)|^{\frac{1}{p'}}\leq 1.$$

Then for $\varepsilon\leq\varepsilon_0$ we have

$$E(\varepsilon)\leq\varepsilon^{p-1}E(2\varepsilon)^{\frac{1}{p}}$$

and (3.23) is a consequence of Lemma 3.6 below.

Step 4: Conclusion. Using the notation of Step 3, we claim that we can find η so small that

$$U_0\leq\left(\frac{\tilde c}{\eta^{\frac{d}{d-1}}}\right)^{-\frac{1}{\vartheta-1}}A^{-\frac{1}{(\vartheta-1)^2}}=\tilde c^{-\frac{1}{\vartheta-1}}A^{-\frac{1}{(\vartheta-1)^2}}\eta^\alpha, \tag{3.24}$$

where α is given in (3.22). Then it is easily seen by induction that $U_i\leq A^{-\frac{i}{\vartheta-1}}U_0$ so that

$$\lim_{i\to+\infty}U_i=0,$$

and equality (3.17), concluding the proof of the theorem.

In order to verify that (3.24) can be achieved for η small enough, we write

$$\begin{aligned}U_0=\left|\Omega\left(\frac{\eta}{2},\eta\right)\right|\left[\int_{\eta/2}^\eta h(s)\,ds\right]^\alpha&=\left|\Omega\left(\frac{\eta}{2},\eta\right)\right|\left[\int_{\Omega\left(\frac{\eta}{2},\eta\right)}|\nabla u|\,dx\right]^\alpha\\&\leq\left|\Omega\left(\frac{\eta}{2},\eta\right)\right|\left[E(\eta)^{\frac{1}{p}}\left|\Omega\left(\frac{\eta}{2},\eta\right)\right|^{\frac{1}{p'}}\right]^\alpha=\left|\Omega\left(\frac{\eta}{2},\eta\right)\right|^{1+\frac{\alpha}{p'}}E(\eta)^{\frac{\alpha}{p'}}\end{aligned}$$

so that thanks to the decay for $E(\eta)$ obtained in Step 3 we deduce

$$U_0\leq\hat C_0\left|\Omega\left(\frac{\eta}{2},\eta\right)\right|^{1+\frac{\alpha}{p'}}\eta^\alpha$$

for some $\hat{C}_0 > 0$. Then (3.24) is achieved if η is so small that

$$\hat{C}_0 \left| \Omega\left(\frac{\eta}{2}, \eta\right)\right|^{1+\frac{\alpha}{p'}} \leq \tilde{c}^{-\frac{1}{\vartheta-1}} A^{-\frac{1}{(\vartheta-1)^2}}.$$

□

In the previous proof, we made use of the following simple lemma.

Lemma 3.6. *Let $\varphi : [0, +\infty[\to [0, +\infty[$ be a monotone function such that for every $r \leq r_0$*

$$\varphi(r) \leq (2r)^\alpha \varphi(2r)^\beta,$$

where $r_0, \alpha > 0$ and $0 < \beta < 1$. Then there exists $C > 0$ such that for $r \leq r_0$

$$\varphi(r) \leq C r^{\frac{\alpha}{1-\beta}}.$$

Proof. Let $r \leq r_0$ and $m \in \mathbb{N}$ such that $\frac{r_0}{2} < 2^m r \leq r_0$. We can write

$$\begin{aligned}\varphi(r) &\leq (2r)^\alpha \varphi(2r)^\beta \leq (2r)^\alpha [(4r)^\alpha \varphi(4r)^\beta]^\beta \\ &\leq \cdots \leq [2^{1+2\beta+3\beta^2+\cdots+m\beta^{m-1}} r^{1+\beta+\beta^2+\cdots+\beta^{m-1}}]^\alpha \varphi(2^m r)^{\beta^m} \\ &\leq \tilde{C} r^{\frac{\alpha}{1-\beta}} r^{-\frac{\beta^m \alpha}{1-\beta}} \leq \hat{C} r^{\frac{\alpha}{1-\beta}} \left(2^{-m-1} r_0\right)^{-\frac{\beta^m \alpha}{1-\beta}} \leq C r^{\frac{\alpha}{1-\beta}},\end{aligned}$$

and the result follows. □

Remark 3.7. An inspection of the previous proof shows that the estimate from below on the support is a consequence of the following inequality (according to the notation introduced above)

$$E(\varepsilon) + c_1 \delta^p \gamma(\delta, \varepsilon) \leq c_2 \varepsilon^p h(\varepsilon) \qquad \text{for a.e. } 0 < \delta < \varepsilon \leq \varepsilon_0,$$

where $c_1, c_2 > 0$, which is a consequence of the comparison between u and $u 1_{\{u \geq \varepsilon\}}$ (ε_0 smaller than the constants appearing in (3.2)).

Remark 3.8 (Bound from below and the Hopf Lemma). In the classical setting with $p = 2$, $f(x, \xi) = |\xi|^2$ and $\beta(x) = \beta$, the bound from below of u on the associated regular domain Ω is a consequence of the classical Hopf lemma in view of the Robin condition at the boundary. Let indeed the subsolution $u \in W^{1,2}(\Omega)$, $u \geq 0$, be associated to Ω. We get easily that u is subharmonic in Ω. Let x_0 be a minimum point of u on $\overline{\Omega} \setminus B$. We can assume $x_0 \in \partial\Omega$, otherwise the bound from below is trivial. Since by Hopf Lemma $\frac{\partial u}{\partial \nu}(x_0) < 0$, the Robin condition entails

$$\beta u(x_0) = -\frac{\partial u}{\partial \nu}(x_0) > 0$$

so that $u \geq u(x_0) > 0$ on Ω.

Remark 3.9. In the case $p = 2$ and $\gamma > 0$, the result of Theorem 3.5 was proved in [6] for subsolutions of

$$u \mapsto \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \beta \int_{J_u} [(u^+)^2 + (u^-)^2] \, d\mathcal{H}^{d-1} + \gamma |\{u > 0\}|, \tag{3.25}$$

where $\beta > 0$. In this paper, based on an iteration technique à la De Giorgi, we also cover the limit case $\gamma = 0$.

We can now show that the free discontinuity problem is well posed.

Theorem 3.10. *Let u be the function given by Proposition 3.3. Then $u \in \mathcal{F}_{B,g}(D)$, and it is a minimizer for the free discontinuity functional F in* (3.6).

Proof. The SBV regularity of u follows by the chain rule formula in BV in view of the lower bound (3.12) and of the fact $u^p \in SBV(\mathbb{R}^d)$. The minimality is a consequence of Proposition 3.3. □

The following result is fundamental to prove some regularity for minimizers of F.

Proposition 3.11 (Almost-quasi minimality). *Let u be a minimizer of F on $\mathcal{F}_{B,g}(D)$ with $0 \le u \le \|g\|_\infty$. Then there exists $\alpha > 0$ with*

$$u > \alpha \qquad \text{a.e. on } supp(u) \tag{3.26}$$

and such that function $(2\beta_1)^{1/p}\alpha u$ is an almost-quasi minimizer of the Mumford-Shah functional

$$MS(u) := \int_D f(x, \nabla u)\, dx + \mathcal{H}^{d-1}(J_u)$$

on D with Dirichlet condition $u = g$ on B according to Definition 2.2.

Proof. The existence of $\alpha > 0$ satisfying (3.26) is a consequence Theorem 3.5. Coming to almost-quasi minimality, let $B_\rho(x) \subset D$ and $v \in SBV_{loc}(D)$ be such that $v = g$ on B and $\{v \neq u\} \subseteq B_\rho(x) \subseteq D$. Let us consider

$$w := (v \wedge \|g\|_\infty) \vee 0.$$

Clearly $w \in SBV(D)$ with $\{w \neq u\} \subseteq B_\rho(x)$ and still $w = g$ on B. Comparing u and w we get

$$\int_{B_\rho(x)} f(x, \nabla u)\, dx + 2\beta_1 \alpha^p \mathcal{H}^{d-1}(J_u \cap \bar{B}_\rho(x)) \\ \le \int_{B_\rho(x)} f(x, \nabla w)\, dx + 2\beta_2 \|g\|_\infty^p \mathcal{H}^{d-1}(J_w \cap \bar{B}_\rho(x)) + \gamma \omega_d r^d$$

so that

$$\int_{B_\rho(x)} f(x, \nabla u)\, dx + 2\beta_1 \alpha^p \mathcal{H}^{d-1}(J_u \cap \bar{B}_\rho(x)) \\ \le \int_{B_\rho(x)} f(x, \nabla v)\, dx + 2\beta_2 \|g\|_\infty^p \mathcal{H}^{d-1}(J_v \cap \bar{B}_\rho(x)) + \gamma \omega_d r^d,$$

and the result follows. □

We complete our analysis of the free discontinuity functional F with the following density result.

Proposition 3.12. *Given $v \in \mathcal{F}_{B,g}(D) \cap L^\infty(\mathbb{R}^d)$ with $\mathcal{H}^{d-1}(J_v) < +\infty$, for every $\varepsilon > 0$ there exists $w \in \mathcal{F}_{B,g}(D) \cap L^\infty(\mathbb{R}^d)$ with $J_w \subset\subset D \setminus \overline{B}$, $\mathcal{H}^{d-1}(J_w) < +\infty$, and such that*

$$F(w) \le F(v) + \varepsilon.$$

Proof. We can assume that $g \in C_c^1(D)$ with $g \ge 0$. Let $\varepsilon_1 > 0$ to be fixed below, and let $U \subset\subset D \setminus \bar{B}$ open such that

$$\mathcal{H}^{d-1}(J_v \cap (D \setminus U)) < \varepsilon_1.$$

Since $D \setminus \bar{B}$ has Lipschitz boundary, for every $x \in \partial D \cup \partial B$ we can find an orthogonal coordinate system $y = (y', y_d)$ with origin at x, two numbers $r, s > 0$, and a Lipschitz function $f^x : \mathbb{R}^{d-1} \to \mathbb{R}$ such that setting

$$V_{r,s}^x := \{|y'| < r, |y_d| < s\}$$

we have $V_{r,s}^x \cap U = \emptyset$ and

$$(D \setminus \bar{B}) \cap V_{r,s}^x = \{y \in V_{r,s}^x \,:\, y_d < f^x(y')\}.$$

We can also assume that for $\xi > 0$ small enough

$$\{|y'| < r, f^x(y') - \xi < y_d < f^x(y') + \xi\} \subseteq V_{r,s}^x.$$

By compactness, we can cover $\partial D \cup \partial B$ with a finite number of neighborhoods $V_1, \dots, V_m$ of the type $V_i := V_{r_i,s_i}^{x_i}$, and associated function f_i. Moreover it is not restrictive to assume that the $\mathcal{H}^{d-1}$-measure of the part of ∂D on which the V_i's overlap is less than ε_1. Let $\psi_1, \dots, \psi_m$ be a partition of unity for ∂D subordinated to the V_i's. We write

$$v = \psi_1 v + \dots + \psi_m v + (1 - \psi_1 - \dots - \psi_m)v := v_1 + \dots + v_m + v_0.$$

Notice that $v_0 = v$ on U, and that v_i has compact support in V_i (so that we can assume that they are defined on the entire $\mathbb{R}^d$ for the operations performed below).

We proceed to approximate each v_i by "pushing inside" the jumps occurring at the boundary. Given $\xi > 0$ small, we consider

$$v_i^\xi(y) := \begin{cases} v_i(y', y_d + \xi) & \text{if } y_d < f_i(y') - \xi \\ (\psi_i g)(y', f_i(y')) & \text{if } y_d \geq f_i(y') - \xi. \end{cases}$$

Let us consider

$$v^\xi := v_1^\xi + \cdots + v_m^\xi + v_0.$$

By construction, $v^\xi \in SBV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with the jump set with finite $\mathcal{H}^{d-1}$-measure, and well contained in D, being v^ξ Lipschitz regular in a neighborhood of ∂D.

Concerning the behaviour of the functional F, for $\xi \to 0$ we get that the volume energy of v^ξ and the measure of its support are arbitrarily closed to that of v. As for the surface energy, which we denote by $\mathcal{E}^s$, thanks to the continuity of $\beta(x)$ we get with obvious notation

$$\mathcal{E}^s(v^\xi) \leq \mathcal{E}^s(v_0, U) + \mathcal{E}^s(v, \partial D \cup \partial B) + C\varepsilon_1,$$

where C depends only on m, g and $\|v\|_\infty$. The last term takes into account the jumps in $(D \backslash \bar{B}) \backslash U$, and the possible interference of the v_i^ξ obtained by the translation of the v_i's. The result thus follows if $C\varepsilon_1 < \varepsilon$ and ξ is small enough. □

We are now in a position to prove the main result of the paper.

Proof of Theorem 3.2. Let u be a minimizer of F on $\mathcal{F}_{B,g}(D)$ with $0 \leq u \leq \|g\|_\infty$: its existence is secured by Proposition 3.10. Thanks to Proposition 3.11 and to Theorem 2.3, we deduce that

$$\mathcal{H}^{d-1}\left((\overline{J_u} \setminus J_u) \cap D\right) = 0, \tag{3.27}$$

i.e., the jump set of u is essentially closed in D. Moreover the lower bound (3.26) entails

$$\mathcal{H}^{d-1}(J_u) < +\infty. \tag{3.28}$$

Let Ω be given by the connected component of $D \setminus \overline{J_u}$ on which u does not vanish (we have just one component by minimality). Recalling (3.28), and since

$$\partial\Omega \subseteq \left(\overline{J_u} \cap D\right) \cup \partial D,$$

we get that $\Omega \in \mathcal{A}_B(D)$. Moreover

$$\int_{\partial\Omega \setminus J_u} [(u^+)^p + (u^-)^p]\, d\mathcal{H}^{d-1} = 0. \tag{3.29}$$

Indeed thanks to (3.27)

$$\mathcal{H}^{d-1}\left((\partial\Omega \setminus J_u) \cap D\right) = 0,$$

while

$$u^\pm = 0 \qquad \mathcal{H}^{d-1}\text{-a.e. on } (\partial\Omega \setminus J_u) \cap \partial D$$

since, being D Lipschitz, we have $u^- = 0$ $\mathcal{H}^{d-1}$-a.e. on ∂D.

Since

$$u_{|\Omega} \in W^{1,p}(\Omega),$$

and taking into account (3.29), we conclude that

$$J(\Omega) \leq \int_\Omega f(x, \nabla u)\, dx + \int_{\partial\Omega} \beta(x)[|u^+|^p + |u^-|^p]\, d\mathcal{H}^{d-1} = F(u). \tag{3.30}$$

Let us prove that Ω is a minimizer of the shape optimization problem (3.4). Let us consider $\tilde{\Omega} \in \mathcal{A}_B(D)$ with associated function $v \in W^{1,p}(\tilde{\Omega}) \cap L^\infty(\tilde{\Omega})$ which realizes $J(\tilde{\Omega})$. By extending v to zero outside $\tilde{\Omega}$ we get $v \in \mathcal{F}_{B,g}(D)$, $J_v \subseteq \partial\tilde{\Omega}$, and

$$\begin{aligned} F(v) = \int_{\mathbb{R}^d} f(x, \nabla v)\, dx + \int_{J_v} \beta(x)[|v^+|^p + |v^-|^p]\, d\mathcal{H}^{d-1} \\ \leq \int_{\tilde{\Omega}} f(x, \nabla v)\, dx + \int_{\partial\tilde{\Omega}} \beta(x)[|v^+|^p + |v^-|^p]\, d\mathcal{H}^{d-1} = J(\tilde{\Omega}) \end{aligned}$$

so that in view of (3.30) we infer $J(\Omega) \le J(\tilde\Omega)$ and the optimality of Ω follows. Moreover we infer $J(\Omega) = F(u)$.

Let us finally prove the "relaxation condition" (3.5). By Proposition 3.12, for every $\varepsilon > 0$ we can find $w \in \mathcal{F}_{B,g}(D) \cap L^\infty(\mathbb{R}^d)$ with J_w well contained in $D \setminus \overline{B}$ and with finite $\mathcal{H}^{d-1}$-measure, and such that

$$F(w) < F(u) + \varepsilon.$$

By the density result Theorem 2.4, truncating and using a cut-off function to accommodate the Dirichlet condition on B, we can find $w_k \in \mathcal{F}_{B,g}(D) \cap L^\infty(\mathbb{R}^d)$ with $J_{w_k} \subset\subset D \setminus \overline{B}$ essentially closed and polyhedral (i.e., given by the union of the intersection with D of a finite number of $(d-1)$-simplexes), w_k is of class $W^{1,p}$ on $D \setminus \bar J_{w_k}$ for every k, and such that

$$\begin{aligned} w_k &\to w \qquad \text{strongly in } L^p(D),\\ \nabla w_k &\to \nabla w \qquad \text{strongly in } L^p(D;\mathbb{R}^d),\end{aligned}$$

and

$$F(w_k) \to F(w),$$

Note that the convergence for the measure of the supports can be obtained by replacing w_k given by Theorem 2.4, if necessary, by $(w_k - \eta_k)_+$ for suitable $\eta_k \to 0$.

If we enlarge the jump set creating small holes H_k with polyhedral boundary, we get that the domain

$$\Omega_k := D \setminus \overline{H_k} \in \mathcal{A}_B(D)$$

is Lipschitz regular. The restriction of w_k to Ω_k is moreover an admissible function to compute $J(\Omega_k)$. We can thus consider the holes so small and k so large that

$$J(\Omega_k) \le \int_{\Omega_k} f(x,\nabla w_k)\,dx + \int_{\partial\Omega_k} \beta(x)|w_k|^p\,d\mathcal{H}^{d-1} \le F(w_k) + \varepsilon \le J(u) + 2\varepsilon.$$

Now the result follows by letting $\varepsilon = \varepsilon_n \to 0$ and selecting the associated $k = k_n$.

□

4. Essential closedness of the jump set of almost-quasi minimizers of free discontinuity functionals

The present section is devoted to the proof of Theorem 2.3. As explained in Section 2, we follow the approach a la De Giorgi-Carriero-Leaci [10] along the lines of [2, Chapter 7]. The main point is to recover a decay lemma for the energy (see Theorem 4.8). This is achieved through a contradiction argument in which the analysis of sequences of almost-quasi minimizers with vanishing jump set play a key role. As in the classical setting, we need to prove that our weaker minimality still entails that they converge to a function without jump which is a local minimizer of the volume energy.

We divide the section in several parts. In Subsection 4.1 we collect some regularity results for local minimizers of integral functionals. In Subsection 4.2 we prove some technical lemmas concerning the behaviour of almost-quasi minimizers on the unit ball with vanishing jump set. In Subsection 4.3 we prove the basic decay lemma, while Subsection 4.4 contains the proof of Theorem 2.3.

4.1. **Regularity results for local minimizers of integral functionals.**

We will need suitable gradient bound estimates for local minimizers of the bulk energy. Let $f : \mathbb{R}^d \to [0,+\infty]$. For every $r \le 1$ and $u \in W^{1,p}(B_r)$, let

$$F(u,B_r) := \int_{B_r} f(\nabla u)\,dx$$

Definition 4.1 (Local minimizers). *Let $r > 0$. We say that $u \in W^{1,p}(B_r)$ is a local minimizer of $F(\cdot,B_r)$ if*

$$F(u,B_r) \le F(v,B_r)$$

for every $v \in W^{1,p}(B_r)$ with $\{u \ne v\} \subset\subset B_r$.

The following result has been proved in [11, Theorem 2.2].

Theorem 4.2 (**Interior gradient bound**). *Assume that $f:\mathbb{R}^d\to\mathbb{R}$ is a continuous function satisfying (H_1) and (H_2). Let $u\in W^{1,p}(B_1)$ be a local minimizer of $F(\cdot,B_1)$. Then u is locally Lipschitz in B_1, and there exists a constant $C_0=C_0(d,p,L,\mu)>0$ such that*

$$\sup_{B_{\rho/2}}|\nabla u|^p\le\frac{C_0}{\rho^d}\int_{B_\rho}|\nabla u|^p\,dx\quad\text{for every }\rho\le 1.$$

We need an analogous result for the case with boundary conditions. Let us introduce, for $\delta\ge 0$,

$$H_\delta:=\{x=(x',x_d)\in\mathbb{R}^d:x_d>-\delta\},$$

and for every $r\le 1$ and $u\in W^{1,p}(B_r)$, let

$$F_{0,\delta}(u,B_r):=\begin{cases}\displaystyle\int_{B_r}f(\nabla u)\,dx & \text{if } u=0 \text{ a.e. in } B_r\setminus H_\delta,\\ +\infty & \text{otherwise.}\end{cases}$$

The definition of local minimizers is adapted to $F_{0,\delta}$, taking into account the boundary condition. The following result has been proved in [3, Theorem 3.8].

Theorem 4.3 (**Boundary gradient bound**). *Assume that $\delta\in[0,1/2]$ and that $f:\mathbb{R}^d\to\mathbb{R}$ is a continuous function satisfying (H_1) and (H_2). Let $u\in W^{1,p}(B_1)$ be a local minimizer of $F_{0,\delta}(\cdot,B_1)$. Then, for each $R_0<1$, there exists a constant $C_0'=C_0'(d,p,L,\mu,R_0,\|\nabla u\|_p)>0$ (independent of δ) such that u is locally Lipschitz continuous in B_1 and*

$$\sup_{B_{\rho/2}}|\nabla u|^p\le C_0'\left(\frac{1}{\rho^d}\int_{B_\rho}|\nabla u|^p\,dx+1\right)\quad\text{for every }\rho\le R_0.$$

4.2. **Some lemmas on the unit ball.** In the footsteps of [2, Chapter 7], we study the behaviour of sequences of functions in $SBV(B_1)$ with vanishing jump set which satisfy a suitable minimality property for the functional (2.1).

More precisely, let $f_n:B_1\times\mathbb{R}^d\to[0,+\infty[$ be a Caratheodory function with

$$f_n(x,\xi)\le L(1+|\xi|^p)\tag{4.1}$$

$$\xi\mapsto f_n(x,\xi)\text{ is convex for a.e. }x\in B_1,\tag{4.2}$$

and such that

$$f_n\to f_\infty\qquad\text{uniformly on the compact sets of }B_1\times\mathbb{R}^d.\tag{4.3}$$

Here $L>0$ and $p>1$. We are interested in the behaviour of a sequence $(u_n)_{n\in\mathbb{N}}$ in $SBV(B_1)$ which satisfy the minimality property

$$\begin{aligned}\int_{B_r}f_n(x,\nabla u_n)\,dx+c_n\mathcal{H}^{d-1}(J_{u_n}\cap\bar B_r)&\\ \le\int_{B_r}f_n(x,\nabla v)\,dx+\Lambda c_n\mathcal{H}^{d-1}(J_v\cap\bar B_r)+D_n(r)&\end{aligned}\tag{4.4}$$

for every $v\in SBV(B_1)$ with $\{v\ne u_n\}\subseteq B_r$ and $r\in[0,1]$. Here $\Lambda\ge 1$, $c_n\ge 0$ and $D_n:[0,1]\to\mathbb{R}^+$ is such that

$$D_n\to 0\qquad\text{pointwise.}\tag{4.5}$$

Following [10], we will make use of a suitable truncation of a function in $SBV(B_1)$. For every $s\in[0,\omega_d]$, let us set

$$u_*(s,B_1):=\inf\{t\in\mathbb{R}\,:\,|\{u<t\}|\ge s\}.$$

If

$$\left(2\gamma_d\mathcal{H}^{d-1}(J_u)\right)^{\frac{d}{d-1}}<\frac{\omega_d}{2},$$

where γ_d is the constant appearing in the relative isoperimetric inequality, let

$$\tau^-(u,B_1):=u_*\left(\left(2\gamma_d\mathcal{H}^{d-1}(J_u)\right)^{\frac{d}{d-1}},B_1\right)$$

and

$$\tau^+(u,B_1) := u_*\left(\omega_d - \left(2\gamma_d\mathcal{H}^{d-1}(J_u)\right)^{\frac{d}{d-1}}, B_1\right).$$

Notice that $\tau^-(u,B_1) \le m \le \tau^+(u,B_1)$ for every *median* m of u, i.e.,

$$|\{x\in B_1 \,:\, u(x)<t\}| \le \frac{\omega_d}{2} \qquad \text{for every } t<m$$

and

$$|\{x\in B_1 \,:\, u(x)>t\}| \le \frac{\omega_d}{2} \qquad \text{for every } t>m.$$

The following result holds true (see [10, Theorem 3.5] and [2, Proposition 7.5]).

Lemma 4.4. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV(B_1)$ such that*

$$\sup_n \int_{B_1} |\nabla u|^p\,dx < +\infty \qquad \text{and} \qquad \lim_{n\to+\infty}\mathcal{H}^{d-1}(J_{u_n}) = 0$$

for some $p>1$, and let m_n be medians of u_n. Then there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $u\in W^{1,p}(B_1)$ such that

$$u_{n_k} - m_{n_k} \to u \qquad \text{a.e. in } B_1.$$

Moreover the truncated function

$$\bar u_{n_k} := \left(u_{n_k} \vee \tau^-(u_{n_k},B_1)\right) \wedge \tau^+(u_{n_k},B_1) \tag{4.6}$$

satisfies

$$\bar u_{n_k} - m_{n_k} \to u \qquad \text{strongly in } L^p(B_1)$$

and

$$|\{u_{n_k} \ne \bar u_{n_k}\}| \le 2\left(2\gamma_d\mathcal{H}^{d-1}(J_{u_{n_k}})\right)^{\frac{d}{d-1}}. \tag{4.7}$$

We begin with the following lemma.

Lemma 4.5. *Let f_n satisfy* (4.1) *and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV(B_1)$ satisfying the minimality property* (4.4) *and such that*

$$\sup_n\left[\int_{B_1}|\nabla u_n|^p\,dx + c_n\mathcal{H}^{d-1}(J_{u_n})\right] < +\infty \qquad \text{and} \qquad \mathcal{H}^{d-1}(J_{u_n})\to 0. \tag{4.8}$$

Then there exists $E\subseteq[0,1]$ with $|[0,1]\setminus E|=0$ such that the truncated functions $\bar u_n$ defined in (4.6) *still satisfy the minimality property* (4.4) *for every $r\in E$, with D_n replaced by $\tilde D_n : E\to\mathbb{R}^+$ such that $\tilde D_n\to 0$ pointwise.*

Proof. Thanks to Lemma 4.4, in view of assumption (4.8), we have

$$c_n|\{u_n\ne\bar u_n\}| \le \gamma_d c_n\left(\mathcal{H}^{d-1}(J_{u_n})\right)^{\frac{d}{d-1}} \to 0.$$

Since

$$|\{u_n\ne\bar u_n\}| = \int_0^1 \mathcal{H}^{d-1}(\{u_n\ne\bar u_n\}\cap\partial B_r)dr$$

we deduce that for a.e. $r\in[0,1]$

$$c_n\mathcal{H}^{d-1}(\{u_n\ne\bar u_n\}\cap\partial B_r)\to 0.$$

Let us consider now $v\in SBV(B_1)$ with $\{v\ne\bar u_n\}\subseteq B_r$, where r satisfies the previous property. Comparing u_n with $w := v1_{B_r} + u_n 1_{B_1\setminus B_r}$ we get

$$\begin{aligned}
&\int_{B_r} f_n(x,\nabla u_n)\,dx + c_n\mathcal{H}^{d-1}(J_{u_n}\cap\bar B_r)\\
&\qquad\le \int_{B_r} f_n(x,\nabla v)\,dx + \Lambda c_n\mathcal{H}^{d-1}(J_v\cap B_r) + \Lambda c_n\mathcal{H}^{d-1}(\{v\ne u_n\}\cap\partial B_r) + D_n(r)\\
&\qquad\le \int_{B_r} f_n(x,\nabla v)\,dx + \Lambda c_n\mathcal{H}^{d-1}(J_v\cap\bar B_r) + \Lambda c_n\mathcal{H}^{d-1}(\{u_n\ne\bar u_n\}\cap\partial B_r) + D_n(r).
\end{aligned}$$

Since in view of (4.1)

$$\int_{B_r} f_n(x,\nabla \bar u_n)\,dx + c_n\mathcal{H}^{d-1}(J_{\bar u_n}\cap \bar B_r) \\ \leq \int_{B_r} f_n(x,\nabla u_n)\,dx + c_n\mathcal{H}^{d-1}(J_{u_n}\cap \bar B_r) + L|\{u_n\neq \bar u_n\}|,$$

we conclude

$$\int_{B_r} f_n(x,\nabla \bar u_n)\,dx + c_n\mathcal{H}^{d-1}(J_{\bar u_n}\cap \bar B_r) \leq \int_{B_r} f_n(x,\nabla v)\,dx + \Lambda c_n\mathcal{H}^{d-1}(J_v\cap \bar B_r) + \tilde D_n(r)$$

where

$$\tilde D_n(r) := L|\{u_n\neq \bar u_n\}| + \Lambda c_n\mathcal{H}^{d-1}(\{u_n\neq \bar u_n\}\cap \partial B_r) + D_n(r)\to 0,$$

and the result follows. □

Lemma 4.6. *Let f_n satisfy* (4.1), (4.2) *and* (4.3). *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV(B_1)$ satisfying the minimality property* (4.4) *such that*

$$\sup_n \int_{B_1} |\nabla u_n|^p\,dx < +\infty, \qquad \mathcal{H}^{d-1}(J_{u_n})\to 0, \tag{4.9}$$

$$\sup_n \left[\int_{B_1} f_n(x,\nabla u_n)\,dx + c_n\mathcal{H}^{d-1}(J_{u_n})\right] < +\infty \tag{4.10}$$

and

$$u_n\to u\in W^{1,p}(B_1) \qquad \text{pointwise a.e.}$$

Then u is a local minimizer in $W^{1,p}(B_1)$ of the functional

$$v\mapsto \int_{B_1} f_\infty(x,\nabla v)\,dx,$$

and for every $r\in[0,1]$

$$\lim_{n\to+\infty}\int_{B_r} f_n(x,\nabla u_n)\,dx = \int_{B_r} f_\infty(x,\nabla u)\,dx \qquad \text{and} \qquad \lim_{n\to+\infty} c_n\mathcal{H}^{d-1}(J_{u_n}\cap \bar B_r) = 0. \tag{4.11}$$

Proof. Let $\bar u_n$ denote the truncation of u_n according to (4.6). We have thanks to (4.7), (4.9) and (4.10)

$$c_n|\{\bar u_n\neq u_n\}| = c_n\int_0^1 \mathcal{H}^{d-1}(\{\bar u_n\neq u_n\}\cap \partial B_r)\,dr\to 0. \tag{4.12}$$

By Helly's theorem, up to a subsequence we can assume that for every $r\in(0,1)$

$$\lim_{n\to+\infty}\left[\int_{B_r} f_n(x,\nabla \bar u_n)\,dx + c_n\mathcal{H}^{d-1}(J_{\bar u_n}\cap \bar B_r)\right] = \alpha(r),$$

where $\alpha:[0,1]\to[0,+\infty[$ is an increasing function.

In view of Lemma 4.4 we get easily

$$\bar u_n\to u \qquad \text{strongly in } L^p(B_1), \tag{4.13}$$

and

$$|\{\bar u_n\neq u_n\}|\to 0. \tag{4.14}$$

Following [12, Theorem 2.6] we have

$$\int_{B_r} f_\infty(x,\nabla u)\,dx \leq \liminf_{n\to+\infty}\int_{B_r} f_n(x,\nabla \bar u_n)\,dx. \tag{4.15}$$

Indeed, by *Chacon biting lemma* (see e.g. [2, Lemma 3.52]), there exists a decreasing family of Borel sets $A_k\subset B_1$ with $|A_k|\to 0$ as $k\to\infty$ and such that

$$(|\nabla u_n|^p 1_{B_1\setminus A_k})_{n\in\mathbb{N}} \quad \text{is equintegrable.}$$

Then for every $m \in \mathbb{N}$

$$\begin{aligned}\liminf_{n\to+\infty} \int_{B_r} f_n(x,\nabla \bar u_n)\,dx &\geq \liminf_{n\to+\infty} \int_{[B_r\setminus A_k]\cap M_{n,m}} f_n(x,\nabla \bar u_n)\,dx\\ &\geq \liminf_{n\to+\infty} \int_{B_r\setminus A_k} f_\infty(x,\nabla \bar u_n)\,dx - \limsup_{n\to+\infty} \int_{[B_r\setminus A_k]\setminus M_{n,m}} L(1+|\nabla \bar u_n|^p)\,dx - e(m)\end{aligned}$$

where

$$M_{n,m} := \{x \in B_1 \,:\, |\nabla \bar u_n| \leq m\}$$

and $e(m) \to 0$. Then, thanks to (4.13) and in view of the lower semicontinuity result [2, Theorem 5.29] we get

$$\liminf_{n\to+\infty} \int_{B_r\setminus A_k} f_\infty(x,\nabla \bar u_n)\,dx \geq \int_{B_r\setminus A_k} f_\infty(x,\nabla u)\,dx,$$

so that (4.15) follows by letting $m \to +\infty$ and then $k \to +\infty$.

In view of (4.15) and (4.14), for every $r \in (0,1)$ we obtain

$$\int_{B_r} f_\infty(x,\nabla u)\,dx \leq \liminf_{n\to+\infty} \int_{B_r} f_n(x,\nabla \bar u_n)\,dx \leq \liminf_{n\to+\infty} \int_{B_r} f_n(x,\nabla u_n)\,dx \tag{4.16}$$

Let us consider $v \in W^{1,p}(B_1)$ and $r \in (0,1)$ such that $\{v \neq u\} \subseteq B_r$. Considering the measures

$$\mu_n := c_n \mathcal{H}^{d-1} \lfloor J_{\bar u_n},$$

we can assume up to a subsequence that

$$\mu_n \overset{*}{\rightharpoonup} \mu \qquad \text{weakly}^* \text{ in the sense of measures} \tag{4.17}$$

for some finite positive measure μ on B_1. Let us consider

$$r < r' < r'' < 1$$

with

$$\mu(\partial B_{r'}) = 0 \qquad \text{and} \qquad r'' \in E,$$

where $E \subseteq [0,1]$ is given by Lemma 4.5. Let $\varphi \in C_c^\infty(B_1)$ be a cut-off function between $B_{r'}$ and $B_{r''}$, and let us compare $\bar u_n$ with $\varphi v + (1-\varphi)\bar u_n$. Since $r'' \in E$ we can write

$$\begin{aligned}&\int_{B_{r''}} f_n(x,\nabla \bar u_n)\,dx + c_n \mathcal{H}^{d-1}(J_{\bar u_n} \cap \bar B_{r''})\\ &\qquad \leq \int_{B_{r''}} f_n(x,\varphi\nabla v + (1-\varphi)\nabla \bar u_n + \nabla\varphi(v-\bar u_n))\,dx\\ &\qquad\qquad + \Lambda c_n \mathcal{H}^{d-1}\left(J_{\bar u_n} \cap (\bar B_{r''} \setminus B_{r'})\right) + \tilde D_n(r'')\\ &\qquad \leq \int_{B_{r'}} f_n(x,\nabla v)\,dx + \int_{B_{r''}\setminus B_{r'}} \tilde L\left(1 + |\nabla v|^p + |\nabla \bar u_n|^p + |\nabla\varphi|^p|v-\bar u_n|^p\right)\,dx\\ &\qquad\qquad + \Lambda c_n \mathcal{H}^{d-1}\left(J_{\bar u_n} \cap (\bar B_{r''} \setminus B_{r'})\right) + \tilde D_n(r''),\end{aligned}$$

where $\tilde L > 0$ is independent of n. Letting $n \to +\infty$ we get using the uniform convergence of f_n to f_∞ and (4.17)

$$\begin{aligned}&\liminf_{n\to+\infty} \left[\int_{B_{r''}} f_n(x,\nabla \bar u_n)\,dx + c_n \mathcal{H}^{d-1}(J_{\bar u_n} \cap \bar B_{r''})\right]\\ &\qquad\qquad \leq \int_{B_{r'}} f_\infty(x,\nabla v)\,dx + \tilde L \int_{B_{r''}\setminus B_{r'}} (1+|\nabla v|^p)\,dx + \Lambda\mu(\bar B_{r''} \setminus B_{r'}).\end{aligned}$$

Letting $r'' \to r'$, and since $\mu(\partial B_{r'}) = 0$ we deduce

$$\alpha(r') \leq \int_{B_{r'}} f_\infty(x,\nabla v)\,dx.$$

Moreover, thanks to (4.16), for $r' \to r$ we obtain

$$\int_{B_r} f_\infty(x, \nabla u)\, dx \le \alpha(r) \le \int_{B_r} f_\infty(x, \nabla v)\, dx.$$

We conclude that u is a local minimizer in $W^{1,p}(B_1)$ for

$$v \mapsto \int_{B_1} f_\infty(x, \nabla v)\, dx,$$

and that for every $r \in (0,1)$ (take $v = u$)

$$\alpha(r) = \int_{B_r} f_\infty(x, \nabla u)\, dx.$$

We note that the previous equality together with (4.15) entails that for every $r \in (0,1)$

$$\lim_{n\to+\infty} \int_{B_r} f_n(x, \nabla \bar u_n)\, dx = \int_{B_r} f_\infty(x, \nabla u)\, dx \qquad \text{and} \qquad \lim_{n\to+\infty} c_n \mathcal{H}^{d-1}(J_{\bar u_n} \cap \bar B_r) = 0. \tag{4.18}$$

In order to obtain (4.11), we need to replace $\bar u_n$ with u_n. Let $\tilde r > r$ be such that according to relation (4.12)

$$c_n \mathcal{H}^{d-1}\left(\{\bar u_n \neq u_n\} \cap \partial B_{\tilde r}\right) \to 0. \tag{4.19}$$

Comparing u_n with $\bar u_n 1_{B_{\tilde r}} + u_n 1_{B_1 \setminus B_{\tilde r}}$ we get

$$\begin{aligned}\int_{B_{\tilde r}} & f_n(x, \nabla u_n)\, dx + c_n \mathcal{H}^{d-1}(J_{u_n} \cap \bar B_{\tilde r}) \\ &\le \int_{B_{\tilde r}} f_n(x, \nabla \bar u_n)\, dx + \Lambda c_n \mathcal{H}^{d-1}(J_{\bar u_n} \cap B_{\tilde r}) + \Lambda c_n \mathcal{H}^{d-1}\left(\{\bar u_n \neq u_n\} \cap \partial B_{\tilde r}\right) + D_n(\tilde r)\end{aligned}$$

so that recalling (4.16), (4.18) and (4.19)

$$\begin{aligned}\int_{B_r} f_\infty(x, \nabla u)\, dx &\le \liminf_{n\to+\infty} \left[\int_{B_r} f_n(x, \nabla u_n)\, dx + c_n \mathcal{H}^{d-1}(J_{u_n} \cap \bar B_r)\right] \\ &\le \limsup_{n\to+\infty} \left[\int_{B_r} f_n(x, \nabla u_n)\, dx + c_n \mathcal{H}^{d-1}(J_{u_n} \cap \bar B_r)\right] \le \int_{B_{\tilde r}} f_\infty(x, \nabla u)\, dx,\end{aligned}$$

from which (4.11) follows letting $\tilde r \to r$. □

We need now to adapt the previous lemma to the case with boundary conditions. Let $\varphi_n : \mathbb{R}^{d-1} \to \mathbb{R}$ be a sequence of continuous functions, and let

$$T_n := \{x = (x', x_d) \in B_1 \,:\, x_d \le \varphi_n(x')\}.$$

Assume that $(\varphi_n)_{n\in\mathbb{N}}$ is locally uniformly converging to the constant function $-\delta$, with $\delta \in [0,1)$. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in $C^1(\bar B_1)$ such that

$$\|g_n\|_\infty + \|\nabla g_n\|_\infty \to 0. \tag{4.20}$$

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV(B_1)$ with $u_n = g_n$ on T_n, which satisfy the minimality property

$$\begin{aligned}\int_{B_r} & f_n(x, \nabla u_n)\, dx + c_n \mathcal{H}^{d-1}(J_{u_n} \cap \bar B_r) \\ &\le \int_{B_r} f_n(x, \nabla v)\, dx + \Lambda c_n \mathcal{H}^{d-1}(J_v \cap \bar B_r) + D_n(r)\end{aligned} \tag{4.21}$$

for every $v \in SBV(B_1)$ with $v = g_n$ on T_n, $\{v \neq u_n\} \subseteq B_r$, $r \in [0,1]$, and with D_n satisfying (4.5). Let finally $F^\infty_{0,\delta} : W^{1,p}(B_1) \to [0, +\infty[$ be given by

$$F^\infty_{0,\delta}(u) := \begin{cases} \int_{B_1} f_\infty(x, \nabla u) & \text{if } u = 0 \text{ on } \{x \in B_1 \,:\, x_d < -\delta\} \\ +\infty & \text{otherwise.}\end{cases}$$

We say that u is a local minimizer of $F^\infty_{0,\delta}$ if for every $v \in W^{1,p}(B_1)$ with $v = 0$ on $\{x \in B_1 \,:\, x_d < -\delta\}$ and $\{v \neq u\} \subset\subset B_1$ we have $F^\infty_{0,\delta}(u) \le F^\infty_{0,\delta}(v)$.

The following lemma holds true.

Lemma 4.7. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV(B_1)$ with $u_n = g_n$ on T_n, and satisfying the almost-quasi minimality property* (4.21)*. Assume that*

$$\sup_n \int_{B_1} |\nabla u_n|^p\,dx < +\infty, \qquad \mathcal{H}^{d-1}(J_{u_n}) \to 0,$$
$$\sup_n \left[\int_{B_1} f_n(x,\nabla u_n)\,dx + c_n\mathcal{H}^{d-1}(J_{u_n})\right] < +\infty$$

and

$$u_n \to u \in W^{1,p}(B_1) \qquad \text{pointwise a.e.}$$

Then u is a local minimizer in $W^{1,p}(B_1)$ of the functional $F^\infty_{0,\delta}$ and for every $r\in[0,1]$

$$\lim_{n\to+\infty}\int_{B_r} f_n(x,\nabla u_n)\,dx = \int_{B_r} f_\infty(x,\nabla u)\,dx \qquad \text{and} \qquad \lim_{n\to+\infty} c_n\mathcal{H}^{d-1}(J_{u_n}\cap \bar B_r) = 0. \tag{4.22}$$

Proof. If we set $v_n := u_n - g_n$, we get easily that for n large enough

$$\bar v_n = 0 \qquad \text{on } T_n.$$

Setting

$$\bar u_n := g_n + \bar v_n$$

we have

$$\bar u_n \to u \qquad \text{strongly in } L^p(B_1)$$

and

$$(1+c_n)|\{\bar u_n \neq u_n\}| \to 0.$$

We can then follow the proofs of Lemma and of Lemma 4.6 to get the conclusion: it suffices to note that every $v\in W^{1,p}(B_1)$ with $v=0$ on $\{x\in B_1 : x_d \le -\delta\}$, in view of (4.20), there exists $v_n\in W^{1,p}(B_1)$ with $v_n = g_n$ on T_n and such that $v_n\to v$ strongly in $W^{1,p}(B_1)$. □

4.3. **The decay lemma.** For every ball $B_\rho(x_0)\subset\subset\Omega'$ let us write

$$F(u,B_\rho(x_0)) := \int_{B_\rho} f(x,\nabla u)\,dx + \mathcal{H}^{d-1}(J_u\cap B_\rho).$$

Let us denote by $Dev_{g,\Lambda}(u,B_\rho(x_0))$ the infimum of those constants D such that

$$\int_{B_\rho(x_0)} f(x,\nabla u)\,dx + \mathcal{H}^{d-1}(J_u\cap \bar B_\rho(x_0)) \le \int_{B_\rho(x_0)} f(x,\nabla v)\,dx + \Lambda\mathcal{H}^{d-1}(J_v\cap \bar B_\rho(x_0)) + D$$

for every $v\in SBV^p(\Omega')$ with $v=g$ on $\Omega'\setminus\overline{\Omega}$ and such that $\{v\neq u\}\subseteq B_\rho(x_0)$.

The decay lemma with boundary conditions which we will use is the following.

Lemma 4.8 (Decay). *Assume that f satisfies assumptions $(H_1)-(H_3)$. For every $\tau\in(0,1)$ and $\tilde\Omega\subset\subset\Omega'$ there exist $\varepsilon(\tau,\tilde\Omega),\vartheta(\tau,\tilde\Omega),\rho(\tau,\tilde\Omega),\chi(\tau,\tilde\Omega)>0$ such that if $u\in SBV^p(\Omega')$ with $u=g$ on $\Omega'\setminus\overline{\Omega}$, $x\in\tilde\Omega$, $\rho<\rho(\tau,\tilde\Omega)$, $B_\rho(x)\subset\subset\Omega'$,*

$$\mathcal{H}^{d-1}(J_u\cap B_\rho(x)) \le \varepsilon(\tau,\tilde\Omega)\rho^{d-1} \qquad \text{and} \qquad Dev_{g,\Lambda}(u,B_\rho(x)) \le \vartheta(\tau,\tilde\Omega)F(u,B_\rho(x))$$

then

$$F(u,B_{\tau\rho}(x)) \le C_1\tau^d \max\{F(u,B_\rho(x)),\chi(\tau,\tilde\Omega)\rho^d[Lip_{B_\rho(x)}g]^p\},$$

where $C_1>0$ depends only on the dimension d and on the constants L,μ,p associated to f.

Proof. It suffices to consider the case $\tau\in(0,1/4)$ (otherwise we can choose $C_1 = 4^d$). Let $C_1>0$ to be fixed below. By contradiction, let us assume that there exist

$$\tau\in(0,1), \qquad \varepsilon_n\to 0, \qquad \vartheta_n\to 0, \qquad \rho_n\to 0, \qquad \chi_n\to+\infty, \qquad x_n\in\tilde\Omega$$

such that $B_{\rho_n}(x_n)\subset\subset\Omega'$,

$$\mathcal{H}^{d-1}(J_u\cap B_{\rho_n}(x_n)) = \varepsilon_n\rho_n^{d-1}, \qquad Dev_{g,\Lambda}(u,B_{\rho_n}(x_n)) = \vartheta_n F(u,B_{\rho_n}(x_n)),$$

and

$$F(u, B_{\tau\rho_n}(x_n)) > C_1\tau^d \max\{F(u, B_{\rho_n}(x_n)), \chi_n \rho_n^d [Lip_{B_{\rho_n}(x_n)} g]^p\}. \tag{4.23}$$

We consider the rescaled function $v_n \in SBV^p(B_1)$ given by

$$v_n(y) := \rho_n^{\frac{1-p}{p}} c_n^{\frac{1}{p}} u(x_n + \rho_n y). \tag{4.24}$$

where

$$c_n := \frac{\rho_n^{d-1}}{F(u_n, B_{\rho_n}(x_n))}.$$

We thus get

$$\int_{B_1} f_n\left(y, \nabla v_n(y)\right)\, dy + c_n \mathcal{H}^{d-1}(J_{v_n}) = 1, \tag{4.25}$$

where

$$f_n(y,\xi) := f\left(x_n + \rho_n y, \xi\right),$$

and

$$\sup_n \int_{B_1} |\nabla v_n|^p \, dy < +\infty \qquad \text{and} \qquad \mathcal{H}^{d-1}(J_{v_n}) \to 0. \tag{4.26}$$

If we define g_n starting from g as in (4.24), we get

$$\int_{B_\tau} f_n\left(y, \nabla v_n(y)\right)\, dy + c_n \mathcal{H}^{d-1}(J_{v_n} \cap \bar{B}_\tau) > C_1\tau^d \max\{1, \chi_n [Lip_{B_1} g_n]^p\} \geq C_1\tau^d, \tag{4.27}$$

so that in particular

$$Lip_{B_1} g_n \leq \frac{1}{[C_1\tau^d \chi_n]^{1/p}} \to 0. \tag{4.28}$$

Finally v_n satisfies the minimality property (4.21), with respect to the region T_n corresponding to $(\Omega' \setminus \overline{\Omega}) \cap B_{\rho_n}(x_n)$, the given datum g_n, and $D_n(r) = \vartheta_n$ for every $r \in [0,1]$.

Up to a subsequence, we may assume that $x_n \to x_\infty \in \overline{\tilde{\Omega}}$, so that

$$f_n(y,\xi) \to f(x_\infty, \xi) \qquad \text{uniformly on compact subsets of } B_1 \times \mathbb{R}^d.$$

It is not restrictive to assume $x_\infty \in \overline{\Omega}$ (otherwise for n large we have $u = g$ on $B_{\rho_n}(x_n)$, and the result is trivial). We now divide three cases.

Case 1 Assume that $T_n = \emptyset$ for n large. Thanks to Lemma 4.4 and Lemma 4.6, and since the energy is invariant under the addition of a constant, we may assume that

$$v_n \to v \in W^{1,p}(B_1) \qquad \text{pointwise a.e.}$$

with v local minimizer of $w \mapsto \int_{B_1} f(x_\infty, \nabla w(y))\, dy$ on $W^{1,p}(B_1)$, and such that every $r \in [0,1]$

$$\lim_{n\to+\infty} \int_{B_r} f_n\left(y, \nabla v_n(y)\right)\, dy = \int_{B_r} f(x_\infty, \nabla v(y))\, dy \qquad \text{and} \qquad \lim_{n\to+\infty} c_n \mathcal{H}^{d-1}(J_{v_n} \cap \bar{B}_r) = 0.$$

In particular

$$\int_{B_1} f(x_\infty, \nabla v(y))\, dy = 1.$$

However, by Theorem 4.2 v is Lipschitz continuous with

$$\sup_{B_{1/2}} |\nabla v|^p \leq \frac{C_0}{\omega_d} \int_{B_1} f(x_\infty, \nabla v)\, dy \leq \frac{C_0}{\omega_d},$$

where $C_0 = C_0(d, p, L, \mu)$. Hence we get

$$\int_{B_\tau} f(x_\infty, \nabla v(y))\, dy \leq L \int_{B_\tau} |\nabla v|^p \, dy \leq L\omega_d \tau^d \sup_{B_{1/4}} |\nabla v|^p \leq LC_0\tau^d.$$

We get a contradiction with (4.27) if we choose $C_1 > LC_0$.

Case 2 Assume that $T_n \neq \emptyset$ for n large. Since $\rho_n \to 0$, we get $x_\infty \in \partial(\Omega' \setminus \overline{\Omega})$. Since $\Omega' \setminus \overline{\Omega}$ has C^1-boundary, there exists a coordinate system such that, up to a subsequence,

$$T_n = \{y = (y', y_d) \in B_1 \,:\, y_d \leq \varphi_n(y')\}$$

for some $\varphi_n \in C^1(\mathbb{R}^{d-1})$ locally uniformly converging to a constant $-\delta$, with $\delta \in [0,1]$ (see e.g. [3, Lemma 6.4]).

Thanks to (4.26) and to Lemma 4.4, there exists $v \in W^{1,p}(B_1)$ such that up to a subsequence (not relabelled)

$$v_n - m_n \to v \qquad \text{a.e. in } B_1,$$

where m_n denotes a median of v_n.

Assume now that $0 \leq \delta \leq \frac{1}{2}$. In view of (4.28), we conclude that v is constant on $\{y = (y', y_d) \in B_1 \,:\, y_d \leq -\delta\}$. It is not restrictive (since the energies are invariant by addition of a constant) to assume that $m_n = 0$,

$$v = 0 \qquad \text{on } \{y = (y', y_d) \in B_1 \,:\, y_d \leq -\delta\}$$

and that

$$v_n \to v \qquad \text{a.e. in } B_1.$$

By Lemma 4.7 we deduce that v is a local minimizer of the functional $F^\infty_{0,\delta}$ associated to the energy density $f(x_\infty, \xi)$,

$$\lim_{n\to+\infty} \int_{B_\tau} f_n\left(y, \nabla v_n(y)\right)\, dy = \int_{B_\tau} f(x_\infty, \nabla v)\, dy, \qquad \lim_{n\to+\infty} c_n \mathcal{H}^{d-1}(J_{v_n} \cap \bar{B}_\tau) = 0$$

and

$$\int_{B_1} f(x_\infty, \nabla v)\, dy = 1.$$

In view of Theorem 4.3, v is locally Lipschitz in B_1, and there exists a constant $C_0' = C_0'(d, p, L, \mu)$ such that

$$\sup_{B_\tau} |\nabla v|^p \leq C_0' \left[\int_{B_1} |\nabla v|^p\, dx + 1\right].$$

Consequently we have

$$\int_{B_\tau} f(x_\infty, \nabla v)\, dy \leq L C_0' \left[\int_{B_1} |\nabla v|^p\, dx + 1\right] \omega_d \tau^d$$

and we get a contradiction with (4.27) if

$$C_1 > L(L+1) C_0' \omega_d.$$

Let now $\frac{1}{2} < \delta \leq 1$. Then for n large enough

$$T_n \cap B_{1/2} = \emptyset.$$

By Lemma 4.6 we infer that v is a local minimizer of F_∞ associated to $f(x_\infty, \xi)$ on $B_{1/2}$,

$$\lim_{n\to+\infty} \int_{B_r} f_n\left(y, \nabla v_n(y)\right)\, dy = \int_{B_r} f(x_\infty, \nabla v)\, dx, \qquad \lim_{n\to+\infty} \mathcal{H}^{d-1}(J_{v_n} \cap \bar{B}_r) = 0$$

for every $0 \leq r \leq 1/2$ and

$$\int_{B_{1/2}} f(x_\infty, \nabla v)\, dx \leq 1.$$

Since v is Lipschitz continuous thanks to Theorem 4.2 with

$$\sup_{B_{1/4}} |\nabla v|^p \leq \frac{\tilde{C}_0}{\omega_d} \int_{B_{1/2}} f(x_\infty, \nabla v)\, dy \leq \frac{\tilde{C}_0}{\omega_d},$$

where $\tilde{C}_0 = \tilde{C}_0(d, p, L, \mu)$, we get as $\tau \leq 1/4$

$$\int_{B_\tau} f(x_\infty, \nabla v(y))\, dy \leq L \int_{B_\tau} |\nabla v|^p\, dy \leq L\omega_d \tau^d \sup_{B_{1/4}} |\nabla v|^p \leq L\tilde{C}_0 \tau^d.$$

We get a contradiction with (4.27) if we choose $C_1 > L\tilde{C}_0$. The proof is now concluded. □

4.4. **Proof of Theorem 2.3.** By comparing u with $u1_{B_\rho(x)}$ for $B_\rho(x) \subset\subset \Omega'$, we get easily that

$$\mathcal{H}^{d-1}(J_u \cap B_\rho(x)) \le \Lambda\omega_d\rho^{d-1} + c_\alpha\rho^\alpha \le (\Lambda\omega_d + 1)\rho^{d-1}$$

for ρ small enough (depending only on α and c_α). Let $\tilde\Omega \subset\subset \Omega'$ be fixed. We claim that we can find $\varepsilon_0, \rho_0 > 0$ depending on $\tilde\Omega$ such that the relations

$$x \in \tilde\Omega, \qquad B_\rho(x) \subset\subset \Omega' \qquad \text{and} \qquad \mathcal{H}^{d-1}(J_u \cap B_\rho(x)) \le \varepsilon_0\rho^{d-1} \quad \text{for some } \rho < \rho_0$$

entails

$$\lim_{r\to 0} \frac{\int_{B_r(x)} |\nabla u|^p\, dx + \mathcal{H}^{d-1}(J_u \cap B_r(x))}{r^{d-1}} = 0. \tag{4.29}$$

Then the thesis follows since thanks to [10, Theorem 3.6] relation (4.29) implies $x \notin J_u$. We can thus achieve inequality (2.2) by reducing ε_0 if necessary.

In order to prove (4.29), it is not restrictive to assume $\rho_0 < 1$. We will use the almost- quasi minimality property of Definition 2.2 only on balls with radius $\rho < \rho_0$, so that we can assume without loss of generality that $\alpha \le 1$.

We proceed as follows. Let $\tau \in (0,1)$ be such that

$$C_1\tau^d \le \tau^{d-\frac{\alpha}{2}}, \tag{4.30}$$

where C_1 is the constant given by the Decay Lemma 4.8. Let $\sigma \in (0,1)$ to be fixed below. We claim that there exists $\rho_0 > 0$ such that if for some $\rho < \rho_0$

$$\mathcal{H}^{d-1}(J_u \cap B_\rho(x)) \le \varepsilon(\sigma)\rho^{d-1}$$

then

$$F(u, B_{\sigma\tau^h\rho}) \le \varepsilon(\tau)\tau^{\frac{\alpha h}{2}}(\sigma\tau^h\rho)^{d-1}. \tag{4.31}$$

Here $\varepsilon(\tau)$ and $\varepsilon(\sigma)$ are the numbers associated to τ, σ and $\tilde\Omega$ according to the Decay Lemma 4.8. From this inequality, relation (4.29) easily follows, and the thesis is proved by choosing $\varepsilon_0 = \varepsilon(\sigma)$.

We prove (4.31) by induction. In what follows, L_g will denote the Lipschitz constant of g on $\tilde\Omega$. For $h = 0$ it reads

$$F(u, B_{\sigma\rho}) \le \varepsilon(\tau)(\sigma\rho)^{d-1}. \tag{4.32}$$

If

$$Dev_{g,\Lambda}(u, B_\rho(x)) \le \vartheta(\sigma)F(u, B_\rho(x)),$$

we get according to Lemma 4.8, and using the almost-quasi minimality property of u in comparison with $g1_{B_\rho(x)} + u1_{\Omega'\setminus B_\rho(x)}$

$$\begin{aligned} F(u, B_{\sigma\rho}(x)) &\le C_1\sigma^d \max\{F(u, B_\rho(x)), \chi(\sigma)\rho^d[Lip_{B_\rho(x)}g]^p\} \\ &\le C_1\sigma^d \max\{\omega_d\rho^d L[Lip_{B_\rho(x)}g]^p + d\omega_d\rho^{d-1} + c_\alpha\rho^{d-1+\alpha}, \chi(\sigma)\rho^d[Lip_{B_\rho(x)}g]^p\} \\ &\le (\rho\sigma)^{d-1}C_1\sigma \max\{\omega_d\rho L L_g^p + d\omega_d + c_\alpha\rho^\alpha, \chi(\sigma)\rho L_g^p\} \le \varepsilon(\tau)(\sigma\rho)^{d-1} \end{aligned}$$

provided that

$$\rho_0 \le \min\left\{\frac{1}{\chi(\sigma)}, 1\right\} \tag{4.33}$$

and

$$C_1\sigma \max\{\omega_d L L_g^p + d\omega_d + c_\alpha, L_g^p\} \le \varepsilon(\tau). \tag{4.34}$$

If on the other hand

$$Dev_{g,\Lambda}(u, B_\rho(x)) > \vartheta(\sigma)F(u, B_\rho(x))$$

then by the minimality property of u

$$F(u, B_{\sigma\rho}(x)) \le F(u, B_\rho(x)) \le \frac{1}{\vartheta(\sigma)} Dev_{g,\Lambda}(u, B_\rho) \le \frac{c_\alpha\rho^{d-1+\alpha}}{\vartheta(\sigma)} \le \varepsilon(\tau)(\sigma\rho)^{d-1}$$

provided that

$$c_\alpha\rho_0^\alpha < \varepsilon(\tau)\vartheta(\sigma)\sigma^{d-1}. \tag{4.35}$$

Relation (4.32) is now completely proved.

Assume now that (4.31) holds true. We want to see that

$$F(u, B_{\sigma\tau^{h+1}\rho}) \le \varepsilon(\tau)\tau^{\frac{\alpha(h+1)}{2}}(\sigma\tau^{h+1}\rho)^{d-1}. \tag{4.36}$$

From (4.31) we infer that

$$\mathcal{H}^{d-1}(J_u \cap B_{\sigma\tau^h\rho}) \le \varepsilon(\tau)(\sigma\tau^h\rho)^{d-1}.$$

If

$$Dev_{g,\Lambda}(u, B_{\sigma\tau^h\rho}(x)) \le \vartheta(\tau)F(u, B_{\sigma\tau^h\rho})$$

then by the Decay Lemma 4.8

$$\begin{aligned} F(u, B_{\sigma\tau^{h+1}\rho}) &\le C_1\tau^d \max\{F(u, B_{\sigma\tau^h\rho}), \chi(\tau)(\sigma\tau^h\rho)^d L_g^p\} \\ &\le C_1\tau^d \max\{\varepsilon(\tau)\tau^{\frac{\alpha h}{2}}(\sigma\tau^h\rho)^{d-1}, \chi(\tau)(\sigma\tau^h\rho)^d L_g^p\} \\ &= C_1\tau^d(\sigma\tau^h\rho)^{d-1}\tau^{\frac{\alpha h}{2}} \max\{\varepsilon(\tau), \chi(\tau)\sigma\tau^{\frac{(2-\alpha)h}{2}}\rho L_g^p\}. \end{aligned}$$

If

$$\chi(\tau)\rho_0 L_g^p \le \varepsilon(\tau), \tag{4.37}$$

and since $\alpha \le 1$, we infer taking into account also (4.30)

$$F(u, B_{\sigma\tau^{h+1}\rho}) \le \varepsilon(\tau)\tau^{d-\frac{\alpha}{2}}(\sigma\tau^h\rho)^{d-1}\tau^{\frac{\alpha h}{2}} = \varepsilon(\tau)\tau^{1-\alpha}\tau^{\frac{\alpha(h+1)}{2}}(\sigma\tau^{h+1}\rho)^{d-1} \le \varepsilon(\tau)\tau^{\frac{\alpha(h+1)}{2}}(\sigma\tau^{h+1}\rho)^{d-1}.$$

If on the contrary

$$Dev_{g,\Lambda}(u, B_{\sigma\tau^h\rho}(x)) > \vartheta(\tau)F(u, B_{\sigma\tau^h\rho})$$

we have using the minimality property of u

$$\begin{aligned} F(u, B_{\sigma\tau^{h+1}\rho}(x)) \le F(u, B_{\sigma\tau^h\rho}(x)) &\le \frac{1}{\vartheta(\tau)} Dev_{g,\Lambda}(u, B_{\sigma\tau^h\rho}(x)) \le \frac{1}{\vartheta(\tau)} c_\alpha(\sigma\tau^h\rho)^{d-1+\alpha} \\ &= \frac{c_\alpha(\sigma\tau^h\rho)^\alpha}{\vartheta(\tau)} \frac{(\sigma\tau^{h+1}\rho)^{d-1}}{\tau^{d-1}} \le \varepsilon(\tau)\tau^{\frac{\alpha(h+1)}{2}}(\sigma\tau^{h+1}\rho)^{d-1} \end{aligned}$$

provided that

$$c_\alpha\rho_0^\alpha \le \varepsilon(\tau)\vartheta(\tau)\tau^{d-1+\frac{\alpha}{2}}. \tag{4.38}$$

Relation (4.36) thus follows.

Summarizing, we conclude that (4.31) holds true provided that σ satisfies (4.34) and ρ_0 satisfies (4.33), (4.35), and (4.38). The proof is now concluded.

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