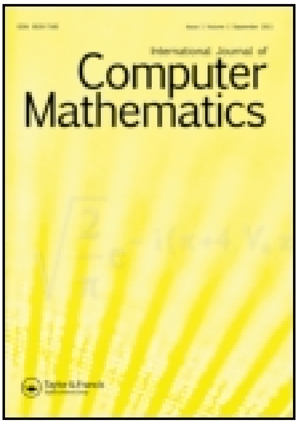

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Shape-preserving curve interpolation

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This work is a contribution towards the graphical display of 2D data when they are convex, monotone and positive. A piecewise rational function in a cubic/cubic form is proposed, which, in each interval, involves four free parameters in its construction. These four free parameters have a direct geometric interpretation, making their use straightforward. Illustrations of their effect on the shape of the rational function are given. Two of these free parameters are constrained to preserve the shape of convex, monotone and positive data, while the other two parameters are utilized for the modification of positive, monotone and convex curves to obtain a visually pleasing curve. The problem of shape preservation of data lying above a line is also discussed. The method that is presented applies equally well to data or data with derivatives. The developed scheme is computationally economical and pleasing. The error of rational interpolating function is also derived when the arbitrary function being interpolated is C^3 in an interpolating interval. The order of approximation is $O(h_i^3)$.

Keywords: spline; positivity; monotony; convexity; visualization

2000 AMS Subject Classifications: 68U05; 65D05; 65D07; 65D18

1. Introduction

In computer graphics environment, a user is always in need of interpolatory schemes which preserve the shape of the data under consideration under different conditions and circumstances. Positivity, monotony and convexity are the fundamental and important shapes which may arise in the data coming from any scientific, business or social environment. For example, the positivity of data can be seen in monthly rainfall amounts, levels of gas discharge in certain chemical reactions, progress of an irreversible process, resistance offered by an electric circuit, volume and density, etc. These are some of the physical quantities which are always positive. The non-negative graphical display of these physical quantities is meaningless. Monotony is another important shape property. There are many physical situations where entities only have

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a meaning when their values are monotone. For example, dose–response curves and surfaces in biochemistry and pharmacology [1], approximations of couples and quasi-couples in statistics [1], empirical option pricing model in finance [1] and approximation of potential functions in physical and chemical systems [1] are always monotone. Similarly, convexity has various applications in different disciplines including telecommunication systems designing, nonlinear programming, engineering, optimization, parameter estimation, approximation theory [3] and others.

Shape control [19], shape design [5] and shape preservation [12–15] are important areas for the graphical presentation of data. This paper pertains to the area of shape preservation. The problem of shape preservation has been discussed by a number of authors. Brodlie and Butt [2] preserved the shape of convex data by piecewise cubic interpolation. In any interval where convexity was lost, they divided the interval into two subintervals by inserting extra knots into that interval. The method that was presented was C^1. They used the same technique in [4] to preserve the shape of positive data. Fuhr and Kallay [6] used a C^1 monotone rational B-spline of degree one to preserve the shape of monotone data. Goodman, Ong and Unsworth [8] presented two interpolating schemes to preserve the shape of data lying on one side of the straight line using a rational cubic function. The first scheme preserved the shape of the data lying above the straight line by scaling the weights by some scale factor. The second scheme preserved the shape of the data by the insertion of a new interpolation point. Goodman [7] surveyed the shape-preserving interpolating algorithms for 2D data. Gregory and Sarfraz [9] introduced a rational cubic spline with one tension parameter in each subinterval, both interpolatory and rational B-spline forms. They also analysed the effect of variation of tension parameter on the shape of the curve. Hussain and Sarfraz [10] used a rational cubic function in its most generalized form to preserve the shape of positive planar data. They used the same rational cubic function in [11] to preserve the shape of monotone data. They developed data-dependent sufficient conditions on free parameters in [10,11] to preserve the shape of planar data. Lamberti and Manni [12] used cubic Hermite in a parametric form to preserve the shape of data. The step lengths were used as tension parameters to preserve the shape of planar functional data. The first-order derivatives at the knots were estimated by a tridiagonal system of equations which assured C^2 continuity at the knots. Schmidt and Heβ [17] developed sufficient conditions on derivatives at the end points of an interval to assure the positivity of the cubic polynomial over the whole interval.

As has been discussed in the previous paragraph, positivity, monotony and convexity are important shapes. These are independent shapes which are found inherited in data, under different conditions and circumstances, in one form or the other. In the past, most of the authors had discussed these shapes independently using different mathematical models and methodologies. This paper intends to discuss the three important shapes within one mathematical model. It introduces a C^1 rational cubic function with four free parameters in its description. These four free parameters have a direct geometric interpretation, making their use straightforward. Illustrations of their effect on the shape of the rational function are given. Two of these free parameters are constrained to preserve the shape of convex, monotone and positive data. The other two parameters are utilized for further modification, if needed, to obtain a visually pleasing curve. This paper discusses the problem of shape preservation of positive data as well as its generalized form when the data are lying above a straight line. It is also extended to the problems of monotony and convexity preservation of data. Each of the positivity-, monotony- and convexity-preserving schemes has been supported with practical demonstrations on various examples of data. It also describes the details of the error of interpolation when the function being interpolated is C^3 in an interpolating interval.

This paper is different from a recently published paper of Sarfraz *et al.* [16] on the same subject. These differences are elaborated as follows:

#	The paper of Sarfraz *et al.* [16]	This paper
1.	It is based on curves and surfaces	It is based on only curves
2.	It is about the visualization of only positive data	It is about the visualization of positive, monotone and convex data
3.	The model, under discussion, used is rational cubic by quadratic	The model, under discussion, used is rational cubic by cubic
4.	The rational function used has two shape parameters in its description	The rational function used has four shape parameters in its description
5.	The two shape parameters are constrained to preserve the shape of only positive data	Two of the shape parameters are constrained to preserve the shape of convex, monotone and positive data, while the other two parameters are utilized for the modification of positive, monotone and convex curves to obtain a visually pleasing curve

This paper, when compared with the paper of Sarfraz *et al.* [16], has done an additional study of two different kinds of data shapes, namely monotone and convex. Thus, based on the above-mentioned points, all the analytical studies, spline models, derivations, error analyses, and figures are different and are explained in the following sections with details.

The remainder of this paper is organized as follows. In Section 2, the C^1 rational cubic function, with four free parameters in its description, is developed. Section 3 discusses the problem of shape preservation of positive data. Section 4 discusses the positivity problem when the data are lying above the straight line. The problems of monotony and convexity preservation of planar data are described in Sections 5 and 6, respectively. An error analysis is developed in Section 7. This section details the error of interpolation when the function being interpolated is C^3 in an interpolating interval. The paper is concluded in Section 8.

2. Rational cubic spline

Rational spline interpolation has an upper hand over polynomial spline interpolation as it can carry more degrees of freedom in its description. This freedom can be utilized for various purposes and objectives to be achieved in diverse real-life problems arising in different disciplines. This section introduces a very general kind of rational cubic spline which has four free parameters in its description. These four free parameters can be used for shape control and shape preservation.

Let $\{(x_i, f_i), i = 0, 1, 2, \ldots, n\}$ be the given set of data points defined over the interval $[a, b]$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. The piecewise rational cubic function with four free parameters is defined over each subinterval $I_i = [x_i, x_{i+1}], i = 0, 1, 2, \ldots, n-1$, as follows:

$$S(x) \equiv S_i(x) = \frac{A_0(1-\theta)^3 + A_1(1-\theta)^2\theta + A_2(1-\theta)\theta^2 + A_3\theta^3}{\alpha_i(1-\theta)^2 + \beta_i(1-\theta)^2\theta + \gamma_i(1-\theta)\theta^2 + \delta_i\theta^2}, \quad (1)$$

where

$$\theta = \frac{x - x_i}{h_i} \quad \text{and} \quad h_i = x_{i+1} - x_i.$$

Thus, we have transformed $x_i \leq x \leq x_{i+1}$ to $0 \leq \theta \leq 1$. The piecewise rational cubic function (1) is C^1 if it satisfies the following interpolatory conditions:

$$S(x_i) = f_i, \quad S(x_{i+1}) = f_{i+1}, \quad S^{(1)}(x_i) = d_i, \quad S^{(1)}(x_{i+1}) = d_{i+1}, \quad (2)$$

where $S^{(1)}(x)$ denotes the derivative with respect to x and d_i denotes the derivative values estimated or given. The imposition of interpolatory conditions (2) on the rational cubic function (1) yields the following values of unknown A_i, $i = 0, 1, 2, 3$:

$$A_0 = \alpha_i f_i,$$
$$A_1 = (\alpha_i + \beta_i)f_i + \alpha_i h_i d_i,$$
$$A_2 = (\gamma_i + \delta_i)f_{i+1} - \delta_i h_i d_{i+1},$$
$$A_3 = \delta_i f_{i+1}.$$

These values of unknown A_i, $i = 0, 1, 2, 3$, reduce the rational cubic function (1) to the following piecewise cubic spline:

$$S(x) \equiv S_i(x) = \frac{p_i(\theta)}{q_i(\theta)}, \tag{3}$$

where

$$p_i(\theta) = \alpha_i f_i(1-\theta)^3 + \{(\alpha_i + \beta_i)f_i + \alpha_i h_i d_i\}(1-\theta)^2\theta + \{(\gamma_i + \delta_i)f_{i+1} - \delta_i h_i d_{i+1}\}(1-\theta)\theta^2 + \delta_i f_{i+1}\theta^3,$$
$$q_i(\theta) = \alpha_i(1-\theta)^2 + \beta_i(1-\theta)^2\theta + \gamma_i(1-\theta)\theta^2 + \delta_i\theta^2.$$

It is interesting to note that when $\alpha_i = \delta_i = 1$ and $\beta_i = \gamma_i = 2$, the piecewise rational cubic spline (3) reduces to a standard cubic Hermite spline.

2.1 *Some observations*

This section illustrates the effect of the free parameters α_i, δ_i, β_i and γ_i on the shape of a curve both mathematically and graphically.

(1) *Point tension:* $\lim_{\beta_i \to \infty} S_i(x) = f_i$ and $\lim_{\gamma_i \to \infty} S_i(x) = f_{i+1}$; that is, for a given interval $I_i = [x_i, x_{i+1}]$, the free parameter β_i controls the shape of the curve near the left-end point of the interval and the free parameter γ_i controls the shape of the curve near the end point of the interval. It is also observed that $\lim_{\gamma_{i-1} \to \infty} S_{i-1}(x) = \lim_{\beta_i \to \infty} S_i(x) = f_i$. Hence, in two adjacent intervals, by simultaneously increasing the right-end free parameter of the left interval and the left-end free parameter of the right interval, the rational cubic function (3) converges to a single point (x_i, f_i).
(2) *Interval tension:* To observe the simultaneous increase in both the free parameters on the rational cubic function (3), (3) is expressed as follows:

$$S_i(x) = f_i(1-\theta) + f_{i+1}\theta + \frac{R}{q_i(\theta)}, \tag{4}$$

where

$$R = (1-\theta)^2\theta\{(\alpha_i + \beta_i)f_i + \alpha_i h_i d_i - \alpha_i f_{i+1}\} + (1-\theta)\theta^2\{(\gamma_i + \delta_i)f_{i+1} - \delta_i h_i d_{i+1} - \delta_i f_i\} - \beta_i f_i(1-\theta)^3\theta - \gamma_i f_{i+1}(1-\theta)\theta^3 - (1-\theta)^2\theta^2(\beta_i f_{i+1} + \gamma_i f_i).$$

From Equation (4), the following observation is made:

$$\lim_{\beta_i, \gamma_i \to \infty} S_i(x) = \lim_{\beta_i, \gamma_i \to \infty} \left\{ f_i(1-\theta) + f_{i+1}\theta + \frac{R}{q_i(\theta)} \right\} = f_i(1-\theta) + f_{i+1}\theta.$$

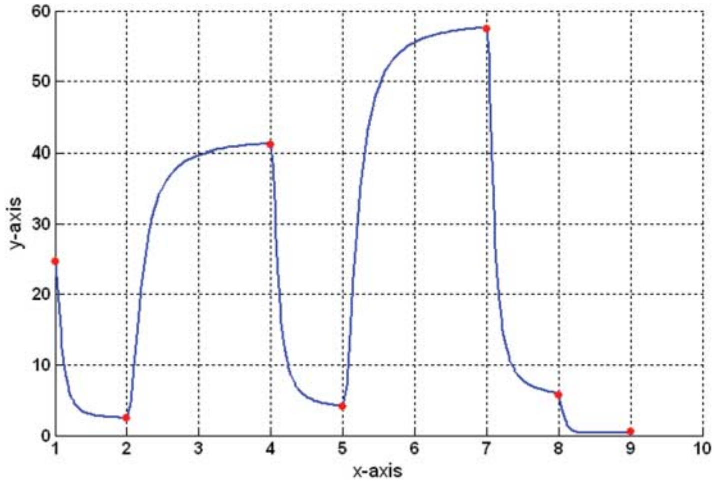

Figure 1. The rational cubic function (3) with $\alpha_i = 1$, $\beta_i = 2$, $\gamma_i = 100$ and $\delta_i = 1$.

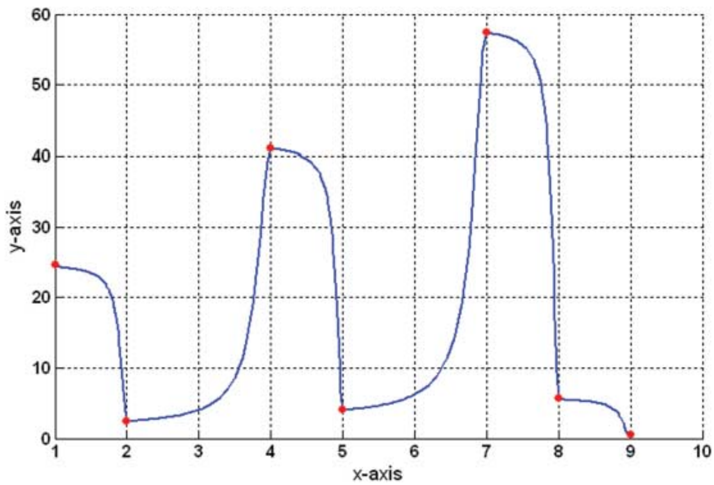

Figure 2. The rational cubic function (3) with $\alpha_i = 1$, $\beta_i = 100$, $\gamma_i = 2$ and $\delta_i = 1$.

Hence, the simultaneous increase in both the free parameters β_i and γ_i reduces the rational cubic function (3) in the interval $I_i = [x_i, x_{i+1}]$ to the straight line $f_i(1-\theta) + f_{i+1}\theta$. These observations are demonstrated graphically in Figures 1–4 for the data given in Table 1.

Figures 1 and 2 demonstrate the point tension effect caused by the free parameters γ_i and β_i, respectively. Figures 3 and 4 demonstrate the effect of the increase in the free parameters α_i and δ_i. Increment in the free parameter α_i demonstrates the tension effect, whereas a simultaneous increase in both the free parameters produces a visually pleasing curve.

3. Positive curve interpolation

The rational cubic spline, discussed in Section 2, carries four degrees of freedom in its description in the form of shape parameters. This freedom can be utilized to achieve the positivity of the data-interpolating curve when positive data are under consideration. This section is dedicated to

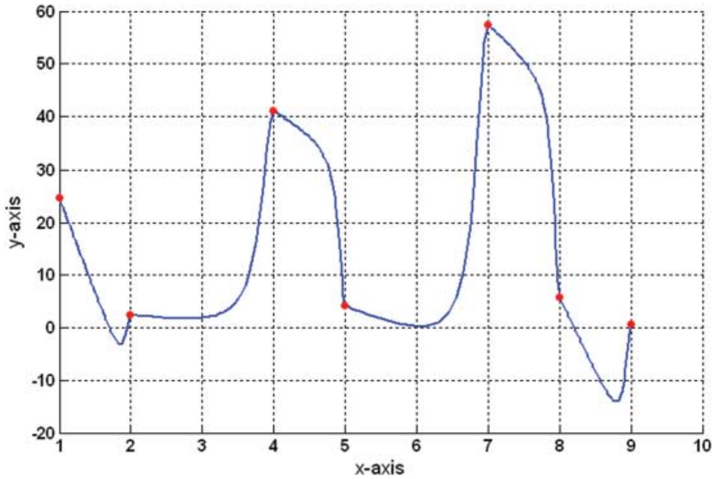

Figure 3. The rational cubic function (3) with $\alpha_i = 100$, $\beta_i = 2$, $\gamma_i = 2$ and $\delta_i = 1$.

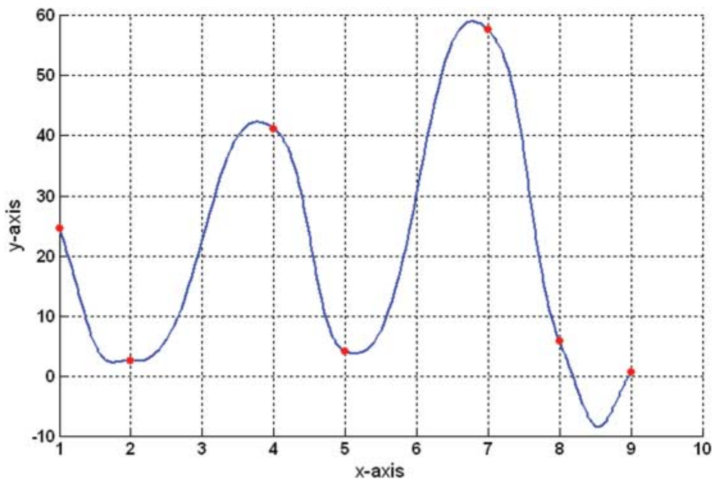

Figure 4. The rational cubic function (3) with $\alpha_i = 100$, $\beta_i = 2$, $\gamma_i = 2$ and $\delta_i = 80$.

Table 1. Positive data of the molal volume of a gas.

x	1	2	4	5	7	8	9
f	24.6162	2.4616	41.0270	4.1027	57.4378	5.7438	0.5744

orient the rational cubic spline, discussed in Section 2, to make it produce a positive curve. The positive curve would be achieved by constraining the shape parameters. The objective would be to utilize two parameters for the positivity-preserving constraints, while the other two parameters would be kept free for shape control to enhance the positive curve further if needed.

Let $\{(x_i, f_i),\ i = 0, 1, 2, \ldots, n\}$ be the positive data defined over the interval $[a, b]$ such that

$$f_i > 0, \quad i = 0, 1, 2, \ldots, n.$$

The piecewise rational cubic function (3) preserves positivity if

$$S_i(x) > 0, \quad i = 0, 1, 2, \ldots, n-1.$$

$S_i(x) > 0$ if

$$p_i(\theta) > 0 \quad \text{and} \quad q_i(\theta) > 0.$$

$q_i(\theta) > 0$ if

$$\alpha_i > 0, \quad \beta_i > 0, \quad \gamma_i > 0, \quad \delta_i > 0.$$

Using the result developed by Schmidt and Heβ in [17], the cubic polynomial $p_i(\theta) > 0$ if

$$(p_i'(0), p_i'(1)) \in R_1 \cup R_2, \tag{5}$$

where

$$R_1 = \left\{(a,b) : a > \frac{-3f_i}{h_i}, b < \frac{3f_{i+1}}{h_i}\right\}, \tag{6}$$

$$R_2 = \left\{\begin{array}{c}(a,b) : 36f_if_{i+1}(a^2+b^2+ab-3\Delta_i(a+b)+3\Delta_i^2) \\ +3(f_{i+1}a - f_ib)(2h_iab - 3f_{i+1}a + 3f_ib) \\ +4h_i(f_{i+1}a^3 - f_ib^3) - h_i^2a^2b^2 > 0\end{array}\right\}. \tag{7}$$

Since $p_i'(0) = -2\alpha_if_i + \beta_if_i + \alpha_ih_id_i$, $p_i'(1) = 2\delta_if_{i+1} - \lambda_if_{i+1} + \delta_ih_id_{i+1}$. The relation (5) is true when

$$(p_i'(0), p_i'(1)) \in R_1,$$

or

$$p_i'(0) > \frac{-3f_i}{h_i}, \quad p_i'(1) < \frac{3f_{i+1}}{h_i}.$$

This leads to the following constraints:

$$\beta_i > -\frac{\alpha_ih_id_i}{f_i}, \quad \gamma_i > \frac{\delta_ih_id_{i+1}}{f_{i+1}}, \quad \alpha_i > 1.5 \quad \text{and} \quad \delta_i > 1.5.$$

Furthermore, $(p_i'(0), p_i'(1)) \in R_2$ if

$$\begin{aligned}\phi(\alpha_i,\beta_i,\gamma_i,\delta_i) &\\ = 36f_if_{i+1}&[\phi_1^2(\alpha_i,\beta_i) + \phi_2^2(\gamma_i,\delta_i) + \phi_1(\alpha_i,\beta_i)\phi_2(\gamma_i,\delta_i) - 3\Delta_i(\phi_1(\alpha_i,\beta_i)+\phi_2(\gamma_i,\delta_i)) + 3\Delta_i^2] \\ &+ 3[f_{i+1}\phi_1(\alpha_i,\beta_i) - f_i\phi_2(\gamma_i,\delta_i)][2h_i\phi_1(\alpha_i,\beta_i)\phi_2(\gamma_i,\delta_i) - 3f_{i+1}\phi_1(\alpha_i,\beta_i) + 3f_i\phi_2(\gamma_i,\delta_i)] \\ &+ 4h_i[f_{i+1}\phi_1^3(\alpha_i,\beta_i) - f_i\phi_2^3(\gamma_i,\delta_i)] - h_i^2\phi_1^2(\alpha_i,\beta_i)\phi_2^2(\gamma_i,\delta_i) \geq 0,\end{aligned} \tag{8}$$

where

$$\phi_1(\alpha_i,\beta_i) = p_i'(0), \quad \phi_2(\gamma_i,\delta_i) = p_i'(1).$$

The constraints on the free parameters can be derived from both Equations (6) and (7). But Equation (7) involves a lot of computation, thus Equation (6) is a reasonable choice. Thus, the above discussion can be summarized as follows:

THEOREM 1 *The piecewise rational cubic interpolant $S(x)$, defined over the interval $[a, b]$, in (3), is positive if the following sufficient conditions are satisfied:*

$$\alpha_i > 1.5, \quad \delta_i > 1.5,$$
$$\beta_i > \text{Max}\left\{0, -\frac{\alpha_i h_i d_i}{f_i}\right\},$$
$$\gamma_i > \text{Max}\left\{0, \frac{\delta_i h_i d_{i+1}}{f_{i+1}}\right\}.$$

Remark 1 The constraints on the shape parameters can be rearranged as follows:

$$\alpha_i > 1.5, \quad \delta_i > 1.5,$$
$$\beta_i = l_i + \text{Max}\left\{0, -\frac{\alpha_i h_i d_i}{f_i}\right\}, \quad l_i > 0,$$
$$\gamma_i = m_i + \text{Max}\left\{0, \frac{\delta_i h_i d_{i+1}}{f_{i+1}}\right\}, \quad m_i > 0.$$

The parameters β_i's and γ_i's are meant for the shape-preserving constraints, whereas the parameters α_i's and δ_i's are meant for shape control to enhance the curve shape further.

3.1 *Demonstration*

In this section, we illustrate the positivity-preserving scheme developed in Section 3 through numerical examples. Let us take the positive data given in Table 1, collected in a chemical experiment, where the x-values are the temperature code and the f-values are the molal volume of the gas in l/mole by the ideal gas law. Figure 5 is produced from the rational cubic function (3) for the value of free parameters $\alpha_i = \delta_i = 1$ and $\beta_i = \gamma_i = 2$. For these values of the free parameters, the rational cubic function (3) reduces to the standard cubic Hermite spline. From Figure 5, it is clear that the cubic Hermite spline failed to preserve the positive shape of the data. Figure 6 is

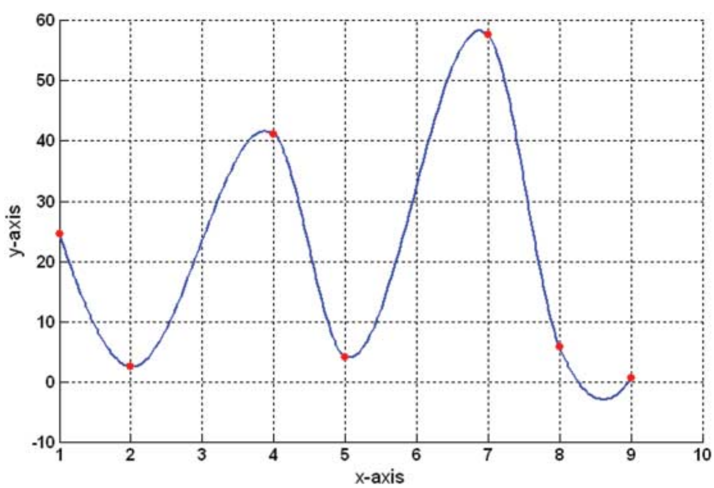

Figure 5. The cubic Hermite spline.

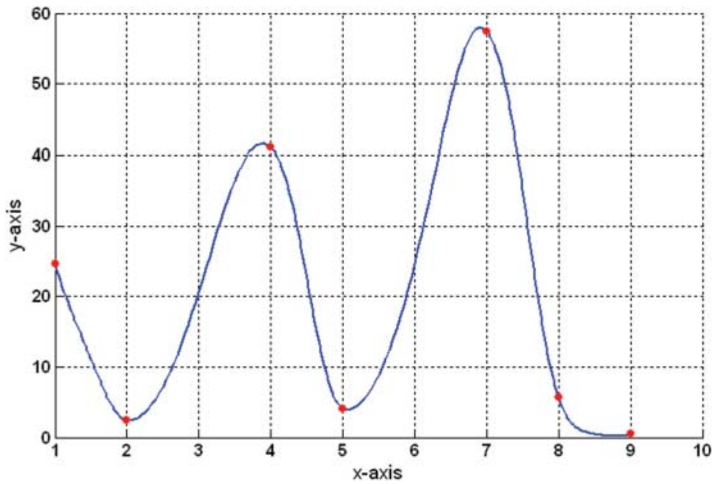

Figure 6. The positive rational cubic function with $\alpha_i = 1.6$ and $\delta_i = 1.6$.

produced by implementing Theorem 1 on the positive data given in Table 1 with $\alpha_i = \delta_i = 1.6$ and $l_i = m_i = 0.1$. In Figure 6, the shape of the positive data has been preserved in a visually pleasing way.

4. Constrained curve interpolation

This section generalizes the curve scheme developed in Section 3 for positive data. It assumes that the data may lie, not just over the line $y = 0$, over any arbitrary line $y = mx + c$. Thus, the freedom of the shape parameters, in the description of the rational cubic spline in Section 2, would be utilized to achieve the interpolating curve when data are under consideration over an arbitrary line $y = mx + c$. Similar to that done in Section 3, the desired curve would be achieved by constraining two shape parameters. The other two parameters would be kept free for shape control to enhance the constrained curve further if needed.

Let $\{(x_i, f_i), i = 0, 1, 2, \ldots, n\}$ be the given set of data points lying above the straight line $y = mx + c$, that is,

$$f_i > mx_i + c, \quad \forall i = 0, 1, 2, \ldots, n.$$

The curve will lie above the straight line if the rational cubic function (3) satisfies the following condition:

$$S(x) > mx + c, \quad \forall x \in [x_0, x_n]. \tag{9}$$

For each subinterval $I_i = [x_i, x_{i+1}]$, the above relation can be expressed as follows:

$$S_i(x) = \frac{p_i(\theta)}{q_i(\theta)} > a_i(1 - \theta) + b_i\theta, \tag{10}$$

where $a_i(1 - \theta) + b_i\theta$ is the parametric equation of the straight line with $a_i = mx_i + c$ and $b_i = mx_{i+1} + c$. Multiplying both sides of Equation (10) with $q_i(\theta)$ by assuming that $\alpha_i > 0$, $\beta_i > 0$,

$\gamma_i > 0$ and $\delta_i > 0$ and after some rearrangement, Equation (10) reduces to

$$U_i(\theta) = \sum_{i=0}^{4} (1-\theta)^{4-i}\theta^i B_i, \tag{11}$$

where

$$\begin{aligned}
B_0 &= \alpha_i(f_i - a_i),\\
B_1 &= (\alpha_i + \beta_i)(f_i - a_i) + \alpha_i h_i d_i + \alpha_i f_i - \alpha_i b_i,\\
B_2 &= (\alpha_i + \beta_i)f_i + \alpha_i h_i d_i - (\alpha_i + \beta_i)b_i + (\gamma_i + \delta_i)f_{i+1} - \delta_i h_i d_{i+1} - (\gamma_i + \delta_i)a_i,\\
B_3 &= (\gamma_i + \delta_i)(f_{i+1} - b_i) - \delta_i h_i d_{i+1} + \delta_i f_{i+1} - \delta_i a_i,\\
B_4 &= \delta_i(f_{i+1} - a_i).
\end{aligned}$$

$U_i(\theta) > 0$ if

$$B_i > 0, \quad i = 0, 1, 2, 3, 4.$$

$B_i > 0, \quad i = 0, 1, 2, 3, 4$, if

$$\begin{aligned}
&\alpha_i > 0, \quad \delta_i > 0,\\
&\beta_i > \text{Max}\left\{0, -\frac{\alpha_i(h_i d_i + f_i - b_i)}{f_i - a_i}\right\},\\
&\gamma_i > \text{Max}\left\{0, -\frac{\delta_i(-h_i d_{i+1} + f_{i+1} - a_i)}{f_{i+1} - b_i}\right\}.
\end{aligned}$$

The above can be summarized as follows:

THEOREM 2 *The piecewise rational cubic interpolant* $S(x)$, *defined over the interval* $[a, b]$, *in* (3), *is positive if the following sufficient conditions are satisfied:*

$$\begin{aligned}
&\alpha_i > 0, \quad \delta_i > 0,\\
&\beta_i > \text{Max}\left\{0, -\frac{\alpha_i(h_i d_i + f_i - b_i)}{f_i - a_i}\right\},\\
&\gamma_i > \text{Max}\left\{0, -\frac{\delta_i(-h_i d_{i+1} + f_{i+1} - a_i)}{f_{i+1} - b_i}\right\}.
\end{aligned}$$

Remark 2 The constraints on the shape parameters can be rearranged as follows:

$$\begin{aligned}
&\alpha_i > 0, \quad \delta_i > 0,\\
&\beta_i = r_i + \text{Max}\left\{0, -\frac{\alpha_i(h_i d_i + f_i - b_i)}{f_i - a_i}\right\},\\
&\gamma_i = s_i + \text{Max}\left\{0, -\frac{\delta_i(-h_i d_{i+1} + f_{i+1} - a_i)}{f_{i+1} - b_i}\right\}.
\end{aligned}$$

The parameters β_i's and γ_i's are meant for the shape-preserving constraints, whereas the parameters α_i's and δ_i's are meant for shape control to enhance the curve shape further.

Table 2. Data set above the straight line $y = x/2 + 1$.

x	2	3	7	8	9	13	14
f	12	4.5	6.5	12	7.5	9.5	18

4.1 *Demonstration*

Consider the data set given in Table 2, which is lying above the straight line $y = x/2 + 1$. Obviously, the interpolating curve to the data given in Table 2 should lie above the straight line $y = x/2 + 1$. But the cubic Hermite does not assure this behaviour, as shown in Figure 7. In other words, the cubic Hermite does not preserve the shape of the data lying above the straight line. This flaw is recovered nicely in Figure 8 by implementing Theorem 2 in the data given in Table 2 with $\alpha_i = \delta_i = 0.05$ and $r_i = s_i = 0.1$.

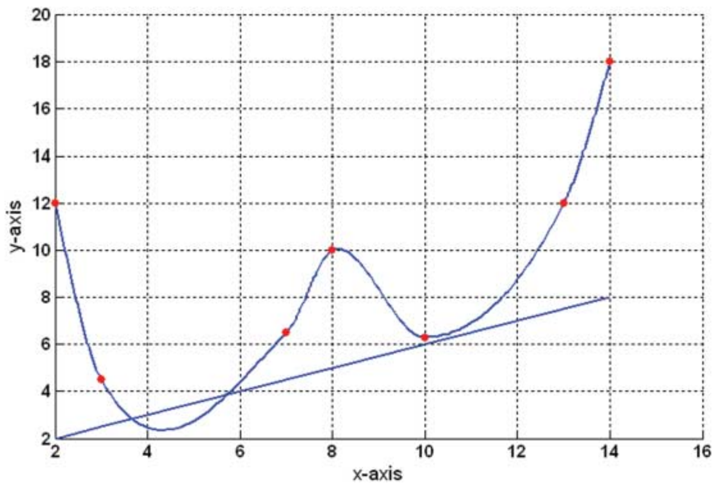

Figure 7. The cubic Hermite spline.

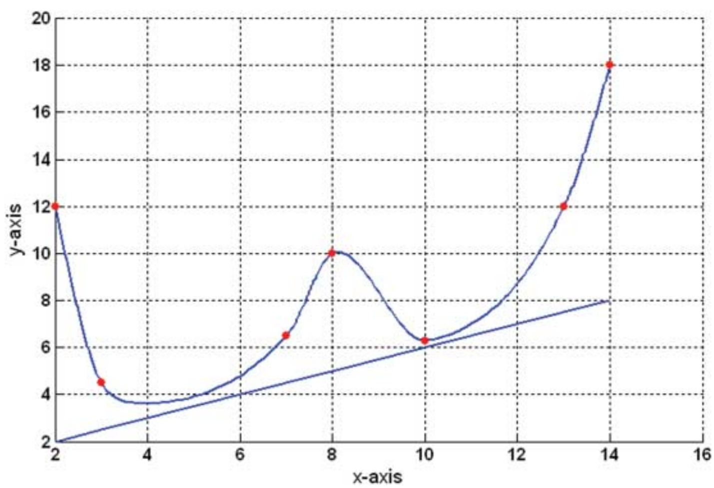

Figure 8. The rational cubic function with $\alpha_i = \delta_i = 0.05$ and $r_i = s_i = 0.1$.

5. Monotone curve interpolation

This section introduces the monotone curve scheme for monotone data. As the rational cubic spline, discussed in Section 2, carries four shape parameters in its description, it is a luxury freedom. This freedom can be utilized to achieve the monotony of the data-interpolating curve when monotone data are under consideration. This section is dedicated to orient the rational cubic spline, discussed in Section 2, to make it produce a monotone curve. The monotone curve would be achieved by constraining the shape parameters. The objective would be to utilize two parameters for the monotony-preserving constraints, while the other two parameters would be kept free for shape control to enhance the monotone curve further if needed.

Let $\{(x_i, f_i), i = 0, 1, 2, \ldots, n\}$ be the monotone data defined over the interval $[a, b]$ such that

$$f_i < f_{i+1}, \quad \Delta_i = \frac{f_{i+1} - f_i}{h_i} > 0, \quad d_i > 0, \quad i = 0, 1, 2, \ldots, n-1.$$

The piecewise rational cubic function (3) preserves monotony if

$$S_i^{(1)}(x) > 0, \quad i = 0, 1, 2, \ldots, n-1,$$

where

$$S_i^{(1)}(x) = \frac{\sum_{i=0}^{5}(1-\theta)^{5-i}\theta^i C_i}{(q_i(\theta))^2}, \tag{12}$$

$$\begin{aligned}
C_0 &= \alpha_i^2 d_i,\\
C_1 &= 2\alpha_i(\gamma_i + \delta_i)\Delta_i - 2\alpha_i\delta_i d_{i+1} + \alpha_i^2 d_i,\\
C_2 &= (\beta_i\gamma_i + \beta_i\delta_i + 3\alpha_i\gamma_i + 6\alpha_i\delta_i)\Delta_i - \delta_i(\beta_i + 3\alpha_i)d_{i+1} - (\gamma_i + \delta_i)\alpha_i d_i,\\
C_3 &= (\beta_i\gamma_i + \alpha_i\gamma_i + 3\beta_i\delta_i + 6\alpha_i\delta_i)\Delta_i - \delta_i(\alpha_i + \beta_i)d_{i+1} - (3\delta_i + \gamma_i)\alpha_i d_i,\\
C_4 &= 2\delta_i(\alpha_i + \beta_i)\Delta_i - 2\alpha_i\delta_i d_i + \delta_i^2 d_{i+1},\\
C_5 &= \delta_i^2 d_{i+1}.
\end{aligned}$$

From Equation (12), $S_i^{(1)}(x) > 0$ if

$$C_i > 0, \quad i = 0, 1, 2, 3, 4, 5.$$

Moreover, $C_i > 0$, $i = 0, 1, 2, 3, 4, 5$, if

$$\gamma_i > \frac{\delta_i d_{i+1}}{\Delta_i},$$

$$\beta_i > \frac{\alpha_i d_i}{\Delta_i}.$$

The above can be summarized as follows:

THEOREM 3 *The piecewise rational cubic interpolant $S(x)$, defined over the interval $[a, b]$, in (3), is monotone if the following sufficient conditions are satisfied:*

$$\alpha_i > 0, \quad \delta_i > 0,$$

$$\beta_i > \mathrm{Max}\left\{0, \frac{\alpha_i d_i}{\Delta_i}\right\},$$

$$\gamma_i > \mathrm{Max}\left\{0, \frac{\delta_i d_{i+1}}{\Delta_i}\right\}.$$

Table 3. Monotone radiochemical data.

x	7.99	8.09	8.19	8.7	9.2	10	12	15	20
f	0	0.276429e−2	0.437498e−3	0.169183	0.469428	0.943740	0.998636	0.999919	0.999994

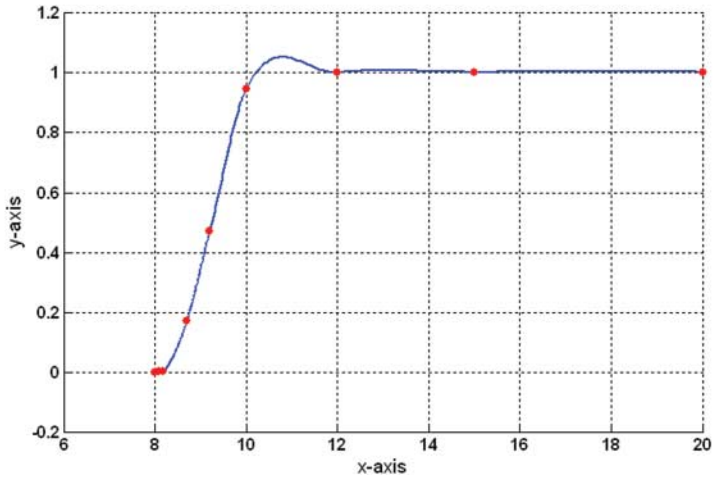

Figure 9. The cubic Hermite spline.

Remark 3 The constraints on the shape parameters can be rearranged as follows:

$$\alpha_i > 0, \quad \delta_i > 0,$$
$$\beta_i = n_i + \text{Max}\left\{0, \frac{\alpha_i d_i}{\Delta_i}\right\}, \quad n_i > 0,$$
$$\gamma_i = o_i + \text{Max}\left\{0, \frac{\delta_i d_{i+1}}{\Delta_i}\right\}, \quad o_i > 0.$$

The parameters β_i's and γ_i's are meant for the shape-preserving constraints, whereas the parameters α_i's and δ_i's are meant for shape control to enhance the curve shape further.

5.1 *Demonstration*

The monotone radiochemical data given in Table 3 are taken from [12]. A non-monotone curve from the monotone data given in Table 3 is produced in Figure 9 by using the cubic Hermite spline. To overcome this remedy, in Figure 10, the monotonicity-preserving scheme proposed in Theorem 3 is implemented on the monotone data given in Table 3. The values assigned to the free parameters are as follows: $\alpha_i = \delta_i = 2$ and $n_i = o_i = 1.9$. From Figure 10, it is clear that the shape of the monotone data given in Table 3 has been preserved.

6. Convex curve interpolation

This section introduces the convexity-preserving curve scheme for convex data. As the rational cubic spline, discussed in Section 2, carries four shape parameters in its description, it is a luxury freedom. This freedom can be utilized to achieve the convexity of the data-interpolating curve

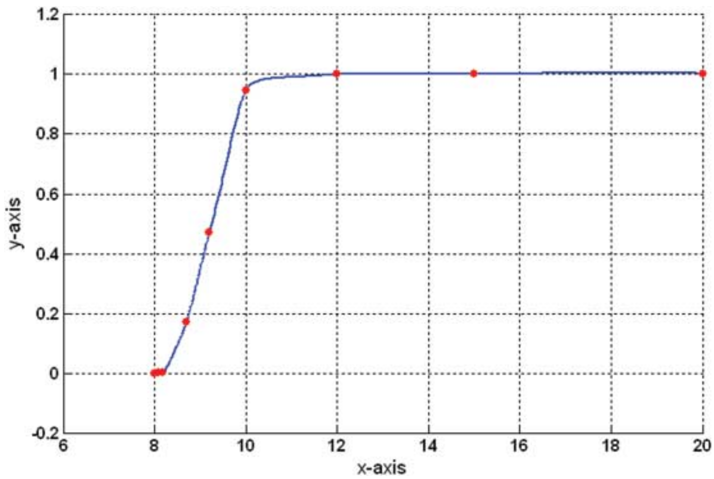

Figure 10. The rational cubic function with $\alpha_i = \delta_i = 2$ and $n_i = o_i = 1.9$.

when convex data are under consideration. This section is dedicated to orient the rational cubic spline, discussed in Section 2, to make it produce a convex curve. The convex curve would be achieved by constraining the shape parameters. The objective would be to utilize two parameters for the convexity-preserving constraints, while the other two parameters would be kept free for shape control to enhance the convex curve further if needed.

Let $\{(x_i,f_i), i = 0, 1, 2, \ldots, n\}$ be the convex data defined over the interval $[a, b]$ such that

$$\Delta_i < \Delta_{i+1}, \quad d_i < d_{i+1}, \quad i = 0, 1, 2, \ldots, n-1.$$

The piecewise rational cubic function (3) preserves convexity if

$$S_i^{(2)}(x) > 0, \quad i = 0, 1, 2, \ldots, n-1,$$

where

$$S_i^{(2)}(x) = \frac{\sum_{i=0}^{7}(1-\theta)^{7-i}\theta^i D_i}{h_i(q_i(\theta))^3}, \tag{13}$$

where

$$\begin{aligned}
D_0 &= \alpha_i C_1 - (\alpha_i + 2\beta_i)C_0,\\
D_1 &= 2\alpha_i C_2 + (\alpha_i - \beta_i)C_1 + (-\alpha_i - \beta_i - 4\gamma_i - 4\delta_i)C_0,\\
D_2 &= 3\alpha_i C_3 + 3\alpha_i C_2 + (-3\gamma_i - 3\delta_i)C_1 + (-3\gamma_i - 9\delta_i)C_0,\\
D_3 &= 4\alpha_i C_4 + (5\alpha_i + \beta_i)C_3 + (\alpha_i + \beta_i - 2\gamma_i - 2\delta_i)C_2 + (-2\gamma_i - 7\delta_i)C_1,\\
D_4 &= (7\alpha_i + 2\beta_i)C_4 + (2\alpha_i + 2\beta_i - \gamma_i - \delta_i)C_3 + (-\gamma_i - 5\delta_i)C_2 - 4\delta_i C_1,\\
D_5 &= (9\alpha_i + 3\beta_i)C_5 + (3\alpha_i + 3\beta_i)C_4 - 3\delta_i C_3 - 3\delta_i C_2,\\
D_6 &= (\gamma_i + \delta_i + 4\alpha_i + 4\beta_i)C_5 + (\gamma_i - \delta_i)C_4 - 2\delta_i C_3,\\
D_7 &= (\delta_i + 2\gamma_i)C_5 - \delta_i C_4.
\end{aligned}$$

$S_i^{(2)}(x) > 0$ if

$$(q_i(\theta))^3 > 0, \quad D_i > 0, \quad i = 0, 1, 2, \ldots, 7.$$

$(q_i(\theta))^3 > 0$ if

$$\alpha_i > 0, \quad \beta_i > 0, \quad \gamma_i > 0, \quad \delta_i > 0.$$

$D_i > 0, \quad i = 0, 1, 2, \ldots, 7$, if

$$\begin{aligned} &\alpha_i > 0, \quad \beta_i > 0, \quad \delta_i > 0, \\ &\gamma_i = \beta_i, \\ &\gamma_i > \text{Max}\left\{0, \frac{\delta_i(d_{i+1} - \Delta_i)}{\Delta_i - d_i}, \frac{\alpha_i(\Delta_i - d_i)}{d_{i+1} - \Delta_i}\right\}. \end{aligned}$$

The above can be summarized as follows:

THEOREM 4 *The piecewise rational cubic interpolant $S(x)$, defined over the interval $[a, b]$, in (3), is convex if the following sufficient conditions are satisfied:*

$$\begin{aligned} &\alpha_i > 0, \quad \beta_i > 0, \quad \delta_i > 0, \\ &\gamma_i = \beta_i, \\ &\gamma_i > \text{Max}\left\{0, \frac{\delta_i(d_{i+1} - \Delta_i)}{\Delta_i - d_i}, \frac{\alpha_i(\Delta_i - d_i)}{d_{i+1} - \Delta_i}\right\}. \end{aligned}$$

Remark 4 The constraints on the shape parameters can be rearranged as follows:

$$\begin{aligned} &\alpha_i > 0, \quad \beta_i > 0, \quad \delta_i > 0, \\ &\gamma_i = \beta_i, \\ &\gamma_i = k_i + \text{Max}\left\{0, \frac{\delta_i(d_{i+1} - \Delta_i)}{\Delta_i - d_i}, \frac{\alpha_i(\Delta_i - d_i)}{d_{i+1} - \Delta_i}\right\}, \quad k_i > 0. \end{aligned}$$

The parameters γ_i's are meant for the shape-preserving constraints, whereas the parameters α_i's, β_i's and δ_i's are meant for shape control to enhance the curve shape further.

6.1 *Demonstration*

To demonstrate the convexity-preserving scheme developed in Section 6, let us consider the data set given in Table 4.

In Figure 11, we have implemented the cubic Hermite spline on the convex data given in Table 4. The non-convex curve shown in Figure 11 clearly indicates that the cubic Hermite spline failed to preserve the convex shape of the data. To demonstrate the usefulness of the convexity-preserving scheme developed in Section 6, Theorem 4 is implemented on the convex data given in Table 4 with $\alpha_i = \delta_i = 0.4$ and $k_i = 0.1$. From Figure 12, it is clear that the shape of the convex data given in Table 4 is preserved smoothly.

Table 4. Convex data set.

x	−4.0	−3.5	−2.0	0	2.0	3.5	4.0
f	5.0	0	−3.5	−4.0	3.5	0	5

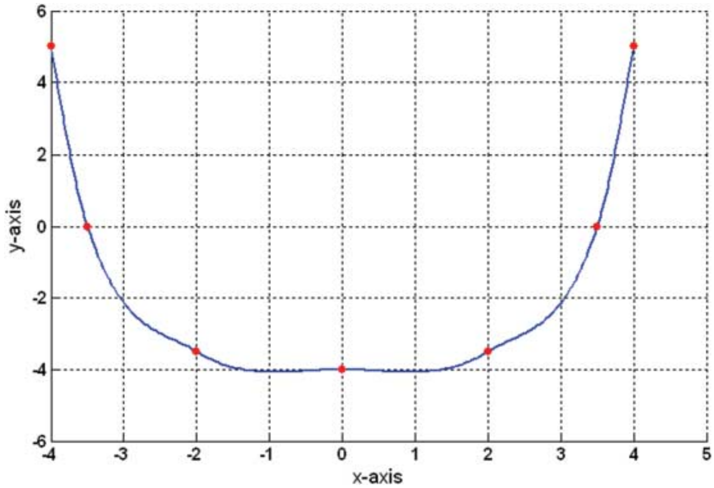

Figure 11. The cubic Hermite spline.

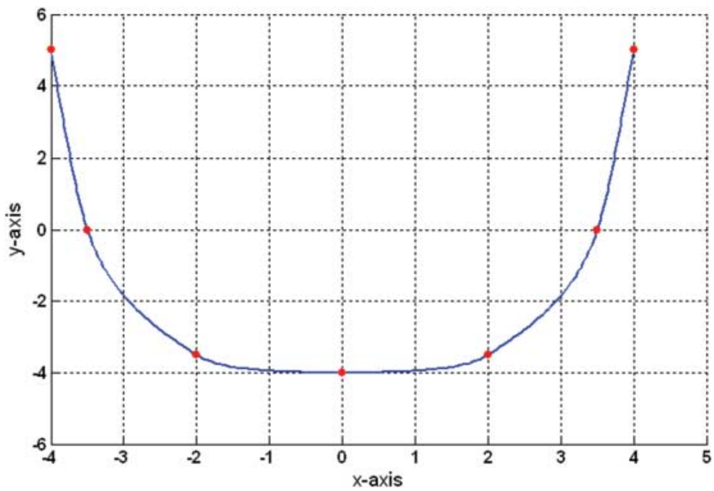

Figure 12. The rational cubic function with $\alpha_i = \delta_i = 0.4$ and $k_i = 0.1$.

7. Error estimation of interpolation

In this section, we estimate the approximation error that occurs when the rational cubic function (3) is used to interpolate data from an arbitrary function, that is, $f(x) \in C^3[x_0, x_n]$. The interpolation scheme is local, so we shall investigate the error in each subinterval $I_i = [x_i, x_{i+1}]$ without loss of generality. The Peano Kernel theorem [18] is used to estimate the error in each subinterval $I_i = [x_i, x_{i+1}]$ as follows:

$$R[f] = f(x) - S(x) = \frac{1}{2}\int_{x_i}^{x_{i+1}} f^{(3)}(\tau) R_x[(x-\tau)_+^2] d\tau. \tag{14}$$

Using the uniform norm, Equation (14) becomes

$$|f(x) - P(x)| \leq \frac{1}{2}\|f^{(3)}(\tau)\| \int_{x_i}^{x_{i+1}} |R_x[(x-\tau)_+^2]| \mathrm{d}\tau, \tag{15}$$

where

$$R_x[(x-\tau)_+^2] = \begin{cases} r(\tau,x), & x_i < \tau < x \\ s(\tau,x), & x < \tau < x_{i+1} \end{cases}$$

is called the Peano Kernel:

$$r(\tau,x) = (x-\tau)^2 - \frac{1}{q_i(\theta)}[(x_{i+1}-\tau)^2\{(1-\theta)\theta^2(\gamma_i+\delta_i)+\theta^3\delta_i\} - 2\delta_i h_i(x_{i+1}-\tau)(1-\theta)\theta^2], \tag{16}$$

$$s(\tau,x) = -\frac{1}{q_i(\theta)}[(x_{i+1}-\tau)^2\{(1-\theta)\theta^2(\gamma_i+\delta_i)+\theta^3\delta_i\} - 2\delta_i h_i(x_{i+1}-\tau)(1-\theta)\theta^2]. \tag{17}$$

In Equation (15), $\int_{x_i}^{x_{i+1}} |R_x[(x-\tau)_+^2]|\mathrm{d}\tau$ can be expressed as follows:

$$\int_{x_i}^{x_{i+1}} |R_x[(x-\tau)_+^2]|\mathrm{d}\tau = \int_{x_i}^{x} |r(\tau,x)|\mathrm{d}\tau + \int_{x}^{x_{i+1}} |s(\tau,x)|\mathrm{d}\tau.$$

For $0 < \delta_i/\gamma_i < 1$, the roots of $r(x,x)$ in [0,1] are $\theta = 0$, $\theta = 1$ and $\theta = 1 - \delta_i/\gamma_i$. For $\delta_i/\gamma_i > 1$, the roots of $r(x,x)$ in [0,1] are $\theta = 0$ and $\theta = 1$.

The roots of $r(\tau,x) = 0$ are

$$\tau_i = x - \frac{h_i\theta(G+(-1)^{i+1}H)}{\alpha_i+\beta_i\theta}, \quad i = 1,2,$$

where

$$G = \gamma_i\theta,$$

$$H = \sqrt{\alpha_i(\gamma_i-\delta_i) + \{\beta_i(\gamma_i-\delta_i)-\gamma_i\alpha_i\}\theta + (\gamma_i^2-\gamma_i\beta_i)\theta^2}.$$

The root of $s(\tau,x) = 0$ are

$$\tau_3 = x_{i+1} - \frac{2\delta_i h_i(1-\theta)}{(\gamma_i+\delta_i)-\gamma_i\theta}, \quad \tau_4 = x_{i+1}.$$

Case 1 For $\delta_i/\gamma_i > 1$, Equation (15) takes the form

$$|f(x) - S(x)| \le \frac{1}{2}\|f^{(3)}(\tau)\| h_i^3 \omega_1(\alpha_i,\beta_i,\gamma_i,\delta_i,\theta),$$

$$\begin{aligned}
\omega_1(\alpha_i,\beta_i,\gamma_i,\delta_i,\theta) &= \int_{x_i}^{x} |r(\tau,x)|\mathrm{d}\tau + \int_{x}^{x_{i+1}} |s(\tau,x)|\mathrm{d}\tau \\
&= -\int_{x_i}^{\tau_1} r(\tau,x)\mathrm{d}\tau + \int_{\tau_1}^{\tau_2} r(\tau,x)\mathrm{d}\tau - \int_{\tau_2}^{x} r(\tau,x)\mathrm{d}\tau \\
&\quad - \int_{x}^{\tau_3} s(\tau,x)\mathrm{d}\tau + \int_{\tau_3}^{x_{i+1}} s(\tau,x)\mathrm{d}\tau \\
&= \frac{(1-\theta)^2\theta^3\{\gamma_i-\delta_i-(\alpha_i+\beta_i\theta)\}}{q_i(\theta)} + \frac{2(1-\theta)^2\theta^3\{6G^2H+2H^3\}}{3q_i(\theta)(\alpha_i+\beta_i\theta)^2} \\
&\quad - \frac{8(1-\theta)^2\theta^4\gamma_i GH}{q_i(\theta)(\alpha_i+\beta_i\theta)^2} - \frac{2(1-\theta)^2\theta^3\{(\gamma_i-\delta_i)-\gamma_i\theta\}H}{(\alpha_i+\beta_i\theta)} \\
&\quad + \frac{8(1-\theta)^3\theta^2\delta_i^3\{(1-\theta)\gamma_i+\delta_i\}}{q_i(\theta)(\gamma_i(1-\theta)+\delta_i\theta)^3}.
\end{aligned}$$

Case 2 For $0 < \delta_i/\gamma_i < 1$ and $0 < \theta < \theta^*$, Equation (15) takes the form

$$
\begin{aligned}
|f(x) - S(x)| &\le \tfrac{1}{2}\|f^{(3)}(\tau)\|h_i^3\omega_2(\alpha_i, \beta_i, \gamma_i, \delta_i, \theta),\\
\omega_2(\alpha_i, \beta_i, \gamma_i, \delta_i, \theta) &= \int_{x_i}^{x} |r(\tau, x)|\mathrm{d}\tau + \int_{x}^{x_{i+1}} |s(\tau, x)|\mathrm{d}\tau\\
&= \int_{x_i}^{\tau_1} r(\tau, t)\mathrm{d}\tau - \int_{\tau_1}^{x} r(\tau, t)\,\mathrm{d}\tau + \int_{x}^{\tau_3} s(\tau, t)\mathrm{d}\tau - \int_{\tau_3}^{x_{i+1}} s(\tau, t)\mathrm{d}\tau\\
&= \frac{-2(1-\theta)^2\theta^3(G+H)^3}{3q_i(\theta)(\alpha_i + \beta_i\theta)^2} + \frac{2(1-\theta)^2\theta^4\{6G^2H + 2H^3\}}{3q_i(\theta)}\\
&\quad - \frac{8(1-\theta)^2\theta^4\gamma_i GH}{q_i(\theta)(\alpha_i + \beta_i\theta)^2} - \frac{4(1-\theta)^2\theta^2\{(\gamma_i - \delta_i) - \gamma_i\theta\}H}{(\alpha_i + \beta_i\theta)}\\
&\quad - \frac{(1-\theta)^3\theta^2\{(\gamma_i + 2\delta_i)\theta - \gamma_i\}}{q_i(\theta)} - \frac{8(1-\theta)^3\theta^2\delta_i^3\{(1-\theta)\gamma_i + \delta_i\}}{q_i(\theta)(\gamma_i(1-\theta) + \delta_i\theta)^3}.
\end{aligned}
$$

Case 3 For $0 < \delta_i/\gamma_i < 1$ and $\theta^* < \theta < 1$, Equation (15) takes the form

$$
\begin{aligned}
|f(x) - S(x)| &\le \tfrac{1}{2}\|f^{(3)}(\tau)\|h_i^3\omega_3(\alpha_i, \beta_i, \gamma_i, \delta_i, \theta),\\
\omega_3(\alpha_i, \beta_i, \gamma_i, \delta_i, \theta) &= \int_{x_i}^{x} |r(\tau, x)|\mathrm{d}\tau + \int_{x}^{x_{i+1}} |s(\tau, x)|\mathrm{d}\tau\\
&= \int_{x_i}^{\tau_1} r(\tau, x)\,\mathrm{d}\tau - \int_{\tau_1}^{\tau_2} r(\tau, x)\mathrm{d}\tau + \int_{\tau_2}^{x} r(\tau, x)\mathrm{d}\tau + \int_{x}^{\tau_3} s(\tau, x)\mathrm{d}\tau\\
&\quad - \int_{\tau_3}^{x_{i+1}} s(\tau, x)\mathrm{d}\tau\\
&= -\omega_1(\alpha_i, \beta_i, \gamma_i, \delta_i, \theta).
\end{aligned}
$$

The above can be summarized as follows:

THEOREM 5 *The error of the rational cubic function* (3), *for* $f(x) \in C^3[x_0, x_n]$, *in each subinterval* $[x_i, x_{i+1}]$ *is*

$$
\begin{aligned}
|f(x) - S(x)| &\le \tfrac{1}{2}\|f^{(3)}(\tau)\|h_i^3 c_i,\\
c_i &= \max_{0\le\theta\le 1} \omega(\alpha_i, \beta_i, \gamma_i, \delta_i, \theta),\\
\omega(\alpha_i, \beta_i, \gamma_i, \delta_i, \theta) &= \begin{cases} \max \omega_1(\alpha_i, \beta_i, \gamma_i, \delta_i, \theta), & \dfrac{\delta_i}{\gamma_i} > 1,\\ \max \omega_2(\alpha_i, \beta_i, \gamma_i, \delta_i, \theta), & 0 < \dfrac{\delta_i}{\gamma_i} < 1, \quad 0 < \theta < \theta^*,\\ \max \omega_3(\alpha_i, \beta_i, \gamma_i, \delta_i, \theta), & 0 < \frac{\delta_i}{\gamma_i} < 1, \quad \theta^* < \theta < 1. \end{cases}
\end{aligned}
$$

8. Conclusion

A C^1 piecewise rational cubic function has been developed to preserve the shape of positive, monotone and convex data. The developed C^1 interpolant involves four free parameters in its construction. Data-dependent sufficient constraints have been developed on two of the free parameters to preserve the shape of the data (positive, monotone and convex), while the other two free parameters have been utilized to refine the shape of the curve at the user's wish. The derivatives have been approximated by the three-point difference approximation formula (arithmetic mean choice) of derivatives. However, these derivatives can be approximated by another means. The choice of the derivative approximation scheme has not affected the shape-preserving interpolation scheme developed in this paper. The proposed curve schemes have been implemented on the practical data sets and have been proved worthy. The order of approximation is $O(h_i^3)$.

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