Tohoku Math. J.
63 (2011), 149–162

SHARED VALUES, PICARD VALUES AND NORMALITY

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(Received November 27, 2009, revised July 12, 2010)

Abstract. It is known that a family of meromorphic functions is normal if each function in the family shares a 3-element set with its derivative. In this paper we consider value distribution and normality problems with regard to 2-element shared sets. First we construct an example, by use of the Weierstrass doubly periodic functions, to show that a 3-element shared set can not be reduced to a 2-element shared set in general. We obtain a new criterion of normal families and new Picard-type theorems. The proofs make use of some results in complex dynamics. More examples are constructed to show that our assumptions are necessary.

1. Introduction and main results. A family $\mathcal{F}$ of meromorphic functions defined in a plane domain $D \subset \boldsymbol{C}$ is said to be normal in D, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges spherically locally uniformly in D to a meromorphic function or ∞. See [10, 20, 22].

It was Schwick [21] who first studied the relation between normality and shared values. He proved that if there exist three distinct finite numbers $a_j,\ j = 1, 2, 3$ such that every function f in a family $\mathcal{F}$ of meromorphic functions satisfies

$$f^{-1}(\{a_j\}; D) = (f')^{-1}(\{a_j\}; D)$$

for $j = 1, 2, 3$, then the family $\mathcal{F}$ is normal in D. Here and in the sequel, $f^{-1}(E; D)$ stands for the set $\{z \in D\,; f(z) \in E\}$. For the case $D = \boldsymbol{C}$, we simply write $f^{-1}(E)$ instead of $f^{-1}(E; \boldsymbol{C})$. Schwick's result has been improved by Pang-Zalcman [18] and Chang-Fang-Zalcman [6]. They proved the following theorem.

THEOREM A. *Let $\mathcal{F}$ be a family of functions meromorphic in D and $a_1,\ a_2$ distinct finite numbers. If $f^{-1}(\{a_j\}; D) = (f')^{-1}(\{a_j\}; D)$ for $j = 1, 2$ and for every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

Recently, Liu-Pang [14] improved Schwick's result as follows.

THEOREM B. *Let $\mathcal{F}$ be a family of functions meromorphic in D and $a,\ b,\ c$ distinct finite numbers. If $f^{-1}(\{a, b, c\}; D) = (f')^{-1}(\{a, b, c\}; D)$ for every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

2000 *Mathematics Subject Classification.* Primary 30D45; Secondary 30D35.

Key words and phrases. Meromorphic functions, normality, shared values and sets.

[1] Supported by NNSF of China (Grant No. 10871094), NSFU of Jiangsu, China (Grant No. 08KJB110001), Qinglan Project of Jiangsu, China, and the SRF for ROCS, SEM.

[2] Supported by NNSF and 973 Projects of China.

Naturally, we may ask whether Theorem B remains true if the 3-element set $\{a, b, c\}$ is replaced by a 2-element set $\{a, b\}$. We first construct an example to show that the answer to this question is negative in general.

EXAMPLE 1. For every $n \in \boldsymbol{N}$, let $\wp_n$ be a doubly periodic function defined by the following differential equation

$$(\wp_n')^2 = 4(\wp_n)^3 + 12\wp_n + 9n^2\,, \quad \wp_n(0) = n\,, \tag{1.1}$$

and let

$$f_n(z) = \frac{1}{2n}\wp_n'\left(\frac{nz}{3}\right) - \frac{1}{2}\,.$$

Then we have

$$n^4(f_n - 1)^2(f_n + 2)^2 = (f_n' - 1)(f_n' + 2)^2\,.$$

Thus all poles of f_n are triple and $f_n^{-1}(\{-2, 1\}) = (f_n')^{-1}(\{-2, 1\})$.

Next we show that the family $\{f_n\}$ is not normal at the origin. By (1.1), we have $|\wp_n'(0)| = \sqrt{4n^3 + 9n^2 + 12n} \le 5n\sqrt{n}$ and hence

$$|f_n(0)| \le \frac{|\wp_n'(0)|}{2n} + \frac{1}{2} \le 3\sqrt{n}\,.$$

Again by (1.1), we also have $\wp_n'' = 6(\wp_n)^2 + 6$, so that

$$|f_n'(0)| = \frac{|\wp_n''(0)|}{6} = n^2 + 1\,.$$

Thus we have

$$f_n^{\#}(0) = \frac{|f_n'(0)|}{1 + |f_n(0)|^2} \ge \frac{n^2 + 1}{1 + 9n} \to \infty\,.$$

It follows from Marty's theorem that $\{f_n\}$ is not normal at the origin.

However, we show that the answer is positive when the quotient a/b lies in some subdomain of $\boldsymbol{C}$. To state our result, we denote

$$\Omega_0 = \{z \in \overline{\boldsymbol{C}}\,;\ z \in \widehat{\Delta}(1, 1) \text{ or } 1/z \in \widehat{\Delta}(1, 1)\}\,,$$

where

$$\widehat{\Delta}(1, 1) = \{z \in \boldsymbol{C}\,;\ |z - 1| < 1 \text{ or } (z - 1)^k = 1 \text{ for some positive integer } k\}\,.$$

That is to say, $\widehat{\Delta}(1, 1)$ is the disk $\Delta(1, 1)$ with some boundary points that are translates of roots of unity. We remark that nonnegative real values and ∞ are in Ω_0, while negative real values are not.

We point out that if two nonzero complex numbers a, b satisfy $|\arg a/b| \le \pi/3$, then $b/a \in \Omega_0$.

THEOREM 1. *Let $\mathcal{F}$ be a family of functions meromorphic in D and a, b distinct nonzero constants such that $b/a \in \Omega_0$. If $f^{-1}(\{a, b\}; D) = (f')^{-1}(\{a, b\}; D)$ for every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

COROLLARY 2. *Let $\mathcal{F}$ be a family of functions meromorphic in D and a, b distinct nonzero numbers such that $|\arg a/b| \le \pi/3$. If $f^{-1}(\{a, b\}; D) = (f')^{-1}(\{a, b\}; D)$ for every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

The proof of Theorem 1 requires the following Picard-type theorems, which are of independent interest. Actually the most hard part of the paper is to prove them. We also make use of some results in complex dynamics.

THEOREM 3. *Let a, b be two distinct nonzero numbers such that $b/a \in \Omega_0$ and f an entire function satisfying $f^{-1}(\{0\}) = (f')^{-1}(\{a, b\})$. Then f is constant.*

If b/a is not in Ω_0, Theorem 3 does not hold in general. This is shown by the examples $f_1(z) = z^2 - z$ and $f_2(z) = \sin z$, each satisfying $f^{-1}(\{0\}) = (f')^{-1}(\{-1, 1\})$. We also note that Theorem 3 does not hold for meromorphic functions, since the function $f(z) = (z^2-1)/z$ satisfies $f^{-1}(\{0\}) = (f')^{-1}(\{1, 2\})$ and $2/1 = 2$ is in Ω_0.

For meromorphic functions, we have the following theorem.

THEOREM 4. *Let a, b be two distinct numbers such that $b/a \in \Omega_0$ and f a meromorphic function satisfying $f^{-1}(\{0\}) = (f')^{-1}(\{a, b\})$.*

(i) *If $ab = 0$ and f is of order less than one, then f is constant.*

(ii) *If $ab \neq 0$ and f is nonconstant and of finite order, then either $f(z) = a(z - z_0) + d(z - z_0)^{-n}$ with $b = (n + 1)a$ or $f(z) = b(z - z_0) + d(z - z_0)^{-n}$ with $a = (n + 1)b$, where d $(\neq 0)$ and z_0 are constants and n is a positive integer.*

(iii) *If $ab \neq 0$ and b/a is not in $\boldsymbol{N} \cup 1/\boldsymbol{N}$, then f is constant.*

Here $1/\boldsymbol{N}$ stands for the set $\{1/n; n \in \boldsymbol{N}\}$.

Examples after Theorem 3 show that the assumption $b/a \in \Omega_0$ in the two results is necessary. More examples can be found in the final section. However, we do not know whether the set Ω_0 can be enlarged. The functions $f(z) = e^z - 1$ and $f(z) = (e^{2z} + 1)/(e^{2z} - 1)$, satisfying $f^{-1}(\{0\}) = (f')^{-1}(\{1, 0\})$, show that Theorem 4(i) is sharp. But we do not know whether the hypothesis that f is of finite order is necessary for Theorem 4(ii).

We note that a Bank-Laine function f is an entire function satisfying $f^{-1}(\{0\}) \subset (f')^{-1}$ $(\{-1, 1\})$. The Bank-Laine functions arise in connection with solutions of second order homogeneous linear differential equations [2], and have been studied in many papers [2, 8, 9, 11, 12, 13].

The plan of the paper is as follows. In Section 2, we state and prove a number of auxiliary results. In Section 3, we give the proofs of Theorems. In the final section, we make a few remarks and construct several examples.

2. Auxiliary results. In this section, we state some known results, and prove the main lemmas that are required in the proofs of our results.

LEMMA 1 ([10, Corollary to Theorem 3.5]). *Let f be a transcendental meromorphic function. Then for every positive integer k, either f or $f^{(k)} - 1$ has infinitely many zeros.*

LEMMA 2 ([4, Corollary 2]). *Let f be a meromorphic function of finite order ρ and E be the set of its critical values. Then the number of asymptotic values of f is at most $2\rho+\mathrm{card}E'$, where E' stands for the derived set of E.*

LEMMA 3 ([18, Lemma 2.2]). *Let f be meromorphic on $\boldsymbol{C}$ such that the set of its finite critical and asymptotic values is bounded. Then there exists a positive number r_0 such that if $|z| > r_0$ and $|f(z)| > r_0$, then*

$$|f'(z)| \geq \frac{|f(z)| \log |f(z)|}{16\pi |z|} .$$

LEMMA 4 (cf. [10, 22]). *Let f be a meromorphic function on the plane such that its spherical derivative $f^{\#}(z) = |f'(z)|/(1 + |f(z)|^2)$ is uniformly bounded. Then f is of order at most 2.*

LEMMA 5 ([17, Lemma 2], cf. [16]). *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$ and $f \in \mathcal{F}$. Then if $\mathcal{F}$ is not normal at z_0, there exist, for each $0 \leq \alpha \leq k$,*

(a) *points $z_n \in D$, $z_n \to z_0$;*

(b) *functions $f_n \in \mathcal{F}$; and*

(c) *positive numbers $\rho_n \to 0$*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) = g_n(\zeta) \to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function in $\boldsymbol{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$.

LEMMA 6 ([7, Lemma 12]). *Let R be a rational function such that $R' \neq 0$ on $\boldsymbol{C}$. Then either $R = az + b$ or $R = a/(z + c)^n + b$, where a $(\neq 0)$, b and c are constant, and n is a positive integer.*

LEMMA 7. *Let f be a transcendental meromorphic function of finite order ρ. If f satisfies $f^{-1}(\{0\}) = (f')^{-1}(\{0, 1\})$, then f' has finitely many zeros.*

PROOF. Set $F = z - f$. Then $F' = 1 - f'$. Since $f^{-1}(\{0\}) = (f')^{-1}(\{0, 1\})$, we have $F(z) = z$ if and only if $F'(z)$ is in $\{0, 1\}$. It follows that all critical values of F are super-attracting fixed points of F. So by Lemma 2 or the Denjoy-Carleman-Ahlfors Theorem [15, p.313], F has at most 2ρ finite asymptotic values.

Now suppose that f' has infinitely many zeros z_n $(n = 1, 2, \dots)$, then we see that these points z_n are rationally indifferent fixed points of F with multiplier 1, and hence there exist parabolic domains U_n $(n = 1, 2, \dots)$ [1, Theorem 2.1] such that $z_n \in \partial U_n$ and $F^j \to z_n$ locally uniformly in U_n as $j \to \infty$. However, we know that every parabolic domain contains at least one singular value [3, Theorem 7]. Thus with at most 2ρ exceptions, U_n contains a critical value and hence a super-attracting fixed point of F. Since $F^j \to z_n$ locally uniformly in U_n, z_n coincides with the super-attracting fixed point, which is a contradiction. This contradiction shows that f' has finitely many zeros. Lemma 7 is proved.

LEMMA 8. *Let f be a nonconstant entire functions of finite order such that $f^{-1}(\{0\}) = (f')^{-1}(\{0, 1\})$. Then f is of the form $f(z) = Ce^{\lambda z} - 1/\lambda$ for some nonzero constants C and λ.*

PROOF. Suppose first that f is transcendental. Then by Lemma 7, $f' = Pe^Q$, where P ($\not\equiv 0$) and Q are a polynomials and Q is non-constant. It follows from $f' = Pe^Q$ that $f' - 1$ has finite many multiple zeros, and hence by the condition $f^{-1}(\{0\}) = (f')^{-1}(\{0, 1\})$,

$$\frac{f}{f' - 1} = Re^H , \tag{2.1}$$

where R ($\not\equiv 0$) is a rational function and H is a polynomial.

Thus we have $f = Re^H(f' - 1) = Re^H(Pe^Q - 1)$ and then

$$f' = [(RP)' + RP(H' + Q')]e^{H+Q} - (R' + RH')e^H .$$

Since $f' = Pe^Q$, we then get

$$Me^Q - Pe^{Q-H} = N , \tag{2.2}$$

where $M = (RP)' + RP(H' + Q')$ and $N = R' + RH'$. If $M \equiv 0$, then $H' + Q' = -(RP)'/(RP) \to 0$ as $z \to \infty$. It follows that $H + Q$ is a constant since $H + Q$ is a polynomial. On the other hand, since $M \equiv 0$, by (2.2), we can see that $Q - H$ is also a constant. Thus Q is a constant, which is a contradiction. Thus $M \not\equiv 0$.

If $N \not\equiv 0$, then by differentiating the both sides of (2.2), we obtain

$$(M' + MQ')e^Q - [P' + P(Q' - H')]e^{Q-H} = N' .$$

It with (2.2) yields that

$$(M' + MQ')e^Q - [P' + P(Q' - H')]\frac{Me^Q - N}{P} = N' ,$$

so that

$$\left(\frac{M'}{M} - \frac{P'}{P} + H'\right)e^Q = \left(\frac{N'}{N} - \frac{P'}{P} + H' - Q'\right)\frac{N}{M} .$$

Since Q is nonconstant, it follows that

$$\frac{M'}{M} - \frac{P'}{P} + H' = \frac{N'}{N} - \frac{P'}{P} + H' - Q' = 0 .$$

Thus $Q' = N'/N - M'/M \to 0$ as $z \to \infty$. Hence Q is a constant, which is a contradiction.

Thus $N = R' + RH' \equiv 0$, and hence $H' = -R'/R \to 0$ as $z \to \infty$. This with the fact that H is a polynomial shows that H is a constant, and then so is R. Thus by (2.1), $f' - 1 = \lambda f$ for some nonzero constant λ. It follows that f is of the form $f(z) = Ce^{\lambda z} - 1/\lambda$.

Now consider the case that f is a polynomial. Since $f^{-1}(\{0\}) = (f')^{-1}(\{0, 1\})$, f' can not be constant. Thus we may say

$$f'(z) = \prod_{j=1}^{m}(z - z_j)^{p_j} , \quad f'(z) - 1 = \prod_{j=m+1}^{n}(z - z_j)^{p_j} , \tag{2.3}$$

where $p_j, m, n\ (\geq m+1 \geq 2)$ are positive integers satisfying

$$\sum_{j=1}^{m} p_j = \sum_{j=m+1}^{n} p_j = \deg(f') = \deg(f) - 1\,. \tag{2.4}$$

Since $f^{-1}(\{0\}) = (f')^{-1}(\{0, 1\})$, it follows from (2.3) that all zeros of f are z_j, $1 \leq j \leq n$, where z_j, $1 \leq j \leq m$, have multiplicity $p_j + 1$ while z_j, $m+1 \leq j \leq n$, are simple zeros. Thus

$$\deg(f) = \sum_{j=1}^{m}(p_j + 1) + n - m = \sum_{j=1}^{m} p_j + n\,.$$

From this with (2.4), we deduce that $n = 1$, which is impossible. Hence Lemma 8 is proved.

LEMMA 9. *Let f be a nonconstant meromorphic function of finite order ρ and a, b distinct nonzero finite values satisfying $|a-b| < |a|$ or $(a-b)^k = a^k$ for some positive integer k. If $f(z) = 0$ if and only if $f'(z)$ is in $\{a, b\}$, then f is a rational function.*

PROOF. We may assume $a = 1$ and then either $|1-b| < 1$ or $(1-b)^k = 1$. First we prove that $f' - b$ has at most finitely many zeros. Set $F = z - f$. Then $F' = 1 - f'$. Since $f(z) = 0$ if and only if $f'(z) \in \{1, b\}$, we have $F(z) = z$ if and only if $F'(z) \in \{0, 1-b\}$. It follows that all critical values of F are super-attracting fixed points of F. So by Lemma 2, F has at most 2ρ finite asymptotic values.

Now assume the contrary that $f' - b$ has infinitely many zeros, say $\{z_n\}$. Then we have $F'(z_n) = 1 - b$ and hence $F(z_n) = z_n$. Since either $0 < |1-b| < 1$ or $(1-b)^k = 1$, z_n are either attracting fixed points or rationally indifferent fixed points of F.

If z_n is an attracting fixed point, then there exists an attracting basin U_n such that $z_n \in U_n$ and $F^j \to z_n$ locally uniformly in U_n as $j \to \infty$ [1, Theorem 2.1]. However, a well-known fact [3, Theorem 7] is that every attracting basin contains at least one singular value. Thus with at most 2ρ exceptions, U_n contains a critical value and hence a super-attracting fixed point of F. Since $F^j \to z_n$ locally uniformly in U_n, z_n coincides with the super-attracting fixed point. It is a contradiction.

If z_n is a rationally indifferent fixed point, then there exists a parabolic domain U_n such that $z_n \in \partial U_n$ and $F^{kj} \to z_n$ locally uniformly in U_n as $j \to \infty$ [1, Theorem 2.1]. However, we know that every parabolic domain contains at least one singular value [3, Theorem 7]. Thus with at most 2ρ exceptions, U_n contains a critical value and hence a super-attracting fixed point of F. Since $F^{kj} \to z_n$ locally uniformly in U_n, z_n coincides with the super-attracting fixed point. Again this is a contradiction.

Thus $f' - b$ has at most 2ρ zeros.

Next we show that f has finitely many zeros. Suppose that f has infinitely many zeros ζ_n. Then $\zeta_n \to \infty$. Set $G = z - f/b$. Then $G' = 1 - f'/b$ has at most 2ρ zeros. It follows that G has at most finitely many critical values and hence has at most 2ρ finite asymptotic values. It follows from $G(\zeta_n) = \zeta_n$ and Lemma 3 that, for sufficiently large n,

$$|G'(\zeta_n)| \geq \frac{|G(\zeta_n)| \log |G(\zeta_n)|}{16\pi |\zeta_n|} = \frac{\log |\zeta_n|}{16\pi} \to \infty .$$

However, since $f(z) = 0$ if and only if $f'(z) \in \{1, b\}$ and $f' - b$ has at most 2ρ zeros, for sufficiently large n, $f'(\zeta_n) = 1$ and hence $G'(\zeta_n) = 1 - 1/b$. This is a contradiction.

Thus f has finitely many zeros. By Lemma 1, f is a rational function. Hence Lemma 9 is proved.

LEMMA 10. *Let P be a nonconstant polynomial of degree k, and a and b distinct nonzero finite values. If $P(z) = 0$ if and only if $P'(z)$ is in $\{a, b\}$, then $k \geq 2$ and either $a + (k-1)b = 0$ or $(k-1)a + b = 0$.*

PROOF. Obviously, we have $k \geq 2$. Let

$$A = \frac{PP''}{(P' - a)(P' - b)} . \tag{2.5}$$

Then A ($\not\equiv 0$) is a rational function. Since $P(z) = 0$ if and only if $P'(z) \in \{a, b\}$, we see that A has no pole. So A is a polynomial. Thus $A(P' - a)(P' - b) = PP''$. Comparing the degrees and the coefficients, we see that A is a constant and $A = (k-1)/k$. Thus by (2.5),

$$\frac{(k-1)(a-b)}{k} \cdot \frac{P'}{P} = \frac{aP''}{P' - a} - \frac{bP''}{P' - b} . \tag{2.6}$$

Since $P(z) = 0$ if and only if $P'(z) \in \{a, b\}$ and $ab \neq 0$, all zeros of P are simple. Thus by (2.6), all zeros of $P' - a$ have the same multiplicity, say m, while all zeros of $P' - b$ have the same multiplicity, say n. Let $P' - a$ and $P' - b$ have s and t distinct zeros, respectively. Then

$$sm = tn = k - 1 . \tag{2.7}$$

Since $P(z) = 0$ if and only if $P'(z) \in \{a, b\}$, we have

$$s + t = k . \tag{2.8}$$

By (2.7) and (2.8), we see that s and t are relatively prime. In fact, if s and t have a common divisor q, then by (2.7) and (2.8), q is a common divisor of k and $k - 1$, which shows $q = 1$. Thus by (2.7), $st \leq k - 1$, and hence $s(k - s) \leq k - 1$. It follows that $s \leq 1$ or $s \geq k - 1$. Thus $s = 1$ or $s = k - 1$.

If $s = 1$, then $t = m = k - 1$ and $n = 1$. Thus $P'(z) = a + k(z - z_0)^{k-1}$, and hence $P(z) = az + C + (z - z_0)^k$. Since $P'(z_0) = a$, by the condition, we get $P(z_0) = 0$, and hence

$$P(z) = a(z - z_0) + (z - z_0)^k = (z - z_0)[a + (z - z_0)^{k-1}] . \tag{2.9}$$

Since each zero of $P' - b = a - b + k(z - z_0)^{k-1}$ is a zero of P, by (2.9), we see that $a = (a - b)/k$. That is, $(k-1)a + b = 0$.

Similarly, for the case $s = k - 1$, we can get $a + (k-1)b = 0$. Thus Lemma 10 is proved.

LEMMA 11. *Let R be a non-polynomial rational function, and a and b distinct finite values. If $R(z) = 0$ if and only if $R'(z) \in \{a, b\}$, then $ab \neq 0$ and either $R(z) = a(z - z_0) + d/(z - z_0)^n$ with $b = (n + 1)a$ or $R(z) = b(z - z_0) + d/(z - z_0)^n$ with $a = (n + 1)b$, where d $(\neq 0)$ and z_0 are constants and n is a positive integer.*

PROOF. Let

$$A = \frac{RR''}{(R' - a)(R' - b)} . \tag{2.10}$$

Then A $(\not\equiv 0)$ is a rational function. Since $R(z) = 0$ if and only if $R'(z) \in \{a, b\}$, we see that A has no pole. So A is a polynomial. Next we show that A is a constant.

Since R is a non-polynomial rational function, we have near $z = \infty$

$$R(z) = a_0 z^p [1 + o(1)] ,$$

where a_0 is a nonzero constant and p is an integer. Thus by (2.10),

$$A = \frac{R}{a - b}\left(\frac{R''}{R' - a} - \frac{R''}{R' - b}\right) = \frac{a_0}{a - b} z^p [1 + o(1)] \cdot O\left(\frac{1}{z}\right) = O(z^{p-1}) .$$

Since A $(\not\equiv 0)$ is a polynomial, it follows that $p - 1 \geq 0$ and further $p = 1$ implies that A is a constant.

Assume $p \geq 2$. Then near ∞,

$$R = a_0 z^p [1 + o(1)], \quad R' = p a_0 z^{p-1} [1 + o(1)] , \quad R'' = p(p - 1) a_0 z^{p-2} [1 + o(1)] .$$

Thus by (2.10), we get $A = ((p - 1)/p) [1 + o(1)]$. It follows that A is a nonzero constant with $A = (p - 1)/p$.

Now we prove that a and b are nonzero values. If $ab = 0$, say $a = 0$, and then $b \neq 0$. By (2.10), we see that $R' \neq 0$. In fact, if R' has a zero z_0 of multiplicity $q \geq 1$, then by the assumption that $R(z) = 0$ if and only if $R'(z)$ is in $\{0, b\}$, z_0 is a zero of R of multiplicity $q + 1$, so that z_0 is a zero of A of multiplicity q, which is impossible. Thus, since $R' \neq 0$ and R' is a non-polynomial rational function, the value 0 is attained by $R'(z)$ at $z = \infty$, which contradicts that $R'(z) = p a_0 z^{p-1} [1 + o(1)]$ with $p \geq 1$. This proves that $ab \neq 0$.

Next we prove that at least one of $R' - a$ and $R' - b$ has no zero on $\boldsymbol{C}$.

Suppose this is not the case. Since $R(z) = 0$ if and only if $R'(z) \in \{a, b\}$ and $ab \neq 0$, all zeros of R are simple. By (2.10),

$$A(a - b)\frac{R'}{R} = \frac{aR''}{R' - a} - \frac{bR''}{R' - b} . \tag{2.11}$$

Calculating the residues of both sides of (2.11) at the zeros of $R' - a$ and $R' - b$ yields that all zeros of $R' - a$ have the same multiplicity, say m, while all zeros of $R' - b$ have the same multiplicity, say n, and

$$A(a - b) = am = -bn . \tag{2.12}$$

Thus by (2.11) and (2.12),

$$mn\frac{R'}{R} = n\frac{R''}{R' - a} + m\frac{R''}{R' - b} . \tag{2.13}$$

It follows that

$$R^{mn} = C(R' - a)^n (R' - b)^m , \tag{2.14}$$

where C is a nonzero constant. Since $R = a_0 z^p [1 + o(1)]$ with $p \geq 1$, by (2.14), we see that

$$R^{mn-(m+n)} = C \left(\frac{R' - a}{R} \right)^n \left(\frac{R' - b}{R} \right)^m \to 0 \tag{2.15}$$

as $z \to \infty$. It follows that $mn - (m + n) = (m - 1)(n - 1) - 1 < 0$, so that either $m = 1$ or $n = 1$. Say $m = 1$. Then by (2.14),

$$R^n = C(R' - a)^n (R' - b) . \tag{2.16}$$

Since R is non-polynomial, it has poles in $\boldsymbol{C}$. Let z_0 be a pole of R of multiplicity $t \geq 1$. Then by (2.16), $nt = (n + 1)(t + 1)$. This is impossible.

Thus either $R' - a \not\equiv 0$ or $R' - b \not\equiv 0$.

If $R' - a \not\equiv 0$, then $(R - az)' \not\equiv 0$, so by Lemma 6,

$$R(z) - az = c + \frac{d}{(z - z_0)^n} ,$$

where d ($\neq 0$), c and z_0 are constants, n is a positive integer. Thus

$$R(z) = \frac{(az + c)(z - z_0)^n + d}{(z - z_0)^n} , \quad R'(z) - b = \frac{(a - b)(z - z_0)^{n+1} - nd}{(z - z_0)^{n+1}} . \tag{2.17}$$

Since $R(z) = 0$ if and only if $R'(z)$ is in $\{a, b\}$ and $R' - a \not\equiv 0$, R and $R' - b$ have the same zeros. It follows from (2.17) that $c = -az_0$ and $(a - b)/a = -n$. Thus $b = (n + 1)a$, and

$$R(z) = a(z - z_0) + \frac{d}{(z - z_0)^n} .$$

Similarly, if $R' - b \not\equiv 0$, then $a = (n + 1)b$ and

$$R(z) = b(z - z_0) + \frac{d}{(z - z_0)^n} .$$

Thus Lemma 11 is proved.

3. Proofs of theorems.

PROOF OF THEOREM 3. Suppose that f is nonconstant. Since $ab \neq 0$ and $b/a \in \Omega_0$, we may say $b/a \in \widehat{\Delta}(1, 1)$. It follows that either $|a - b| < |a|$ or $(a - b)^k = a^k$ for some positive integer k.

If f is of finite order, then by Lemma 9, f is a polynomial. Thus by Lemma 10, either $a = -pb$ or $b = -pa$, where $p \in \boldsymbol{N}$. Each case contradicts either $|a - b| < |a|$ or $(a - b)^k = a^k$.

Thus f is of infinite order. It follows from Lemma 4 that for some sequence $z_n \to \infty$, $f^{\#}(z_n) \to \infty$. Thus by Marty's theorem, the family $\{g_n(z) = f(z_n + z)\}$ of entire functions is not normal at $z = 0$. Since $f^{-1}(\{0\}) = (f')^{-1}(\{a, b\})$, we see that $|g_n'(z)| \leq |a| + |b|$ whenever $g_n(z) = 0$. Thus by Lemma 5, there exist points $z_n^* \to 0$, positive numbers $\rho_n \to 0$ and a subsequence of $\{g_n\}$, say $\{g_n\}$ itself w.l.g., such that $g_n(z_n^* + \rho_n \zeta)/\rho_n \to g(\zeta)$ locally uniformly on $\boldsymbol{C}$, where g is a nonconstant entire function satisfying $g^{\#}(\zeta) \le |a| + |b| + 1$. By Lemma 4, g is of finite order. Using Hurwitz's theorem, we can see that $g^{-1}(\{0\}) = (g')^{-1}(\{a, b\})$. Thus by Lemmas 9 and 10, g is constant, which is a contradiction. Theorem 3 is proved.

PROOF OF THEOREM 4. (i) Since $ab = 0$, we may say $a = 1$ and $b = 0$. Now suppose that f is nonconstant. By Lemma 7, f' has finitely many zeros. However, for a transcendental meromorphic function f of order less than one, its derivative has infinitely many zeros [4, Theorem 4 (a)]. So f is a rational function. By Lemma 8, f is not a polynomial. That is, f is a non-polynomial rational function and has the property $f^{-1}(\{0\}) = (f')^{-1}(\{0, 1\})$. By Lemma 11, such a rational function does not exist. Hence Theorem 4 (i) is proved.

Next we prove (ii) and (iii). Since $b/a \in \Omega_0$, we may say $b/a \in \widehat{\Delta}(1, 1)$. Thus either $|a - b| < |a|$ or $(a - b)^k = a^k$ for some positive integer k. Now (ii) follows from Lemmas 9 and 11. To prove (iii), we assume f is nonconstant and consider two cases.

If f is of finite order, then by (ii), we have either $a = pb$ or $b = pa$ for some $p \in \boldsymbol{N}$, which is ruled out by the assumption.

So f is of infinite order. Next, by the same argument used in the proof of Theorem 3, we can get a nonconstant meromorphic function g of finite order which satisfies $g^{-1}(\{0\}) = (g')^{-1}(\{a, b\})$. By the above case, this is impossible. Hence (iii) is also proved.

The proof of Theorem 4 is completed.

PROOF OF THEOREM 1. Suppose that $\mathcal{F}$ is not normal at some point $z_0 \in D$. Then the family $\{g = f - a;\ f \in \mathcal{F}\}$ is also not normal at z_0. Since $f(z)$ is in $\{a, b\}$ if and only if $f'(z)$ is in $\{a, b\}$, we have $|g'(z)| \le |a| + |b|$ whenever $g(z) = 0$. Thus by Lemma 5, there exist points $z_n \to z_0$, positive numbers $\rho_n \to 0$ and functions $g_n = f_n - a$ such that

$$h_n(\zeta) = (\rho_n)^{-1} g_n(z_n + \rho_n \zeta) \to h(\zeta)$$

locally uniformly with respect to the spherical metric, where h is a nonconstant meromorphic function in $\boldsymbol{C}$ such that $h^{\#}(\zeta) \le h^{\#}(0) = |a| + |b| + 1$. In particular, by Lemma 4, h is of order at most two.

We claim that

(i) $h(\zeta) = 0$ if and only if $h'(\zeta)$ is in $\{a, b\}$,

(ii) each pole of h (if exists) is multiple, and

(iii) if h has a pole in $\boldsymbol{C}$, then a/b is a negative rational number.

Proof of (i). Let $h(\zeta_0) = 0$. Then since $h(\zeta) \not\equiv 0$, it follows from Hurwitz's theorem that there exist points $\zeta_n \to \zeta_0$ such that $h_n(\zeta_n) = 0$. It yields that $g_n(z_n + \rho_n \zeta_n) = 0$ and hence $f_n(z_n + \rho_n \zeta_n) = a$. Thus by the condition, $h_n'(\zeta_n) = g_n'(z_n + \rho_n \zeta_n) = f_n'(z_n + \rho_n \zeta_n)$ is in $\{a, b\}$. Since $h_n'(\zeta_n) \to h'(\zeta_0)$, it follows that $h'(\zeta_0)$ is in $\{a, b\}$. This proves that $h(\zeta) = 0$ only if $h'(\zeta)$ is in $\{a, b\}$.

Next, we prove that $h(\zeta) = 0$ if $h'(\zeta)$ is in $\{a, b\}$. Let $h'(\zeta_0)$ be in $\{a, b\}$. We may assume that $h'(\zeta_0) = a$. Since $h^{\#}(0) = |a| + |b| + 1$, we have $h'(\zeta) - a \not\equiv 0$. Thus by Hurwitz's theorem, there exist points $\zeta_n \to \zeta_0$ such that $h_n'(\zeta_n) = a$, so that $g_n'(z_n + \rho_n \zeta_n) = a$ and hence

$f_n'(z_n + \rho_n\zeta_n) = a$. Thus by the condition, $f_n(z_n + \rho_n\zeta_n)$ is in $\{a, b\}$, so that $g_n(z_n + \rho_n\zeta_n)$ is in $\{0, b - a\}$, and hence $h_n(\zeta_n)$ is in $\{0, (\rho_n)^{-1}(b - a)\}$. Since h is holomorphic at ζ_0, it follows that $h(\zeta_0) = 0$. The proof of claim (i) is completed.

Proof of (ii). Let ζ_0 be a pole of h. Then ζ_0 is a zero of $H = 1/h$. Since $H_n = 1/h_n \to 1/h = H$ locally uniformly with respect to the spherical metric, there exists a closed neighborhood $\overline{U}(\zeta_0)$ of ζ_0 on which H and H_n for sufficiently large n are holomorphic, and $H_n \to H$ uniformly. So $H_n - \rho_n/(b - a) \to H$ uniformly. Since $H \not\equiv 0$ and $H(\zeta_0) = 0$, by Hurwitz's theorem, there exist points $\zeta_n \to \zeta_0$ such that $H_n(\zeta_n) - \rho_n/(b - a) = 0$, so that $g_n(z_n+\rho_n\zeta_n) = b-a$ and hence $f_n(z_n+\rho_n\zeta_n) = b$. By the condition, $h_n'(\zeta_n) = f_n'(z_n+\rho_n\zeta_n)$ is in $\{a, b\}$, so that $|h_n'(\zeta_n)| \le |a| + |b|$. Thus by $H_n' = -h_n'/(h_n)^2 = -h_n'(H_n)^2$, we have

$$|H_n'(\zeta_n)| = |h_n'(\zeta_n)||H_n(\zeta_n)|^2 \le \frac{|a| + |b|}{|b - a|^2}(\rho_n)^2 \to 0\,.$$

It follows that $H'(\zeta_0) = 0$. Thus ζ_0 is a multiple zero of H, and hence a multiple pole of h. Thus the claim (ii) is proved.

Proof of (iii). Let ζ_0 be a pole of h. By (ii), it has multiplicity $m \ge 2$. Then as showed in the proof of (ii), ζ_0 is a zero of H of multiplicity m, and hence there exist m points $\zeta_n^{(j)}$, $j = 1, 2, \ldots, m$ such that $\zeta_n^{(j)} \to \zeta_0$ and $H_n(\zeta_n^{(j)}) = \rho_n/(b - a)$. As showed in the proof of (ii), we see that $h_n'(\zeta_n^{(j)})$ is in $\{a, b\}$ and hence

$$H_n'(\zeta_n^{(j)}) = -h_n'(\zeta_n^{(j)})[H_n(\zeta_n^{(j)})]^2 \text{ is in } \left\{-\frac{(\rho_n)^2 a}{(b - a)^2}, -\frac{(\rho_n)^2 b}{(b - a)^2}\right\}.$$

It follows that $\zeta_n^{(j)}$ are distinct and thus are simple zeros of $H_n(\zeta) - \rho_n/(b - a)$. Choosing a subsequence if necessary, we may assume that k points $\{\zeta_n^{(j)}\}$, $1 \le j \le k$, satisfy $H_n'(\zeta_n^{(j)}) = -(\rho_n)^2 a/(b - a)^2$ and the other points $\{\zeta_n^{(j)}\}$, $k + 1 \le j \le m$, satisfy $H_n'(\zeta_n^{(j)}) = -(\rho_n)^2 b/(b - a)^2$, where k is independent of n. Thus by Cauchy's residue theorem,

$$\begin{aligned}
\operatorname{Res}\left(\frac{1}{H}, \zeta_0\right) &= \lim_{n\to\infty} \sum_{j=1}^{m} \operatorname{Res}\left(\frac{1}{H_n - \rho_n/(b - a)}, \zeta_n^{(j)}\right) \\
&= \lim_{n\to\infty} \sum_{j=1}^{m} \frac{1}{H_n'(\zeta_n^{(j)})} \\
&= \lim_{n\to\infty} \left(\frac{k}{-(\rho_n)^2 a/(b - a)^2} + \frac{m - k}{-(\rho_n)^2 b/(b - a)^2}\right) \\
&= \lim_{n\to\infty} -\frac{(b - a)^2}{(\rho_n)^2}\left(\frac{k}{a} + \frac{m - k}{b}\right).
\end{aligned}$$

It follows that $k/a + (m - k)/b = 0$, so that a/b is a negative rational number and the residue of h at ζ_0 is zero. Thus (iii) is also proved.

Since negative rational numbers do not lie in Ω_0, so by (iii), h is an entire function. Thus by Theorem 3, h is constant, which is a contradiction.

The proof of Theorem 1 is completed.

4. Examples and remarks.

a. Related to Theorem 4, we propose the following conjecture.

CONJECTURE 1. *Let a, b be distinct non-zero finite values such that $b/a \notin \boldsymbol{Q}^-$, where $\boldsymbol{Q}^-$ stands for the set of negative rational numbers. Then there are no transcendental meromorphic functions satisfying $f^{-1}(\{0\}) = (f')^{-1}(\{a, b\})$.*

The following examples show that if Conjecture 1 is true, then it is sharp in general.

EXAMPLE 2. The function $f(z) = \sin z$ satisfies $f^{-1}(\{0\}) = (f')^{-1}(\{-1, 1\})$.

EXAMPLE 3. Let $\wp$ be the Weierstrass doubly periodic function satisfying $(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4\wp^3 - 12\wp$, and let $f(z) = \wp'(z/3)/2$. Then f satisfies the differential equation $f^4 = (f' - 2)^2(f' + 1)$, and so $f^{-1}(\{0\}) = (f')^{-1}(\{-1, 2\})$.

EXAMPLE 4. Let $\wp$ be the Weierstrass doubly periodic function satisfying $(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4\wp^3 + 1$, and let $f(z) = \wp'(z/2)/\wp^2(z/2)$. Then f satisfies the equation $f^6 = (f' - 3)^3(f' + 1)$, and so $f^{-1}(\{0\}) = (f')^{-1}(\{-1, 3\})$.

EXAMPLE 5. Let $\wp$ be the Weierstrass doubly periodic function satisfying $(\wp')^2 = 4\wp^3 - g_2\wp - g_3 = 4\wp^3 - 20$, and let $f(z) = \wp(z/5)\wp'(z/5)/2$. Then f satisfies the differential equation $f^6 = (f' - 3)^3(f' + 2)^2$, and so $f^{-1}(\{0\}) = (f')^{-1}(\{-2, 3\})$.

b. Theorem B is not true in general if the 3-element set is replaced by a 2-element set, as shown by Example 1 and the following examples.

EXAMPLE 6. Let a be a nonzero value. For every positive integer n, let

$$f_n(z) = \frac{a}{2}\left(1 + \frac{1}{n}\right)e^{nz} + \frac{a}{2}\left(1 - \frac{1}{n}\right)e^{-nz}.$$

Then one can verify that $n^2[(f_n)^2 - a^2] = (f_n')^2 - a^2$, so that $f_n^{-1}(\{-a, a\}) = (f_n')^{-1}(\{-a, a\})$. But $\{f_n\}$ is not normal at the origin.

EXAMPLE 7. Let b be a nonzero value. For every $n \in \boldsymbol{N}$, let

$$f_n(z) = \frac{b(n + 2 - \sqrt{n^2 + 4})}{2n} - \frac{b(\sqrt{n^2 + 4} - 2)}{n(e^{nz} - 1)}.$$

Then we have

$$f_n'(z) = \frac{b(\sqrt{n^2 + 4} - 2)e^{nz}}{(e^{nz} - 1)^2} \neq 0,$$

and

$$b[f_n'(z) - b] = (\sqrt{n^2 + 4} + 2)f_n(z)[f_n(z) - b].$$

It follows that $f_n^{-1}(\{0, b\}) = (f_n')^{-1}(\{0, b\})$. However, the family $\{f_n\}$ is not normal at the origin.

EXAMPLE 8. For every $n \in \boldsymbol{N}$, let h_n be a doubly periodic function defined by the following differential equation

$$(h_n')^2 = \frac{1}{4n}[(h_n)^4 - 4h_n] + 1\,, \quad h_n(0) = \sqrt{n}\,, \tag{4.1}$$

and let

$$f_n(z) = 1 + 2h_n'(nz)\,.$$

Then we have

$$n^3(f_n + 1)^3(f_n - 3)^3 = (f_n' + 1)(f_n' - 3)^3\,.$$

Thus all poles of f_n are double and $f_n^{-1}(\{-1, 3\}) = (f_n')^{-1}(\{-1, 3\})$.

Next we show that the family $\{f_n\}$ is not normal at the origin. By (4.1), we see that

$$|h_n'(0)| = \sqrt{\frac{n^2 - 4\sqrt{n}}{4n} + 1} \le \frac{\sqrt{n+4}}{2}$$

and hence $|f_n(0)| \le 1 + \sqrt{n+4}$. Also by (4.1), $h_n'' = [(h_n)^3 - 1]/2n$, so that $|f_n'(0)| = 2n|h_n''(0)| = \sqrt{n^3} - 1$. Thus we have

$$f_n^{\#}(0) = \frac{|f_n'(0)|}{1 + |f_n(0)|^2} \ge \frac{\sqrt{n^3} - 1}{1 + (1 + \sqrt{n+4})^2} \to \infty\,.$$

It follows from Marty's theorem that $\{f_n\}$ is not normal at the origin.

c. By the similar argument of proving [7, Theorem B], we can prove the following result.

THEOREM 5. *Let $\mathcal{F}$ be a family of meromorphic functions in D, $k \ge 2$ an integer and a, b two distinct nonzero complex values. If for every $f \in \mathcal{F}$, all poles of f have multiplicity at least k, and $f^{-1}(\{a, b\}) = (f')^{-1}(\{a, b\})$, then $\mathcal{F}$ is normal in D, provided that k, a, b satisfy one of the following conditions*:

(i) $k \ge 4$ *and* $a + b \ne 0$,

(ii) $k \ge 3$ *and* $(a + b)(a + 2b)(2a + b) \ne 0$, *and*

(iii) $k \ge 2$ *and* $(a + b)(a + 2b)(2a + b)(a + 3b)(3a + b) \ne 0$.

The above Examples 1 and 6 through 8 show that the conditions of Theorem 5 about a and b are sharp. In view of Theorems 1 and 5, we propose the following conjecture.

CONJECTURE 2. *Let $\mathcal{F}$ be a family of functions meromorphic in D, and a, b two distinct nonzero finite values. If $f^{-1}(\{a, b\}) = (f')^{-1}(\{a, b\})$ for every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D, so long as $b/a \notin \boldsymbol{Q}^-$.*

Acknowledgment. The authors are grateful to the referee for his valuable suggestions and comments, which help a lot in improving the paper.

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