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Sharp a priori estimates for a class of nonlinear elliptic equations with lower order terms

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Abstract We prove sharp a priori estimates for second-order quasi-linear elliptic operators in divergence form with a first-order term. Such estimates are the first step of a standard procedure which allows to prove existence results for Dirichlet problems related to these operators.

Keywords A priori estimates · Existence · Nonlinear elliptic operators

Mathematics Subject Classification (2000) 35J25 · 35J60

1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, $N \geq 2$. We consider the model problem

$$\begin{cases} -\Delta_p u = G(x, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Delta_p u = \text{div}\,(|\nabla u|^{p-2}\nabla u)$ denotes the p-laplacian operator, $1 < p < N$, and $G(x, \xi)$ is a function which, for $p - 1 < q \leq p$, satisfies the growth condition

$$|G(x, \xi)| \leq \beta|\xi|^q + f(x) \tag{1.2}$$

with β a positive constant and f a summable function.

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Our aim was to study problems in the form (1.1) when the exponent q in (1.2) varies in the given interval, and we are mainly interested to find sharp conditions on f which guarantee the existence of a solution.

Problems like (1.1) have been extensively studied in the last years, and before describing the main results of the paper, we discuss briefly some of the known results. Firstly, we comment on the range given on q. In the case $0 \le q \le p-1$, the main questions appear to be solved (see, for instance, [4,12,13] and the references therein). On the other hand, the case $q > p$ requires a different approach, and it appears to be not completely understood (see, for instance, [19] and the references therein).

So, we will confine ourselves to the case $p-1 < q \le p$. The first results in the literature concern the limit case $q = p$, and they are mainly devoted to find bounded solutions to problem (1.1). In this context, it has been pointed out that if suitable sign conditions on G are made, then the existence of a bounded weak solution can be proved when $f \in L^r(\Omega)$, $r > N/p$ (see, for instance, [18] and the references therein). On the other hand, if no sign conditions are assumed, then a smallness hypothesis on f is required in order to have a bounded solution to problem (1.1) (see, for instance, [27,36,40,41]). We remark that the qualitative results described above can be immediately stated for the full range $p-1 < q \le p$ because if inequality (1.2) holds true, then a similar one with q substituted by p holds true. So, many papers have investigated the case $p-1 < q \le p$, without sign conditions, looking for sharp conditions to be put on f in order to get existence for problem (1.1). Such conditions have involved both summability and smallness assumptions on f.

A first result in this direction is contained in [28], where the existence of a solution which is not necessarily bounded has been investigated, in the case $q = p$. It is proven that there exists a constant $C(\beta, N, p)$ such that, if $f \in L^{N/p}(\Omega)$, with

$$\|f\|_{L^{N/p}} < C(\beta, N, p), \tag{1.3}$$

then a weak solution u to problem (1.1) exists such that $\exp(\frac{\beta}{p-1}|u|) - 1 \in W_0^{1,p}(\Omega)$. It is also shown that the constant $C(\beta, N, p)$ is sharp, in the sense that if (1.3) is not satisfied, then a problem in the form (1.1) can be exhibited such that it does not have a solution which satisfies the condition $\exp(\frac{\beta}{p-1}|u|) - 1 \in W_0^{1,p}(\Omega)$. Similar results in the case $p-1 < q \le p$ have been proved in [14,15,20,21,25,26,29,32]. In particular, in [29], an existence result for problem (1.1) is given under a sharp smallness condition on the norm of f in $L^\infty(\Omega)$, while in [32] the existence of a solution u such that a suitable power of it belongs to $W_0^{1,p}(\Omega)$ is proved under a smallness condition on the norm of f in $L^\gamma(\Omega)$, with $\gamma = \max\{1, \frac{N(q-p+1)}{q}\}$. Finally, in [30] (see also [15]), it is considered the case $q = p$ and f belonging to the Marcinkiewicz space $M^{N/p}(\Omega)$. Again, the existence of a solution u to problem (1.1) such that $\exp(\frac{\beta}{p-1}|u|) - 1 \in W_0^{1,p}(\Omega)$ is proven under a sharp smallness condition on the norm of f in $M^\gamma(\Omega)$.

Our aim was to consider the general case $p-1 < q \le p$, looking for sharp hypotheses to be put on the function f in (1.2) in order to have a solution to problem (1.1). More precisely, we consider a problem in the form

$$\begin{cases} -\text{div}\,(A(x,u,\nabla u)) = H(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

where

$$A : (x,s,z) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow A(x,s,z) \in \mathbb{R}^N$$
$$H : (x,s,z) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow H(x,s,z) \in \mathbb{R}$$

are Carathéodory functions which satisfy the ellipticity condition

$$A(x, s, z) \cdot z \geq |z|^p \,, \tag{1.5}$$

the monotonicity condition

$$(A(x, s, z) - A(x, s, z')) \cdot (z - z') > 0 \,, \quad z \neq z' \tag{1.6}$$

and the growth conditions

$$|A(x, s, z)| \leq a_0 |z|^{p-1} + a_1 |s|^{p-1} + a_2 \,, \quad a_0, a_1, a_2 > 0 \,, \tag{1.7}$$

$$H(x, s, z) \operatorname{sign}(s) \leq \beta |z|^q + f(x) \,, \quad \beta > 0, \tag{1.8}$$

with $1 < p < N$, $p-1 < q \leq p$, for a.e. $x \in \mathbb{R}^N$, for every $s \in \mathbb{R}$, and for every $z, z' \in \mathbb{R}^N$. In order to describe the type of result one can prove, we have to consider three different cases

(a) $p - 1 < q < \frac{N(p-1)}{N-1}$;
(b) $\frac{N(p-1)}{N-1} < q \leq p$;
(c) $q = \frac{N(p-1)}{N-1}$.

In case *(a)* we assume that $f \in L^1(\Omega)$, and we obtain results similar to those given in [32], the main difference being the sharpness on the smallness condition we assume on the norm of f. We refer the reader to Sect. 5 for the precise statements in this case.

Here, we illustrate in more details only the case *(b)*, and we refer the reader to Sect. 5 for the precise statements in the limit case *(c)* which has been partially treated in [32] and for which the introduction of suitable Lorentz–Zygmund spaces is necessary.

Consider problem (1.4) under assumptions (1.5)–(1.8), with $\frac{N(p-1)}{N-1} < q < p$ (for the case $q = p$ a similar result holds true) and $f \in M^\gamma(\Omega)$, $\gamma = \frac{N(q-p+1)}{q}$. Suppose

$$\|f\|_{M^\gamma} < \frac{\gamma}{N} \left(\frac{N\omega_N^{1/N}}{\gamma'} \right)^{\frac{N}{\gamma}} \left(\frac{p-1}{\beta q} \right)^{\frac{N(p-1)}{\gamma q}} ,$$

where ω_N denotes the measure of the unit ball in $\mathbb{R}^N$. Then, a solution to problem (1.4) exists such that

$$\|u\|_{M^{\frac{q\gamma}{p-q}}} \leq \frac{X_0^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}}} \frac{q\gamma}{p-q} \,,$$

where $X_0 \geq 0$ is the smallest nonnegative solution to the equation

$$\frac{\beta}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} X^{\frac{q}{p-1}} - \frac{X}{\gamma'} + \|f\|_{M^\gamma} = 0.$$

Let us comment on the notion of solution we use. We will prove that there exists a solution to problem (1.4) which is limit of approximations ([22,24]). This means that a solution to problem (1.4) is a measurable function u such that:

1. $T_k(u) \in W_0^{1,p}(\Omega)$, for every $k > 0$, where $T_k(s) = \min\{|s|, k\}\operatorname{sign}(s)$ denotes the usual truncation function;
2. u is a solution in the sense of distribution to problem (1.4), i.e.,

$$\int_\Omega A(x, u, \nabla u) \cdot \nabla \phi \, dx = \int_\Omega H(x, u, \nabla u) \phi \, dx \,,$$

for every $\phi \in C_0^\infty(\Omega)$;

3. u is the a.e. limit of a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of weak solutions to the approximated problem

$$\begin{cases} -\text{div}\,(A(x, u_n, \nabla u_n)) = T_n(H(x, u_n, \nabla u_n)) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Observe that, since the right-hand side in the Eq. (1.9) is bounded, in view of (1.5)–(1.7), a classical result (see [37,39]) implies the existence of a bounded weak solution to problem (1.9).

In order to prove the existence of a solution to problem (1.4), we use a standard strategy (see, for instance, [9,23] and Sect. 5 below). Firstly, one proves that any bounded weak solution to problem (1.9) satisfies suitable a priori estimates. Making use of such estimates one can prove that, up to subsequence, $\{u_n\}_{n\in\mathbb{N}}$ converges to a measurable function u, in such a way that it is possible to pass to the limit in the weak formulation of (1.9), yielding that u satisfies (1.4) in the sense of distribution.

We recall that equivalent notions of solution are the renormalized solution ([23,38]) and the entropy solution ([9]).

The method we use in order to obtain the described result is based on the fact that, using also symmetrization techniques (see, for instance, [47,48]), it is possible to obtain sharp a priori estimates for bounded solutions to approximated problems in the form (1.9). These estimates are obtained under sharp smallness assumption on f given above. The use of the estimates on the approximate solution and on its gradient allows us to pass to the limit as described above.

The paper is organized as follows. In Sect. 2, some preliminary results on rearrangements and the definition of some Lorentz–Zygmund spaces are recalled. In Sect. 3, some model radial problems satisfying assumptions (1.5)–(1.8) are considered. Such examples shed light on the sharpness of the existence results obtained in the paper, as regards both the smallness assumption on f and the estimate on the norm of the solution. In Sect. 4, we prove the main pointwise estimates for bounded solutions to approximated problems, and in Sect. 5, we prove the existence results. Both Sects. 4 and 5 are split in subsections where the different cases *(a)*, *(b)*, *(c)*, described above, are considered.
Part of these results have been announced in [5] and [43].

2 Preliminary results

We begin by recalling some properties of rearrangements. If u is a measurable function defined in Ω and

$$\mu(t) = |\{x \in \Omega : |u(x)| \geq t\}|, \quad t \geq 0$$

is its distribution function, then

$$u^*(s) = \sup\{t \geq 0 : \mu(t) > s\}, \quad s \in (0, |\Omega|),$$

is the decreasing rearrangement of u and $u_*(s) = u^*(|\Omega| - s)$ is the increasing rearrangement of u.

If ω_N is the measure of the unit ball of $\mathbb{R}^N$ and $\Omega^\sharp$ is the ball of $\mathbb{R}^N$ centered at the origin with the same measure as Ω,

$$u^\sharp(x) = u^*(\omega_N |x|^N), \; u_\sharp(x) = u_*(\omega_N |x|^N), \qquad x \in \Omega^\sharp,$$

denote the spherically decreasing and increasing rearrangements of u, respectively. We recall the well-known Hardy-Littlewood inequality ([34])

$$\int_{\Omega^\sharp} u^\sharp(x)\, v_\sharp(x)\, dx \le \int_\Omega |u(x)\, v(x)|\, dx \le \int_{\Omega^\sharp} u^\sharp(x)\, v^\sharp(x)\, dx. \tag{2.1}$$

For any $q \in (0, +\infty)$, the Lorentz space $L^{q,r}(\Omega)$ is the collection of all measurable funtions u such that $\|u\|_{q,r}$ is finite, where we use the notation

$$\|u\|_{L^{q,r}} = \left(\int_0^{+\infty} \left[u^*(s)\, s^{1/q}\right]^r \frac{ds}{s} \right)^{1/r}$$

if $r \in]0, \infty[$,

$$\|u\|_{L^{q,\infty}} = \sup_{s>0} u^*(s)\, s^{1/q} = \sup_{t>0} t\, \mu(t)^{1/q} \tag{2.2}$$

if $r = \infty$.

These spaces give in some sense a refinement of the usual Lebesgue spaces. Indeed, $L^{q,q}(\Omega) = L^q(\Omega)$ and $L^{q,\infty}(\Omega) = M^q(\Omega)$ is the Marcinkiewicz space L^q-weak. The following embeddings hold true (see [35,45])

$$L^{q,r_1}(\Omega) \subset L^{q,r_2}(\Omega), \tag{2.3}$$

if $r_1 < r_2$, and

$$L^{q_1,r}(\Omega) \subset L^q(\Omega), \tag{2.4}$$

if $q < q_1$.

We finally recall some Zygmund spaces which will be used in what follows (see [10]). The Zygmund space $L(\log L)^{N-1}$ consists of all measurable functions u such that

$$\int_0^{|\Omega|} u^*(s) \log^{N-1}\left(\frac{\mathcal{M}}{s}\right) ds < +\infty,$$

for a constant $\mathcal{M} > |\Omega|$, while $L^{1,\infty}(\log L)^N$ denotes the Lorentz–Zygmund space which is the collection of the measurable functions u for which

$$\sup_{0<s<|\Omega|} s \log^N\left(\frac{\mathcal{M}}{s}\right) u^*(s) < +\infty,$$

for some constant $\mathcal{M} > |\Omega|$.

We explicitly recall that both these spaces are included in $L^1(\Omega)$ and both of them contain the Marcinkiewicz space $M^t(\Omega)$ with $t > 1$. On the other hand, these spaces are not subset one of the other; examples of functions belonging to one and not to the other could be exhibited.

3 Some remarks in the radial case

In this section, we consider radial solutions to some problems in a ball which, as we will see, are in some sense extremal cases as regards the existence of solutions. We start with the case $p - 1 < q < \frac{N(p-1)}{N-1}$, and we consider the following Dirichlet problem

$$\begin{cases} -\Delta_p v = \beta|\nabla v|^q + K\delta_0 & \text{in } B_R \\ v = 0 & \text{on } \partial B_R\,, \end{cases} \tag{3.1}$$

where β and K are positive constants, δ_0 is the Dirac mass centered at the origin, and B_R is the ball centered at the origin with radius R.

A direct computation proves that if

$$\kappa > \kappa_0 \equiv \frac{\beta(q-p+1)}{N(p-1)-q(N-1)} \tag{3.2}$$

with

$$\kappa = \left(\frac{N\omega_N}{K}\right)^{\frac{q-p+1}{p-1}} R^{-\frac{N(p-1)-q(N-1)}{p-1}} \tag{3.3}$$

the radial function

$$\phi_\kappa(x) = R^{\frac{q-p}{q-p+1}} \int_{\frac{|x|}{R}}^{1} \frac{1}{t^{\frac{N-1}{p-1}}\left(\kappa - \kappa_0 t^{\frac{N(p-1)-q(N-1)}{p-1}}\right)^{\frac{1}{q-p+1}}}\,dt. \tag{3.4}$$

is a renormalized solution to (3.1) (see, e.g., [23]). In other words, combining (3.2) and (3.3), we can say that the function ϕ_κ in (3.4) is a renormalized solution to (3.1) under the smallness assumption on the datum K

$$K < K_1 \equiv N\omega_N \left(\frac{N(p-1)-q(N-1)}{\beta(q-p+1)}\right)^{\frac{p-1}{q-p+1}} R^{-\frac{N(p-1)-q(N-1)}{q-p+1}}. \tag{3.5}$$

The smallness assumption (3.5) is sharp in the sense that for every $K' > K_1$, it is possible to find a problem in the form

$$\begin{cases} -\Delta_p v = \beta|\nabla v|^q + f & \text{in } B_R \\ v = 0 & \text{on } \partial B_R\,, \end{cases} \tag{3.6}$$

where f is a bounded function with $\|f\|_{L^1} = K'$ for which no bounded weak solution exists, as shown by the following example.

Example 3.1 For the sake of simplicity we consider the case $N = 3,\ p = 2,\ q = 5/4,\ \beta = 1,\ R = 1$, but the calculation could be carried on in a similar way for all the values of the parameters under the required conditions. It is immediate to observe that the condition $p - 1 < q < \frac{N(p-1)}{N-1}$ is satisfied and that $K_1 = 64\pi$ in the smallness condition (3.5). For $n \in \mathbb{N}$, we put

$$f_n(x) = \begin{cases} 0 & \dfrac{1}{n} < |x| \le 1 \\ \dfrac{96\, n^6 |x|^3}{(1 - n^{3/2}|x|^2)^5} & 0 \le |x| \le \dfrac{1}{n}. \end{cases} \tag{3.7}$$

The function f_n is bounded and a direct computation gives

$$\|f_n\|_{L^1} = \frac{16\pi(4 - n^{-1/2})}{(1 - n^{-1/2})^4}. \tag{3.8}$$

Then

$$\|f_n\|_{L^1} > 64\pi, \qquad \lim_{n\to+\infty} \|f_n\|_{L^1} = 64\pi. \tag{3.9}$$

On the other hand, using the results contained in [29, Sect. 3], it follows that a nonnegative bounded weak solution to problem

$$\begin{cases} -\Delta v = |\nabla v|^{5/4} + f_n & \text{in } B_1 \\ v = 0 & \text{on } \partial B_1 , \end{cases} \tag{3.10}$$

exists if and only if the problem

$$\begin{cases} -\Delta_5 z = \dfrac{f_n}{4^4}(z+1)^4 & \text{in } B_1 \\ z = 0 & \text{on } \partial B_1 , \end{cases} \tag{3.11}$$

admits a nonnegative weak solution. We observe also that the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_5 w = \lambda \dfrac{f_n}{4^4}|w|^3 w & \text{in } B_1 \\ w = 0 & \text{on } \partial B_1 , \end{cases} \tag{3.12}$$

admits the first eigenvalue $\lambda_1 = 1$ with eigenfunction

$$w(x) = \begin{cases} 1 - |x|^{1/2} & \dfrac{1}{n} < |x| \le 1 \\ (1 - n^{3/2}|x|^2)^{1/4}(1 - n^{-1/2})^{3/4} & 0 \le |x| \le \dfrac{1}{n}, \end{cases}$$

and using Theorem 2.4 in [2] it is possible to show that problem (3.11) does not have a weak solution. This implies that also problem (3.10) cannot have a weak solution.

When $\frac{N(p-1)}{N-1} < q \le p$ we consider the problem

$$\begin{cases} -\Delta_p v = \beta|\nabla v|^q + f & \text{in } B_R \\ v = 0 & \text{on } \partial B_R , \end{cases} \tag{3.13}$$

where

$$f(x) = \frac{K}{\omega_N^{\frac{1}{\gamma}}|x|^{\frac{N}{\gamma}}} \tag{3.14}$$

with $\gamma = N(q-p+1)/q$ and $K \ge 0$. Looking for solutions to (3.13), it is not difficult to see that the radial solutions $v = v(|x|)$ are such that

$$v'(r) = -\left(\frac{X}{N\omega_N^{\frac{1}{\gamma}}}\right)^{\frac{1}{p-1}} \frac{1}{r^{\frac{N}{q\gamma}}}, \tag{3.15}$$

where X is a nonnegative constant which satisfies the following equation

$$\frac{\beta}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} X^{\frac{q}{p-1}} - \frac{X}{\gamma'} + \|f\|_{M^\gamma} = 0. \tag{3.16}$$

The above equation admits at least one nonnegative solution if and only if $\|f\|_{M^\gamma}$ satisfies the following smallness assumption

$$\|f\|_{M^\gamma} \le K_2 \equiv \frac{\gamma}{N}\left(\frac{N\omega_N^{1/N}}{\gamma'}\right)^{\frac{N}{\gamma}} \left(\frac{p-1}{\beta q}\right)^{\frac{N(p-1)}{\gamma q}}. \tag{3.17}$$

In conclusion, using (3.15) it is possible to verify that, if

$$\psi(x)=\begin{cases}\dfrac{q-p+1}{p-q}\left(|x|^{-\frac{p-q}{q-p+1}}-R^{-\frac{p-q}{q-p+1}}\right) & \text{if } \frac{N(p-1)}{N-1}<q<p\\ \log\left(\dfrac{R}{|x|}\right) & \text{if } q=p,\end{cases} \tag{3.18}$$

the function

$$v(x)=\left(\frac{X}{N\omega_N^{\frac{1}{\gamma}}}\right)^{\frac{1}{p-1}}\psi(x), \tag{3.19}$$

is a weak solution to problem (3.13) under the condition (3.17).

The smallness assumption (3.17) is sharp in the sense that for every $\varepsilon>0$, it is possible to find a problem in the form (3.6) where f is a bounded function with $\|f\|_{M^\gamma}=K'>K_2$ and $K'-K_2<\varepsilon$, for which no bounded weak solution exists, as shown by the following example.

Example 3.2 For the sake of simplicity we consider the case $\beta=1,\ R=1$, with $p,\ q$ and N satisfying $\frac{N(p-1)}{N-1}<q\le p$, but the calculation could be carried on in a similar way for all the values of the parameters under the required conditions. In the present case, if $\gamma=N(q-p+1)/q$, we have

$$K_2\equiv\frac{\gamma}{N}\left(\frac{N\omega_N^{1/N}}{\gamma'}\right)^{\frac{N}{\gamma}}\left(\frac{p-1}{q}\right)^{\frac{N(p-1)}{\gamma q}}$$

in the smallness condition (3.17).

For $n\in\mathbb{N}$, we put

$$f_n(x)=\frac{K'}{\omega_N^{\frac{1}{\gamma}}}g_n(x),\qquad \text{with}\quad g_n(x)=\min\{|x|^{-\frac{N}{\gamma}},n\}, \tag{3.20}$$

where $K'>K_2$. The function f_n is bounded and a direct computation gives

$$\|f_n\|_{M^\gamma}=K'. \tag{3.21}$$

On the other hand, using the results contained in [29, Sect. 3], it follows that a nonnegative bounded weak solution to problem

$$\begin{cases}-\Delta_p v=|\nabla v|^q+f_n & \text{in } B_1\\ v=0 & \text{on } \partial B_1,\end{cases} \tag{3.22}$$

exists if and only if the problem

$$\begin{cases}-\Delta_m z=\left(\dfrac{q-p+1}{p-1}\right)^{\frac{p-1}{q-p+1}}\dfrac{K'}{\omega_N^{1/\gamma}}g_n(z+1)^{m-1} & \text{in } B_1\\ z=0 & \text{on } \partial B_1,\end{cases} \tag{3.23}$$

where $m=\frac{q}{q-p+1}$, admits a nonnegative weak solution. We observe that the given conditions on q imply $p\le m<N$.

Let us consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_m w = \lambda\, g_n |w|^{m-2} w & \text{in } B_1 \\ w = 0 & \text{on } \partial B_1. \end{cases} \tag{3.24}$$

Using, for example, Theorem 2.4 in [31], the first eigenvalue $\lambda_1(n)$ for problem (3.24) is such that $\lambda_1(n)$ is strictly decreasing with respect to n and

$$\lim_{n\to+\infty} \lambda_1(n) = \left(\frac{(N-1)q - N(p-1)}{q}\right)^{\frac{q}{q-p+1}}. \tag{3.25}$$

This means that for every n we can choose

$$K' = \lambda_1(n) \left(\frac{p-1}{q-p+1}\right)^{\frac{p-1}{q-p+1}} \omega_N^{1/\gamma}$$

and using Theorem 2.4 in [2] it is possible to show that problem (3.23) does not have a weak solution. This implies that also problem (3.22) cannot have a weak solution. We finally observe that, in view of the monotonicity of $\lambda_1(n)$, $K' > K_2$ and, in view of (3.25), K' can be arbitrarily close to K_2.

When $\frac{N(p-1)}{N-1} < q \le p$ we also observe that under the smallness assumption (3.17) satisfied as a strict inequality, that is, $\|f\|_{M^\gamma} < K_2$, problem (3.13) admits two distributional solutions in the form (3.19). Indeed, in such a case, Eq. (3.16) admits two nonnegative solutions $0 \le X_0 < X_1$ and the functions

$$v^{(i)}(x) = \left(\frac{X_i}{N\omega_N^{\frac{1}{\gamma}}}\right)^{\frac{1}{p-1}} \psi(x) \qquad i = 0, 1, \tag{3.26}$$

are distributional solutions to problem (3.13).

A difference between $v^{(0)}$ and $v^{(1)}$ can be explained as follows. A classical procedure to find a solution to problem (3.13) consists in building an approximate problem with the source term which is a truncation f_n of f in (3.14) for which a bounded weak solution v_n exists. Such a sequence of solutions converges to $v^{(0)}$ and not to $v^{(1)}$. This phenomenon is made explicit in the following example.

Example 3.3 We consider problem (3.13) in the case $N = 3$, $p = q = 2$, $\beta = 1$, $R = 1$, $K > 0$. In the present case $\gamma = 3/2$ and

$$K_2 = \left(\frac{\pi}{6}\right)^{\frac{2}{3}}$$

in the smallness condition (3.17). Furthermore, $\|f\|_{M^{3/2}} = K$, condition $\|f\|_{M^{3/2}} < K_2$ becomes

$$K < \left(\frac{\pi}{6}\right)^{\frac{2}{3}}, \tag{3.27}$$

and $X_i = (4\sqrt{3}\pi)^{2/3} y_i$, $i = 0, 1$, where $y_0 = \frac{1-\sqrt{1-4c}}{2}$, $y_1 = \frac{1+\sqrt{1-4c}}{2}$ and $c = K\left(\frac{3}{4\pi}\right)^{\frac{2}{3}}$.

For $n \in \mathbb{N}$, we put

$$f_n(x) = K\left(\frac{3}{4\pi}\right)^{\frac{2}{3}} \min\{|x|^{-2}, n^2\}, \tag{3.28}$$

where K satisfies (3.27). The function f_n is bounded and a direct computation gives

$$\|f_n\|_{M^{3/2}} = K. \tag{3.29}$$

On the other hand, using the results contained in [29, Sect. 3], a direct computation proves that the only radial bounded weak solution to problem

$$\begin{cases} -\Delta v_n = |\nabla v_n|^2 + f_n & \text{in } B_1 \\ v_n = 0 & \text{on } \partial B_1 , \end{cases} \tag{3.30}$$

is given by

$$v_n(x) = \begin{cases} \log\left(\dfrac{\sin(n\sqrt{c}|x|)}{|x|}\right) + M_n & 0 \le |x| \le \dfrac{1}{n} \\ \log\left(\dfrac{A_n}{|x|^{y_0}} + \dfrac{1-A_n}{|x|^{y_1}}\right) & \dfrac{1}{n} < |x| \le 1 , \end{cases} \tag{3.31}$$

where

$$A_n = \frac{\sqrt{c} - y_0 \tan\sqrt{c}}{\sqrt{c} - y_0 \tan\sqrt{c} - n^{-\sqrt{1-4c}}(\sqrt{c} - y_1 \tan\sqrt{c})}$$

and

$$M_n = \log\left(\frac{\sqrt{1-4c}}{n^{y_1}\cos\sqrt{c}(\sqrt{c} - y_0\tan\sqrt{c} - n^{-\sqrt{1-4c}}(\sqrt{c} - y_1\tan\sqrt{c}))}\right).$$

It is immediate to observe that $0 \le v_n(x) \le v^{(0)}(x)$ and v_n converges to $v^{(0)}$ with

$$v^{(0)}(x) = y_0 \log\left(\frac{1}{|x|}\right).$$

It is clear that also the second solution $v^{(1)}(x) = y_1 \log\left(\frac{1}{|x|}\right)$ can be approximated by a sequence of bounded weak solutions to problems in the form (3.30), but the norm of the approximated source term will not be smaller than K_2. Indeed, if $\bar{c}$ is the solution of $\sqrt{\bar{c}} = y_0 \tan\sqrt{\bar{c}}$ in the interval $(0, \pi^2/4)$, the function

$$\bar{v}_n(x) = \begin{cases} \log\left(\dfrac{\sin(n\sqrt{\bar{c}}|x|)}{n^{y_0}|x| \sin\sqrt{\bar{c}}}\right) & 0 \le |x| \le \dfrac{1}{n} \\ \log\left(\dfrac{1}{|x|^{y_1}}\right) & \dfrac{1}{n} < |x| \le 1, \end{cases} \tag{3.32}$$

is the bounded weak solution to problem (3.30) where the source term f_n is replaced by

$$\bar{f}_n(x) = \begin{cases} \bar{c}\, n^2 & 0 \le |x| \le \dfrac{1}{n} \\ \left(\dfrac{3}{4\pi}\right)^{\frac{2}{3}} \dfrac{K}{|x|^2} & \dfrac{1}{n} < |x| \le 1. \end{cases} \tag{3.33}$$

Using the definition of $\bar{c}$ and recalling that $y_0 < 1/2$, it is immediate to observe that $\bar{c} > 1/4$ and then

$$\|\bar{f}_n\|_{M^{3/2}} = \bar{c}\left(\frac{4\pi}{3}\right)^{\frac{2}{3}} > \left(\frac{\pi}{6}\right)^{\frac{2}{3}} = K_2.$$

We conclude this section observing that in the limit case $q = \frac{N(p-1)}{N-1}$ one could consider problems in the form (3.13) when f belongs to spaces like $L^{1,\infty}(\log L)^N$ or $L(\log L)^{N-1}$. Also, in such cases, one can exhibit solutions which are obtained as limit of bounded solutions to approximate problems under sharp smallness assumptions on the source terms. The sharpness of the smallness assumptions could be established via examples similar to those discussed above.

4 Pointwise estimates

One of the main aims of symmetrization approach to the study of properties of solutions to differential problems is to give comparison results for such solutions in terms of the solution to a "symmetrized problem" which is in the same form as the original one. Several papers have addressed the question of giving comparison results for solutions to problem in the form (1.4) in terms of the following one

$$\begin{cases} -\Delta_p v = \beta|\nabla v|^q + f^\sharp & \text{in } \Omega^\sharp \\ v = 0 & \text{su } \partial\Omega^\sharp. \end{cases} \tag{4.1}$$

Results in this direction can be found, for example, in [29], where bounded solutions to problem (1.4) are considered, and in [40], where a weak solution to (1.4) is estimated in terms of the maximal solution to (4.1), when it exists (i.e., for certain values of q).

In the present section, we obtain pointwise comparison results in terms of a symmetrized problem where the source term is in general not equidistribuited with the source term f which appears in (1.4). However, the estimates we obtain allow us to give sharp conditions on f in order to state existence results for problem (1.4).

Let us recall the following result (see [29, Lemma 4.1]).

Lemma 4.1 *Let us suppose that* (1.5)–(1.8) *hold true with* $1 < p < N$ *and*

$$p - 1 < q \le p.$$

Let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a weak solution to problem* (1.4) *with* $f \in L^\infty(\Omega)$. *We have, a.e. in* $(0, |\Omega|)$, *that*

$$(N\omega_N^{1/N})^{\frac{p}{p-1}} s^{\frac{p(N-1)}{N(p-1)}} [(-u^*)'(s)] \le \left[\int_0^s f^*(\sigma) \exp\left(\frac{\beta}{(N\omega_N^{1/N})^{q-p}} \int_\sigma^s \frac{[(-u^*)'(r)]^{q-p+1}}{r^{\frac{(p-q)(N-1)}{N}}} dr\right) d\sigma\right]^{\frac{1}{p-1}}. \tag{4.2}$$

Depending on the value of q, the proof of comparison results will follow different lines, so we distinguish three cases in separate subsections.

4.1 The case $p - 1 < q < \frac{N(p-1)}{N-1}$

We prove a comparison result which states that a solution to problem (1.4) can be compared with a solution to a symmetrized problem whose datum is not a rearrangement of the datum

f, but a measure, i.e., we compare a solution to problem (1.4) with the solution to problem (3.1) when B_R coincides with the ball $\Omega^\sharp$ and K is the norm of f in $L^1(\Omega)$.

The following result holds.

Theorem 4.1 *Let us suppose that* (1.5)–(1.8) *hold true with* $1 < p < N$ *and*

$$p-1 < q < \frac{N(p-1)}{N-1}. \tag{4.3}$$

Let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a weak solution to problem* (1.4) *with* $f \in L^\infty(\Omega)$. *If the norm of* f *in* $L^1(\Omega)$ *is sufficiently small, that is,*

$$\|f\|_{L^1} < K_1 , \tag{4.4}$$

where

$$K_1 = N\omega_N \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{N(p-1)-q(N-1)}{N(q-p+1)}} \left(\frac{N(p-1)-q(N-1)}{\beta(q-p+1)}\right)^{\frac{p-1}{q-p+1}} , \tag{4.5}$$

then

$$u^*(s) \le z_1(s), \qquad s \in [0, |\Omega|], \tag{4.6}$$

where

$$z_1(s) = \frac{\|f\|_{L^1}^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}}} \int_s^{|\Omega|} \frac{\sigma^{-\frac{p(N-1)}{N(p-1)}}}{\left[1-\left(\frac{\|f\|_{L^1}}{K_1}\right)^{\frac{q-p+1}{p-1}} \left(\frac{\sigma}{|\Omega|}\right)^{1-\frac{q(N-1)}{N(p-1)}}\right]^{\frac{1}{q-p+1}}} d\sigma. \tag{4.7}$$

Remark 4.1 We explicitly observe that the function $z_1(s)$ defined in (4.7) is the decreasing rearrangement of the function ϕ_κ defined in (3.4) with $K = \|f\|_{L^1}$ and that the value of K_1 given in (4.5) coincides with the value given in (3.5) when R is the radius of the ball $\Omega^\sharp$. The considerations made in Sect. 3 and in Example 3.1 show that the smallness assumption (4.4) is sharp.

Proof of Theorem 4.1 We can use Lemma 4.1. If we put

$$U(s) = \int_0^s f^*(\sigma) \exp\left(\frac{\beta}{(N\omega_N^{1/N})^{q-p}} \int_\sigma^s \frac{[(-u^*)'(r)]^{q-p+1}}{r^{\frac{(p-q)(N-1)}{N}}} dr\right) d\sigma , \tag{4.8}$$

inequality (4.2) becomes

$$(N\omega_N^{1/N})^{\frac{p}{p-1}} s^{\frac{p(N-1)}{N(p-1)}} [(-u^*)'(s)] \le (U(s))^{\frac{1}{p-1}}, \qquad \text{a.e. } s \in (0, |\Omega|). \tag{4.9}$$

We observe that the integral which appears as argument of the exponential function in (4.8) is finite. Indeed, since $u \in W_0^{1,p}(\Omega)$, the gradient of u belongs to $L^q(\Omega)$, then, by Hölder and Pólya-Szëgo inequalities, we get

$$\int_0^{|\Omega|} \frac{[(-u^*)'(r)]^{q-p+1}}{r^{\frac{(p-q)(N-1)}{N}}} dr = \frac{1}{N^{q-p+1}\omega_N^{\frac{N-p+q}{N}}} \int_{\Omega^\sharp} \frac{|\nabla u^\sharp|^{q-p+1}}{|x|^{N-1}} dx$$

$$\leq \frac{1}{N^{q-p+1}\omega_N^{\frac{N-p+q}{N}}} \left(\int_\Omega |\nabla u|^q dx \right)^{\frac{q-p+1}{q}} \left(\int_{\Omega^\sharp} |x|^{-\frac{q(N-1)}{p-1}} dx \right)^{\frac{p-1}{q}}$$

where the last term is finite being $\frac{q(N-1)}{p-1} < N$.

Moreover, we have:

$$U'(s) = f^*(s) + \frac{\beta}{(N\omega_N^{1/N})^{q-p}} \frac{[(-u^*)'(s)]^{q-p+1}}{s^{\frac{(p-q)(N-1)}{N}}} U(s), \qquad \text{a.e. } s \in (0, |\Omega|),$$

and then, applying again (4.2), the function $U(s)$ satisfies

$$\begin{cases} U'(s) \leq f^*(s) + \dfrac{\beta}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} \dfrac{U^{\frac{q}{p-1}}}{s^{\frac{q(N-1)}{N(p-1)}}}, & \text{a.e. } s \in (0, |\Omega|), \\ U(0) = 0. \end{cases} \tag{4.10}$$

On the other hand, a direct computation gives

$$(N\omega_N^{1/N})^{\frac{p}{p-1}} s^{\frac{p(N-1)}{N(p-1)}} [(-z_1)'(s)] = (W(s))^{\frac{1}{p-1}}, \qquad s \in (0, |\Omega|), \tag{4.11}$$

where

$$W(s) = \frac{\|f\|_{L^1}}{\left[1 - \left(\dfrac{\|f\|_{L^1}}{K_1} \right)^{\frac{q-p+1}{p-1}} \left(\dfrac{s}{|\Omega|} \right)^{1-\frac{q(N-1)}{N(p-1)}} \right]^{\frac{p-1}{q-p+1}}}$$

is the solution to problem

$$\begin{cases} W'(s) = \dfrac{\beta}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} \dfrac{W^{\frac{q}{p-1}}}{s^{\frac{q(N-1)}{N(p-1)}}}, \ s \in (0, |\Omega|), \\ W(0) = \|f\|_{L^1}. \end{cases} \tag{4.12}$$

The following inequality holds true

$$U(s) \leq W(s), \qquad s \in [0, |\Omega|]. \tag{4.13}$$

If $\|f\|_{L^1} = 0$ there is nothing to prove. If $\|f\|_{L^1} > 0$ we argue by contradiction. Since $W(0) > U(0) = 0$, if (4.13) does not hold, there exists a value $\bar{s}$, with $0 < \bar{s} < |\Omega|$, such that $U(\bar{s}) = W(\bar{s})$ and $U(s) < W(s)$ for $0 \leq s < \bar{s}$. It would follow

$$U(\bar{s}) \leq \int_0^{\bar{s}} f^*(\sigma)d\sigma + \frac{\beta}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} \int_0^{\bar{s}} \frac{U^{\frac{q}{p-1}}}{\sigma^{\frac{q(N-1)}{N(p-1)}}} d\sigma$$

$$< \|f\|_{L^1} + \frac{\beta}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} \int_0^{\bar{s}} \frac{W^{\frac{q}{p-1}}}{\sigma^{\frac{q(N-1)}{N(p-1)}}} d\sigma = W(\bar{s})$$

and we have a contradiction. Thus, (4.13) holds true.

From (4.9), (4.11), and (4.13) we get

$$(-u^*)'(s) \leq (-z_1)'(s), \qquad \text{a.e. } s \in (0, |\Omega|)\,,$$

and the proof is complete. □

Remark 4.2 It is natural to ask if the decreasing rearrangement of a weak solution to problem (1.4) can be related to the solution v to problem (4.1). To this aim, we observe that the function

$$V(s) = \int_0^s f^*(\sigma) \exp\left(\beta(N\omega_N^{1/N})^{q-p} \int_\sigma^s \frac{[(-v^*)'(r)]^{q-p+1}}{r^{\frac{(p-q)(N-1)}{N}}} dr\right) d\sigma\,, \qquad (4.14)$$

solves problem

$$\begin{cases} V'(s) = f^*(s) + \dfrac{\beta}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} \dfrac{V^{\frac{q}{p-1}}}{s^{\frac{q(N-1)}{N(p-1)}}} & \text{in } (0, |\Omega|) \\ V(0) = 0. \end{cases} \qquad (4.15)$$

This fact allows us to apply the comparison result given in [40]. Thus, under the assumptions of Theorem 4.1, we have

$$u^*(s) \leq v^*(s) \qquad s \in (0, |\Omega|).$$

Furthermore, as in the proof of Theorem 4.1, we can prove

$$V(s) \leq W(s)\,, \qquad s \in (0, |\Omega|)\,,$$

which leads to the following inequality

$$v^*(s) \leq z_1^*(s)\,, \qquad s \in (0, |\Omega|).$$

Remark 4.3 As already observed in [40], the initial value problem (4.15) has an explicit solution also when we choose

$$f^*(s) = c^{\frac{q}{p-1}} s^{-\frac{q(N-1)}{N(p-1)}},$$

with a suitable nonnegative constant c, so that $f \in M^{\frac{N(p-1)}{q(N-1)}}(\Omega)$. We put

$$F(t) = \int_0^t \frac{d\tau}{1 + \beta\tau^{\frac{q}{p-1}}}$$

and

$$T(\beta) = \int_0^\infty \frac{d\tau}{1 + \beta\tau^{\frac{q}{p-1}}},$$

where $T(\beta)$ is finite in view of the hypotheses on β and $q > p-1$. Then, if $|\Omega|$ satisfies the condition

$$\frac{c^{\frac{q-p+1}{p-1}}}{N\omega_N^{1/N}} \frac{N(p-1)}{[q-N(q-p+1)]} |\Omega|^{1-\frac{q(N-1)}{N(p-1)}} < T(\beta), \tag{4.16}$$

the solution to (4.15) is

$$V(s) = cN\omega_N^{1/N} F^{-1}\left(\frac{c^{\frac{q-p+1}{p-1}}}{N\omega_N^{1/N}} \frac{N(p-1)}{[q-N(q-p+1)]} s^{1-\frac{q(N-1)}{N(p-1)}} \right).$$

The condition (4.16) can be read as a smallness assumption on the norm of f in $M^{\frac{N(p-1)}{q(N-1)}}(\Omega)$.

4.2 The case $\frac{N(p-1)}{N-1} < q \le p$

In this case too, it is possible to obtain an estimate similar to (4.6) by proving a comparison result which states that a solution to problem (1.4) can be compared with the solution to a problem in the form (3.13) whose datum is not in general a rearrangement of the datum of problem (1.4), and the constant K has a suitable value which depends on the datum of (1.4).

Theorem 4.2 *Let us suppose that* (1.5)–(1.8) *hold true with* $1 < p < N$ *and*

$$\frac{N(p-1)}{N-1} < q \le p. \tag{4.17}$$

Let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a weak solution to problem* (1.4) *with* $f \in L^\infty(\Omega)$. *If the norm of* f *in* $M^\gamma(\Omega)$, *with* $\gamma = \frac{N(q-p+1)}{q}$, *is sufficiently small, that is,*

$$\|f\|_{M^\gamma} < K_2\,, \tag{4.18}$$

with

$$K_2 = \frac{\gamma}{N}\left(\frac{N\omega_N^{1/N}}{\gamma'}\right)^{\frac{N}{\gamma}} \left(\frac{p-1}{\beta q}\right)^{\frac{N(p-1)}{\gamma q}}, \tag{4.19}$$

then

$$u^*(s) \le z_2(s), \tag{4.20}$$

where, denoted by $X_0 \ge 0$ *the smallest nonnegative solution to the equation*

$$\frac{\beta}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} X^{\frac{q}{p-1}} - \frac{X}{\gamma'} + \|f\|_{M^\gamma} = 0, \tag{4.21}$$

we have put

$$z_2(s) = \begin{cases} \dfrac{X_0^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}}} \dfrac{q\gamma}{p-q}\left(s^{-\frac{p-q}{q\gamma}} - |\Omega|^{-\frac{p-q}{q\gamma}}\right) & \textit{if} \quad \frac{N(p-1)}{N-1} < q < p \\[2ex] \dfrac{X_0^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}}} \log\left(\dfrac{|\Omega|}{s}\right) & \textit{if} \quad q = p. \end{cases} \tag{4.22}$$

Remark 4.4 We explicitly observe that the function $z_2(s)$ defined in (4.22) is the decreasing rearrangement of the function $v^{(0)}$ defined in (3.26) and that the value of K_2 given in (4.19) coincides with the value given in (3.17) when R is the radius of the ball $\Omega^\sharp$. The considerations made in Sect. 3 and in Examples 3.2, 3.3, show that the smallness assumption (4.18) is sharp.

Proof of Theorem 4.2 As in the proof of Theorem 4.1 we can use Lemma 4.1. Defining $U(s)$ as in (4.8) we get again (4.9), that is,

$$(N\omega_N^{1/N})^{\frac{p}{p-1}} s^{\frac{p(N-1)}{N(p-1)}} [(-u^*)'(s)] \leq (U(s))^{\frac{1}{p-1}}, \qquad \text{a.e. } s \in (0, |\Omega|), \tag{4.23}$$

and the function $U(s)$ satisfies

$$\begin{cases} U'(s) \leq \dfrac{\|f\|_{M^\gamma}}{s^{1/\gamma}} + \dfrac{\beta}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} \dfrac{U^{\frac{q}{p-1}}}{s^{\frac{q(N-1)}{N(p-1)}}} & \text{a.e. } s \in (0, |\Omega|) \\ U(0) = 0. \end{cases} \tag{4.24}$$

We now put $W(s) = X_0 s^{1/\gamma'}$. Observe that by (4.17) $1 < \gamma < +\infty$ and $1 < \gamma' < +\infty$. We want to show that

$$U(s) \leq W(s), \qquad s \in (0, |\Omega|). \tag{4.25}$$

If $\|f\|_{M^\gamma} = 0$, there is nothing to prove because $X_0 = 0$ and then $U(s) = W(s) = 0$. So, we suppose that $\|f\|_{M^\gamma} > 0$. Let us observe that $W(s)$ solves the problem

$$\begin{cases} W'(s) = \dfrac{\|f\|_{M^\gamma}}{s^{1/\gamma}} + \dfrac{\beta}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} \dfrac{W^{\frac{q}{p-1}}}{s^{\frac{q(N-1)}{N(p-1)}}} & \text{in } (0, |\Omega|) \\ W(0) = 0. \end{cases} \tag{4.26}$$

On the other hand, it holds

$$\lim_{s \to 0^+} \frac{U(s)}{W(s)} = 0. \tag{4.27}$$

Indeed, using the definition of $U(s)$ and the boundedness of u and f, we have

$$\begin{aligned} U(s) &= \int_0^s f^*(\sigma) \exp\left(\beta(N\omega_N^{1/N})^{q-p} \int_\sigma^s \frac{[(-u^*)'(r)]^{q-p+1}}{r^{\frac{(p-q)(N-1)}{N}}} dr\right) d\sigma \\ &\leq s\,\|f\|_{L^\infty} \exp\left(\beta(N\omega_N^{1/N})^{q-p} \|u\|_{L^\infty}^{q-p+1} (N\,|\Omega|^{\frac{1}{N}})^{p-q}\right) \end{aligned}$$

and, by definition of W, (4.27) follows. This means that for a certain $\delta > 0$ we have

$$U(s) < W(s), \qquad 0 < s < \delta. \tag{4.28}$$

Now, in order to prove (4.25), we can argue by contradiction. In view of (4.28), if (4.25) does not hold, there exists a value $\bar{s}$, with $0 < \bar{s} < |\Omega|$, such that $U(\bar{s}) = W(\bar{s})$ and $U(s) < W(s)$ for $0 < s < \bar{s}$. Taking in mind that $W(s)$ solves (4.26), it follows

$$U(\bar{s}) \leq \int_0^{\bar{s}} \frac{\|f\|_{M^\gamma}}{\sigma^{1/\gamma}} d\sigma + \frac{\beta}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} \int_0^{\bar{s}} \frac{U^{\frac{q}{p-1}}}{\sigma^{\frac{q(N-1)}{N(p-1)}}} d\sigma$$

$$< \int_0^{\bar{s}} \frac{\|f\|_{M^\gamma}}{\sigma^{1/\gamma}} d\sigma + \frac{\beta}{(N\omega_N^{1/N})^{\frac{q}{p-1}}} \int_0^{\bar{s}} \frac{W^{\frac{q}{p-1}}}{\sigma^{\frac{q(N-1)}{N(p-1)}}} d\sigma = W(\bar{s})$$

and we have a contradiction. Thus, (4.25) holds true.

From (4.23) and (4.25), we then get

$$(-u^*)'(s) \leq \frac{X_0^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}} s^{1+\frac{p-q}{q\gamma}}} \qquad \text{a.e. } s \in (0, |\Omega|),$$

and (4.20) follows. □

4.3 The limit case $q = \frac{N(p-1)}{N-1}$

In this case, we prove two pointwise estimates for solutions to problem (1.4) depending on different assumptions on the summability of f. We begin with the following result.

Theorem 4.3 *Let us suppose that (1.5)–(1.8) hold true with $1 < p < N$ and*

$$q = \frac{N(p-1)}{N-1}. \tag{4.29}$$

Let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution to problem (1.4) with $f \in L^\infty(\Omega)$ and such that for some constants $\mathcal{M} > |\Omega|$ and $\mathcal{C}_f \geq 0$ we have

$$f^*(s) \leq \frac{\mathcal{C}_f}{s \log^N(\mathcal{M}/s)} \qquad s \in (0, |\Omega|). \tag{4.30}$$

If

$$\mathcal{C}_f < K_3 \tag{4.31}$$

with

$$K_3 = \frac{\omega_N (N-1)^{2N-1}}{\beta^{N-1}},$$

then

$$u^*(s) \leq \frac{Y_0^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}}} \int_s^{|\Omega|} \frac{1}{\sigma^{\frac{p(N-1)}{N(p-1)}} \log^{\frac{N-1}{p-1}}(\mathcal{M}/\sigma)} d\sigma, \quad s \in (0, |\Omega|), \tag{4.32}$$

where $Y_0 \geq 0$ is the smallest nonnegative solution to the equation

$$\frac{\beta}{(N\omega_N^{1/N})^{\frac{N}{N-1}}} Y^{\frac{N}{N-1}} - (N-1)Y + \mathcal{C}_f = 0. \tag{4.33}$$

Remark 4.5 Let us observe that the assumption (4.30) means that the datum f belongs to the Lorentz–Zygmund space $L^{1,\infty}(\log L)^N$ whose definition is given Sect. 2.

Proof of Theorem 4.3 As in the proof of Theorem 4.1 we can use Lemma 4.1. Defining $U(s)$ as in (4.8) we get again (4.9), that is,

$$(N\omega_N^{1/N})^{\frac{p}{p-1}} s^{\frac{p(N-1)}{N(p-1)}} [(-u^*)'(s)] \le (U(s))^{\frac{1}{p-1}}, \qquad \text{a.e. } s \in (0, |\Omega|), \tag{4.34}$$

and the function $U(s)$ satisfies

$$\begin{cases} U'(s) \le \dfrac{\mathcal{C}_f}{s \log^N(\mathcal{M}/s)} + \dfrac{\beta}{(N\omega_N^{1/N})^{\frac{N}{N-1}}} \dfrac{U^{\frac{N}{N-1}}}{s} & \text{a.e. } s \in (0, |\Omega|) \\ U(0) = 0. \end{cases} \tag{4.35}$$

We now put $W(s) = Y_0 \log^{1-N}(\mathcal{M}/s)$. We want to show that

$$U(s) \le W(s), \qquad s \in (0, |\Omega|). \tag{4.36}$$

If $\mathcal{C}_f = 0$, there is nothing to prove because $Y_0 = 0$ and then $U(s) = W(s) = 0$. So, we suppose that $\mathcal{C}_f > 0$. Let us observe that $W(s)$ solves the problem

$$\begin{cases} W'(s) = \dfrac{\mathcal{C}_f}{s \log^N(\mathcal{M}/s)} + \dfrac{\beta}{(N\omega_N^{1/N})^{\frac{N}{N-1}}} \dfrac{W^{\frac{N}{N-1}}}{s} & \text{a.e. } s \in (0, |\Omega|) \\ W(0) = 0. \end{cases} \tag{4.37}$$

On the other hand, using the definition of $U(s)$ and the boundedness of u and f, as in the previous proof, we get

$$\lim_{s \to 0^+} \frac{U(s)}{W(s)} = 0. \tag{4.38}$$

This means that for a certain $\delta > 0$ we have

$$U(s) < W(s), \qquad 0 < s < \delta. \tag{4.39}$$

Now, in order to prove (4.36), we can argue by contradiction. In view of (4.39), if (4.36) does not hold, there exists a value $\bar{s}$, with $0 < \bar{s} < |\Omega|$, such that $U(\bar{s}) = W(\bar{s})$ and $U(s) < W(s)$ for $0 < s < \bar{s}$. Taking in mind that $W(s)$ solves (4.37), it follows

$$\begin{aligned} U(\bar{s}) &\le \int_0^{\bar{s}} \frac{\mathcal{C}_f}{\sigma \log^N(\mathcal{M}/\sigma)} d\sigma + \frac{\beta}{(N\omega_N^{1/N})^{\frac{N}{N-1}}} \int_0^{\bar{s}} \frac{U^{\frac{N}{N-1}}}{\sigma} d\sigma \\ &< \int_0^{\bar{s}} \frac{\mathcal{C}_f}{\sigma \log^N(\mathcal{M}/\sigma)} d\sigma + \frac{\beta}{(N\omega_N^{1/N})^{\frac{N}{N-1}}} \int_0^{\bar{s}} \frac{W^{\frac{N}{N-1}}}{\sigma} d\sigma = W(\bar{s}) \end{aligned}$$

and we have a contradiction. Thus, (4.36) holds true.

From (4.34) and (4.36), we then get

$$(-u^*)'(s) \le \frac{Y_0^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}} s^{\frac{p(N-1)}{N(p-1)}} \log^{\frac{N-1}{p-1}}(\mathcal{M}/s)} \qquad \text{a.e. } s \in (0, |\Omega|),$$

and (4.32) follows. □

Now, we prove our second pointwise estimate.

Theorem 4.4 *Let us suppose that* (1.5)–(1.8) *hold true with* $1 < p < N$ *and* (4.29). *Let* $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a weak solution to problem* (1.4) *with* $f \in L^\infty(\Omega)$ *and such that for a constant* $\mathcal{M} > |\Omega|$, *it holds*

$$\int_0^{|\Omega|} f^*(s)\ \log^{N-1}(\mathcal{M}/s)\, ds < K_4\,, \tag{4.40}$$

with

$$K_4 = \frac{\omega_N (N-1)^{N-1} N^N}{\beta^{N-1}}.$$

Then

$$u^*(s) \le \frac{K_4^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}}} \int_s^{|\Omega|} \frac{1}{\sigma^{\frac{p(N-1)}{N(p-1)}} \log^{\frac{N-1}{p-1}}(\mathcal{M}/\sigma)}\, d\sigma, \quad s \in (0, |\Omega|). \tag{4.41}$$

Remark 4.6 Observe that $Z = K_4$ is the positive solution to the equation

$$\frac{\beta}{(N\omega_N^{1/N})^{\frac{N}{N-1}}}\, Z^{\frac{N}{N-1}} - (N-1)Z = 0.$$

Remark 4.7 Let us observe that the assumption (4.40) means that the datum f belongs to the Zygmund space $L(\log L)^{N-1}$ whose definition is given in Sect. 2.

Remark 4.8 Smallness condition (4.40) is sharp in the following sense. For the sake of simplicity we will refer to the case $p = 2$. Denoting by R_Ω the radius of $\Omega^\sharp$, for every fixed $R > R_\Omega$ the function

$$u_R(|x|) = (N-1)^{N-1} \int_{|x|}^{R_\Omega} \frac{dt}{t^{N-1}\left[\log\left(\frac{R}{t}\right)\right]^{N-1}} \tag{4.42}$$

satisfies the following homogeneous equation in $\Omega^\sharp - \{0\}$

$$-\Delta u - |\nabla u|^{\frac{N}{N-1}} = 0. \tag{4.43}$$

Now, for $\varepsilon > 0$, define the functions

$$u_{R,\varepsilon}(|x|) = \begin{cases} u_R(|x|) & \text{if } |x| > \varepsilon \\ a_\varepsilon(\varepsilon^2 - |x|^2) + u_R(\varepsilon) & \text{if } |x| \le \varepsilon \end{cases}$$

with

$$a_\varepsilon = \frac{(N-1)^{N-1}}{2\varepsilon^N \log^{N-1}\left(\frac{R}{\varepsilon}\right)}.$$

Such functions satisfy the homogeneous Dirichlet problem in $\Omega^\sharp$ for the equation

$$-\Delta u - |\nabla u|^{\frac{N}{N-1}} = f_\varepsilon$$

with

$$f_\varepsilon(x) = \begin{cases} 0 & \text{if } |x| > \varepsilon \\ \dfrac{N(N-1)^{N-1}}{\varepsilon^N \log^{N-1}\left(\frac{R}{\varepsilon}\right)} - \dfrac{(N-1)^N |x|^{\frac{N}{N-1}}}{\varepsilon^{\frac{N^2}{N-1}} \log^N\left(\frac{R}{\varepsilon}\right)} & \text{if } r \le \varepsilon. \end{cases}$$

It is easy to verify that

$$\lim_{\varepsilon\to 0} \int_{\Omega^\sharp} f_\varepsilon(|x|) \log^{N-1}\left(\frac{R}{|x|}\right) dx = N\omega_N (N-1)^{N-1}.$$

This means that the data weakly converge in a weighted L^1 space to a kind of Dirac mass. Actually, the function (4.42) is solution in $\Omega^\sharp$ to an equation which, in contrast to (4.43), is not homogeneus, but it has a datum that is *concentrated* as a measure in the origin.

Proof of Theorem 4.4 As in the previous proofs, we can use Lemma 4.1. Defining $U(s)$ as in (4.8) we get again (4.34) and the function $U(s)$ satisfies

$$\begin{cases} U'(s) \le f^*(s) + \dfrac{\beta}{(N\omega_N^{1/N})^{\frac{N}{N-1}}} \dfrac{U^{\frac{N}{N-1}}}{s} & \text{a.e. } s \in (0, |\Omega|) \\ U(0) = 0. \end{cases}$$

Now, we put $\widetilde{U}(s) = U(s)\log^{N-1}(\mathcal{M}/s)$, and we observe that $\widetilde{U}(s)$ satisfies

$$\begin{cases} \widetilde{U}'(s) \le f^*(s) \log^{N-1}\left(\dfrac{\mathcal{M}}{s}\right) + \dfrac{\beta}{(N\omega_N^{1/N})^{\frac{N}{N-1}}} \dfrac{\widetilde{U}^{\frac{N}{N-1}}}{s \log\left(\dfrac{\mathcal{M}}{s}\right)} - (N-1) \dfrac{\widetilde{U}}{s \log\left(\dfrac{\mathcal{M}}{s}\right)} \\ \widetilde{U}(0) = 0. \end{cases} \tag{4.44}$$

We claim that

$$\widetilde{U}(s) \le K_4. \tag{4.45}$$

Indeed, since $\widetilde{U}(0) = 0 < K_4$, there exists a $\delta > 0$ such that

$$\widetilde{U}(s) < K_4, \qquad \text{for} \quad 0 < s < \delta. \tag{4.46}$$

In order to prove (4.45), we can argue by contradiction. Since (4.46) holds true, if (4.45) does not hold, there exists a value $\bar{s}$ with $0 < \bar{s} < |\Omega|$ such $\widetilde{U}(\bar{s}) = K_4$ and $\widetilde{U}(s) < K_4$, for $0 < s < \bar{s}$. Therefore, since the function

$$\Psi(Z) = \frac{\beta}{(N\omega_N^{1/N})^{\frac{N}{N-1}}} Z^{\frac{N}{N-1}} - (N-1)Z$$

is convex in $[0, K_4]$ and $\Psi(0) = \Psi(K_4) = 0$, for $0 < \varepsilon < \bar{s}$, we have

$$\begin{aligned} \widetilde{U}(\bar{s}) - \widetilde{U}(\varepsilon) &\le \int_\varepsilon^{\bar{s}} f^*(s) \log^{N-1}(\mathcal{M}/s)\, ds + \int_\varepsilon^{\bar{s}} \frac{\Psi(\widetilde{U}(s))}{s \log(\mathcal{M}/s)}\, ds \\ &< \int_\varepsilon^{\bar{s}} f^*(s) \log^{N-1}(\mathcal{M}/s)\, ds. \end{aligned}$$

Letting ε go to zero and recalling assumption (4.40), we have $\widetilde{U}(\bar{s}) < K_4$, that is a contradiction. Thus, (4.45) is proved.

From definition of $\widetilde{U}(s)$, we deduce that

$$U(s) \le \frac{K_4}{\log^{N-1}(\mathcal{M}/s)}$$

and by (4.34) we obtain

$$(-u^*)'(s) \le \frac{K_4^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}} s^{\frac{p(N-1)}{N(p-1)}} \log^{\frac{N-1}{p-1}}(\mathcal{M}/s)} \qquad \text{a.e. } s \in (0, |\Omega|).$$

This implies (4.41). □

5 A priori estimates and existence results

In this section, we prove a priori estimates and we deduce existence results for solutions to problem (1.4).

As in the previous section, depending on the value of q, the proofs will follow different lines, so we distinguish three cases in separate subsections.

5.1 The case $p - 1 < q < \frac{N(p-1)}{N-1}$

The first result is already proved in [32]; here, we give a different proof.

Theorem 5.1 *Let us suppose that (1.5)–(1.8) hold true with $1 < p < N$ and (4.3). Let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak solution to problem (1.4) with $f \in L^\infty(\Omega)$. If the norm of f in $L^1(\Omega)$ satisfies (4.4), then*

$$\|u\|_{M^{\frac{N(p-1)}{N-p}}} \le C\,, \tag{5.1}$$

$$\|\nabla u\|_{M^{\frac{N(p-1)}{N-1}}} \le C\,, \tag{5.2}$$

where C is a positive constant which depends only on p, q, N, $|\Omega|$, β and $\|f\|_{L^1}$.

Proof We observe that it is possible to apply Theorem 4.1 and, taking into account (4.6) and (4.7), we have

$$u^*(s) \le \frac{\|f\|_{L^1}^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}}} \frac{1}{\left[1 - \left(\frac{\|f\|_{L^1}}{K_1}\right)^{\frac{q-p+1}{p-1}}\right]^{\frac{1}{q-p+1}}} \frac{N(p-1)}{N-p} \frac{1}{s^{\frac{N-p}{N(p-1)}}}\,, \tag{5.3}$$

and (5.1) immediately follows.

Let us now prove (5.2). Denoting by $\mu(t)$ the distribution function of u, we define the function

$$\varphi(x) = \operatorname{sign}(u(x)) \int_0^{|u(x)|} \nu_s(\mu(t))\,dt,$$

where

$$\nu_s(r) = \begin{cases} r^\alpha, & \text{if } 0 \le r \le s, \\ s^\alpha, & \text{if } r > s, \end{cases}$$

with $s \in [0, |\Omega|]$ and $\alpha > 0$ such that

$$\frac{N-p}{N(p-1)} < \alpha < \frac{p-q}{q}. \tag{5.4}$$

Let us observe that the above condition on α can be imposed because of the assumption $q < \frac{N(p-1)}{N-1}$ and that $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. In particular, the norm of φ in $L^\infty(\Omega)$ can be estimated making use of (5.3) and (5.4)

$$\|\varphi\|_{L^\infty} = \int_0^{+\infty} \nu_s(\mu(t))\, dt = \alpha \int_0^s r^{\alpha-1} u^*(r)\, dr \le c\, s^{\alpha - \frac{N-p}{N(p-1)}}. \tag{5.5}$$

Here and in what follows, c denotes a positive constant, which can change line by line and depends only on the data. The same notation will be used also in the next proofs.

Using $\varphi(x)$ as test function in (1.4) and assumptions (1.5)–(1.8), we have

$$\int_\Omega \nu_s\,(\mu(|u(x)|))\, |\nabla u|^p\, dx \le \beta \int_\Omega |\nabla u|^q |\varphi|\, dx + \int_\Omega |f||\varphi|\, dx. \tag{5.6}$$

From Hölder and Young inequalities we get

$$\begin{aligned} \int_\Omega |\nabla u|^q |\varphi|\, dx &\le \left(\int_\Omega \nu_s(\mu(|u(x)|)) |\nabla u|^p\, dx \right)^{\frac{q}{p}} \left(\int_\Omega \frac{|\varphi|^{\frac{p}{p-q}}}{[\nu_s(\mu(|u(x)|))]^{\frac{q}{p-q}}}\, dx \right)^{1-\frac{q}{p}} \\ &\le \left(1 - \frac{q}{p}\right) \int_\Omega \nu_s(\mu(|u(x)|)) |\nabla u|^p\, dx + c \left(s^{\alpha+1-\frac{N-p}{N(p-1)}\frac{p}{p-q}} + s^\alpha \right)^{1-\frac{q}{p}} \end{aligned} \tag{5.7}$$

where we have used the fact that, as a consequence of (5.3), it holds

$$\int_\Omega \frac{|\varphi|^{\frac{p}{p-q}}}{[\nu_s(\mu(|u(x)|))]^{\frac{q}{p-q}}}\, dx \le c\, (s^{\alpha+1-\frac{N-p}{N(p-1)}\frac{p}{p-q}} + s^\alpha).$$

On the other hand, taking into account (5.5) and using Hardy inequality, it results

$$\int_\Omega |f||\varphi|\, dx \le c\, s^{\alpha-\frac{N-p}{N(p-1)}} \|f\|_{L^1}. \tag{5.8}$$

Collecting (5.6), (5.7), (5.8), and using (4.3), we get

$$\int_\Omega \nu_s\,(\mu(|u(x)|))\, |\nabla u|^p\, dx \le c\, s^{\alpha-\frac{N-p}{N(p-1)}}. \tag{5.9}$$

Proceeding as in [3], we put

$$D(s) = |\nabla u|^*(s), \qquad s \in]0, |\Omega|[,$$

and we estimate from below the integral on the left-hand side of (5.9) using Hardy–Littlewood inequality. Observing that v_s is an increasing function, we obtain

$$\int_\Omega v_s\left(\mu(|u(x)|)\right)|\nabla u|^p\,dx \geq \int_0^{|\Omega|} v_s(r)D^p(r)\,dr \\ \geq \int_0^s v_s(r)D^p(r)\,dr = D^p(s)\frac{s^{\alpha+1}}{\alpha+1}.$$

The use of (5.9) immediately gives (5.2). □

The previous a priori estimates allow to prove existence results for problem (1.4).

Theorem 5.2 *Let us suppose that* (1.5)–(1.8) *hold true with* $1 < p < N$ *and* (4.3). *If* $f \in L^1(\Omega)$ *and it satisfies* (4.4)*, then there exists at least a solution* u *to the problem* (1.4) *obtained as a limit of approximations which satisfies* (5.1) *and* (5.2).

The proof of the above result uses arguments similar to those contained, for example, in [9,23]. The novelty of Theorem 5.2 relies on the sharpness of the smallness assumption (4.4) on f. Here, we only sketch the scheme of the proof, and some missing details can be found in [32].

Consider a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to the approximated problem

$$\begin{cases} -\mathrm{div}\,(A(x,u_n,\nabla u_n)) = T_n(H(x,u_n,\nabla u_n)) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{5.10}$$

whose existence is assured by a classical result (see [37,39]). By a priori estimates, we can prove that, up to subsequence, $\{u_n\}_{n\in\mathbb{N}}$ converges a.e. to a finite measurable function u such that $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. This implies (see [9], Lemma 2.1) the existence of a measurable function v, the approximated gradient of u, such that

$$\nabla T_k(u) = v\chi_{|u|\leq k}, \quad \text{a.e. in } \Omega.$$

We denote $v = \nabla u$. Observe that v could be not in $\left(L^1_{loc}\right)^N$, but, if it is a summable vector function, it coincides with the distributional gradient of u. Moreover, proceeding as in [9], we can prove that

$$\nabla u_n \to \nabla u \quad \text{a.e. in } \Omega.$$

Finally, since u_n satisfies the equality

$$\int_\Omega A(x,u_n,\nabla u_n)\cdot\nabla\phi\,dx = \int_\Omega T_n(H(x,u_n,\nabla u_n))\phi\,dx\,, \tag{5.11}$$

for every $\phi \in C_0^\infty(\Omega)$, Vitali's Theorem allows to pass to the limit in (5.11), and this yields that u satisfies (1.4) in the sense of distribution. Such a solution is known as solution obtained as a limit of approximations ([24]).

5.2 The case $\frac{N(p-1)}{N-1} \leq q < p-1+\frac{p}{N}$

We begin this section by proving a priori estimates.

Theorem 5.3 *Let us suppose that (1.5)–(1.8) hold true with* $1 < p < N$ *and*

$$\frac{N(p-1)}{N-1} < q < p-1+\frac{p}{N}. \tag{5.12}$$

Let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a weak solution to the problem (1.4) with* $f \in L^\infty(\Omega)$*. If the norm of* f *in* $M^\gamma(\Omega)$*, with* $\gamma = \frac{N(q-p+1)}{q}$*, satisfies (4.18), then*

$$\|u\|_{M^{\frac{\gamma q}{p-q}}} \leq C\,, \tag{5.13}$$

$$\|\nabla u\|_{L^{t,p}} \leq C\,, \tag{5.14}$$

for every $t < N(q-p+1)$*, where* C *is a positive constant which depends only on* $p,\ q,\ N,\ |\Omega|,\ \beta$ *and by* $\|f\|_{M^\gamma}$*.*

Proof The estimate (5.13) is consequence of (4.20) since it gives

$$u^*(s) \leq \frac{X_0^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}}}\frac{q\gamma}{p-q}s^{-\frac{p-q}{q\gamma}}.$$

Now, we prove (5.14). Consider the function

$$\psi(x) = \operatorname{sign}(u(x)) \int_0^{|u(x)|} [\mu(t)]^\alpha\, dt, \tag{5.15}$$

with

$$0 < \frac{p}{N(q-p+1)} - 1 < \alpha < \frac{p-q}{N(q-p+1)}.$$

Let us explicitly observe that the choise of α is possible since $q > \frac{N(p-1)}{N-1}$.

Since $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and therefore, it is a test function for problem (1.4). By assumptions (1.5) and (1.8), we get

$$\int_\Omega |\nabla u|^p [\mu(u(x))]^\alpha\, dx \leq \beta \int_\Omega |\nabla u|^q |\psi|\, dx + \int_\Omega |f\psi|\, dx. \tag{5.16}$$

We begin by evaluating the first integral on the right-hand side. By Hölder inequality, we get

$$\int_\Omega |\nabla u|^q |\psi(x)|\, dx \leq \left(\int_\Omega |\nabla u|^p [\mu(|u(x)|)]^\alpha\, dx\right)^{\frac{q}{p}} \left(\int_\Omega \frac{|\psi(x)|^{\frac{p}{p-q}}}{[\mu(|u(x)|)]^{\frac{\alpha q}{p-q}}}\, dx\right)^{1-\frac{q}{p}}. \tag{5.17}$$

By co-area formula and classical properties of rearrangements, we deduce

$$\int_\Omega \frac{|\psi(x)|^{\frac{p}{p-q}}}{[\mu(|u(x)|)]^{\frac{\alpha q}{p-q}}}\, dx \leq \int_0^{|\Omega|} \left(\int_0^{u^*(s)} [\mu(t)]^\alpha\, dt\right)^{\frac{p}{p-q}} \frac{1}{s^{\frac{\alpha q}{p-q}}}\, ds. \tag{5.18}$$

On the other hand, by (5.13), since $\alpha < \frac{p-q}{N(q-p+1)}$, it results

$$\int_0^{u^*(s)} [\mu(t)]^\alpha \, dt \le c\, s^{\alpha - \frac{p-q}{N(q-p+1)}}. \tag{5.19}$$

Therefore, by (5.18), since $\alpha > \frac{p}{N(q-p+1)} - 1$, we deduce

$$\int_\Omega \frac{|\psi(x)|^{\frac{p}{p-q}}}{[\mu(|u(x)|)]^{\frac{\alpha q}{p-q}}} \, dx \le c \int_0^{|\Omega|} s^{\alpha - \frac{p}{N(q-p+1)}} \, ds \le c \tag{5.20}$$

and, finally, by (5.17),

$$\int_\Omega |\nabla u|^q |\psi| \, dx \le c \left(\int_\Omega |\nabla u|^p [\mu(|u(x)|)]^\alpha \, dx \right)^{\frac{q}{p}}. \tag{5.21}$$

Now, we evaluate the integral

$$\int_\Omega |f\psi| \, dx.$$

By co-area formula, integrating by parts and using again (5.19), we get

$$\int_\Omega |f\psi(x)| \, dx \le \int_0^{|\Omega|} \frac{1}{s^{\frac{q}{N(q-p+1)}}} \left(\int_0^{u^*(s)} [\mu(t)]^\alpha \, dt \right) ds \le c. \tag{5.22}$$

By (5.16), (5.21), and (5.22), since $q < p$, we deduce the following estimate

$$\int_\Omega |\nabla u|^p [\mu(|u(x)|)]^\alpha \, dx \le c.$$

Therefore, by Hardy inequality, the conclusion follows. □

Now, we prove a priori estimates in the limit case $q = \frac{N(p-1)}{N-1}$.

Theorem 5.4 *Let us assume that* (1.5)–(1.8) *hold true with* $1 < p < N$ *and* (4.29)*. Let* $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a weak solution to the problem* (1.4) *with* $f \in L^\infty(\Omega)$*. If* f *satisfies either* (4.30) *and* (4.31)*, or* (4.40)*, then* u *satisfies* (4.32) *or* (4.41)*, respectively, and*

$$\int_0^{|\Omega|} (|\nabla u|^*(s))^q \log^\tau(\mathcal{M}/s) \, ds \le C, \tag{5.23}$$

for every $0 < \tau < N - 1$*, where* $\mathcal{M}$ *is a positive constant larger then* $|\Omega|$*,* C *is a positive constant which depends only on* p*,* q*,* N*,* $|\Omega|$*,* β *and* f*.*

Proof By Theorems 4.3 and 4.4 we have just to prove (5.23). Let us begin by assuming (4.40). Consider the function

$$\psi(x) = \operatorname{sign}(u(x)) \int_0^{|u(x)|} [\mu(t)]^\alpha \log^\delta \left(\frac{\mathcal{M}}{\mu(t)} \right), dt, \tag{5.24}$$

with

$$\alpha = \frac{N-p}{N(p-1)}, \qquad \frac{N-1}{p-1} - 1 < \delta < \frac{p(N-1)}{p-1} - 1. \tag{5.25}$$

Since $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and therefore, it is a test function for problem (1.4). By assumptions (1.5) and (1.8) we get

$$\int_\Omega |\nabla u|^p [\mu(|u(x)|)]^\alpha \log^\delta \left(\frac{\mathcal{M}}{\mu(|u(x)|)} \right) dx \le \beta \int_\Omega |\nabla u|^q |\psi| \, dx + \int_\Omega |f\psi| \, dx. \tag{5.26}$$

We begin by evaluating the first integral on the right-hand side. By Hölder inequality, we get

$$\int_\Omega |\nabla u|^q |\psi(x)| \, dx \le \left(\int_\Omega |\nabla u|^p [\mu(|u(x)|)]^\alpha \log^\delta \left(\frac{\mathcal{M}}{\mu(|u(x)|)} \right) dx \right)^{\frac{q}{p}} \times \left(\int_\Omega \frac{|\psi(x)|^{\frac{p}{p-q}}}{[\mu(|u(x)|)]^{\frac{\alpha q}{p-q}} \left[\log \left(\frac{\mathcal{M}}{\mu(|u(x)|)} \right) \right]^{\frac{\delta q}{p-q}}} \, dx \right)^{1-\frac{q}{p}}. \tag{5.27}$$

By co-area formula and classical properties of rearrangements, we deduce

$$\int_\Omega \frac{|\psi(x)|^{\frac{p}{p-q}}}{[\mu(|u(x)|)]^{\frac{\alpha q}{p-q}} \left[\log \left(\frac{\mathcal{M}}{\mu(|u(x)|)} \right) \right]^{\frac{\delta q}{p-q}}} \, dx \le \int_0^{|\Omega|} \left(\int_0^{u^*(s)} [\mu(t)]^\alpha \left[\log^\delta \left(\frac{\mathcal{M}}{\mu(t)} \right) \right] dt \right)^{\frac{p}{p-q}} \frac{1}{s^{\frac{\alpha q}{p-q}} \left[\log \left(\frac{\mathcal{M}}{s} \right) \right]^{\frac{\delta q}{p-q}}} \, ds. \tag{5.28}$$

On the other hand, since $\alpha = \frac{N-p}{N(p-1)}$, and $\delta > \frac{N-1}{p-1} - 1$, by using (4.41), it results

$$\int_0^{u^*(s)} [\mu(t)]^\alpha \left[\log \left(\frac{\mathcal{M}}{\mu(t)} \right) \right]^\delta dt \le \frac{c}{\log^{\frac{N-p}{p-1} - \delta} (\mathcal{M}/s)}. \tag{5.29}$$

Therefore, by (5.28), since $\delta < \frac{p(N-1)}{p-1} - 1$, we deduce

$$\int_\Omega \frac{|\psi(x)|^{\frac{p}{p-q}}}{[\mu(|u(x)|)]^{\frac{\alpha q}{p-q}} \left[\log \left(\frac{\mathcal{M}}{\mu(|u(x)|)} \right) \right]^{\frac{\delta q}{p-q}}} \, dx \le c \tag{5.30}$$

and, finally, by (5.27),

$$\int_\Omega |\nabla u|^q |\psi| \, dx \le c \left(\int_\Omega |\nabla u|^p [\mu(|u(x)|)]^\alpha \log^\delta \left(\frac{\mathcal{M}}{\mu(|u(x)|)} \right) dx \right)^{\frac{q}{p}}. \tag{5.31}$$

Now we evaluate the integral

$$\int_\Omega |f\psi|\,dx. \tag{5.32}$$

By co-area formula, integrating by parts and using Hardy inequality, since $\delta < \frac{p(N-1)}{p-1} - 1$, we get

$$\begin{aligned}\int_\Omega |f\psi(x)|\,dx &\le \int_0^{|\Omega|} f^*(s)\left(\int_0^{u^*(s)} \mu(t)^\alpha \log^\delta\left(\frac{\mathcal{M}}{\mu(t)}\right)dt\right)ds\\ &\le \int_0^{|\Omega|} f^*(s)\frac{1}{\left[\log\left(\frac{\mathcal{M}}{s}\right)\right]^{\frac{N-p}{p-1}-\delta}}\,ds \le c.\end{aligned} \tag{5.33}$$

By (5.26), (5.31), and (5.33), since $q < p$, we deduce the following estimate

$$\int_\Omega |\nabla u|^p[\mu(u(x))]^\alpha \log^\delta\left(\frac{\mathcal{M}}{\mu(|u(x)|)}\right)dx \le c. \tag{5.34}$$

The fact that the integral in (5.34) is finite for a given $\mathcal{M} > |\Omega|$ is equivalent to say that it is finite for every $\mathcal{M} > |\Omega|$. So, we can suppose that $\mathcal{M}$ is large enough, in such a way that the function $t^\alpha \log^\delta(\mathcal{M}/t)$ is an increasing function with respect to t. Therefore, by Hardy inequality, we get

$$\int_0^{|\Omega|} \left[|\nabla u|^*(s)\right]^p s^\alpha \log^\delta(\mathcal{M}/s)\,ds \le c.$$

Now, by Hölder inequality, we get

$$\begin{aligned}\int_0^{|\Omega|} (|\nabla u|^*(s))^q \log^\tau\left(\frac{\mathcal{M}}{s}\right)ds &\le \left(\int_0^{|\Omega|}\left[|\nabla u|^*(s)\right]^p s^\alpha \log^\delta\left(\frac{\mathcal{M}}{s}\right)ds\right)^{\frac{q}{p}}\\ &\quad\times\left(\int_0^{|\Omega|}\frac{1}{s\left[\log\left(\frac{\mathcal{M}}{s}\right)\right]^{\delta\frac{q}{p-q}-\tau\frac{p}{p-q}}}\,ds\right)^{1-\frac{q}{p}}\end{aligned} \tag{5.35}$$

Taking into account (5.25), for every $\tau \in (0, N-1)$ we can choose δ in such a way that $\delta\frac{q}{p-q} - \tau\frac{p}{p-q} - 1 > 0$, so that the last integral is finite and the conclusion follows.

Assume now that (4.30) and (4.31) hold true. Using (4.32), we can proceed as above, obtaining (5.31). In order to evaluate (5.32), we can use (4.30), obtaining

$$\int_\Omega |f\psi|\,dx \le c\int_0^{|\Omega|}\frac{1}{s\left[\log\left(\frac{\mathcal{M}}{s}\right)\right]^{p\frac{N-1}{p-1}-\delta}}\,ds \le c\,,$$

and the proof continues as in the previous case. □

The previous a priori estimates allow to prove existence results for problem (1.4).

Theorem 5.5 *Assume* (1.5)–(1.8) *hold true with* $1 < p < N$ *and*

$$\frac{N(p-1)}{N-1} < q < p - 1 + \frac{p}{N}.$$

If $f \in M^{\gamma}(\Omega)$, *with* $\gamma = \frac{N(q-p+1)}{q}$, *and it satisfies* (4.18), *then there exists at least a solution to the problem* (1.4) *obtained as a limit of approximations which satisfies* (5.13) *and* (5.14).

Proof As described at the end of Subsect. 5.2, we consider a weak solution $u_n \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ to the approximated problem (5.10). By the a priori estimates obtained in Theorem 5.3, we deduce that $|\nabla u_n|^q$ is bounded in $L^{\frac{r}{q}}(\Omega)$, with $q < r < t (< N(q-p+1))$. Therefore, by growth assumption (1.8) on H, we deduce that $T_n(H(x, u_n, \nabla u_n)$ is bounded in $L^{\frac{r}{q}}(\Omega)$. Moreover, for every fixed $k > 0$, $T_k(u_n)$ can be used as test function in (5.11) and we get

$$\int_{\Omega} |\nabla T_k(u_n)|^p \, dx \leq k \int_{\Omega} T_n(H(x, u_n, \nabla u_n)) \, dx. \tag{5.36}$$

This implies that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, for every $k > 0$. Since the right-hand side in (5.10) is bounded in $L^1(\Omega)$, we can apply a well-known compactness result (see [15,16]), which implies that a function u exists such that, up to extracting a subsequence,

$$u_n \to u \quad \text{and} \quad \nabla u_n \to \nabla u \quad \text{a.e. in } \Omega \tag{5.37}$$

with $u \in M^{\frac{\gamma q}{p-q}}(\Omega)$ and $|\nabla u| \in L^{t,p}(\Omega)$.
We deduce that $A(x, u_n, \nabla u_n)$ converges pointwise to $A(x, u, \nabla u)$ and $T_n(H(x, u_n, \nabla u_n))$ converges pointwise to $H(x, u, \nabla u)$. By Vitali's theorem, we can pass to the limit in (5.11). This proves that u is a solution obtained as a limit of approximations to (1.4). □

Theorem 5.6 *Assume* (1.5)–(1.8) *hold true with* $1 < p < N$ *and*

$$q = \frac{N(p-1)}{N-1}.$$

Suppose that f *satisfies one of the following conditions:*

(1) $f \in L^{1,\infty}(\log L)^N$ *and* (4.30) *and* (4.31) *hold true;*
(2) $f \in L(\log L)^{N-1}$ *and* (4.40) *hold true.*

Then there exists at least a solution to the problem (1.4) *obtained as a limit of approximations which satisfies* (4.32), *in case (1) holds, or* (4.41), *in case (2) holds. Furthermore, such a solution satisfies* (5.23) *in both cases.*

Proof As described at the end of Subsect. 5.2, we consider a weak solution $u_n \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ to the approximated problem (5.10). By the a priori estimates, we deduce that $|\nabla u_n|^q$ is bounded in $L(\log L)^{\tau}$, with $0 < \tau < N-1$, and, therefore, in $L^1(\Omega)$. Therefore, by growth assumption (1.8) on H, we deduce that $T_n(H(x, u_n, \nabla u_n)$ is bounded in $L^1(\Omega)$. Indeed, we have

$$\int_{\Omega} |T_n(H(x, \nabla u_n))| \, dx \leq \beta \int_{\Omega} |\nabla u_n|^q \, dx \leq c.$$

Moreover, for every fixed $k > 0$, $T_k(u_n)$ can be used as test function in (5.11) and we get

$$\int_\Omega |\nabla T_k(u_n)|^p \, dx \le k \int_\Omega T_n(H(x, u_n, \nabla u_n)) \, dx \le kC. \tag{5.38}$$

This implies that $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$. Since the right-hand side in (5.10) is bounded in $L^1(\Omega)$, by a priori estimates we can apply a well-known compactness result (see [15,16]), which implies that a function u exists such that, up to extracting a subsequence,

$$u_n \to u \quad \text{and} \quad \nabla u_n \to \nabla u \quad \text{a.e. in } \Omega. \tag{5.39}$$

Moreover, u satisfies (4.32), when (1) holds, or (4.41), when (2) holds; in both cases, $|\nabla u|$ satisfies (5.23).
We deduce that $A(x, u_n, \nabla u_n)$ converges pointwise to $A(x, u, \nabla u)$ and $T_n(H(x, u_n, \nabla u_n))$ converges pointwise to $H(x, u, \nabla u)$. Moreover, these sequences are equi-integrable. Indeed, since $p - 1 < q$, by growth condition on H and a priori estimates,

$$\begin{aligned}
\int_E |T_n(H(x, u_n, \nabla u_n))| \, dx &\le \int_E |H(x, u_n, \nabla u_n)| \, dx \\
&\le \frac{1}{\log^\tau(\mathcal{M}/|E|)} \int_0^{|E|} (|\nabla u_n|^*)^q \log^\tau(\mathcal{M}/s) \, dx \\
&\quad + \frac{\beta}{\log^{N-1}(\mathcal{M}/|E|)} \int_0^{|E|} f^*(s) \log^{N-1}(\mathcal{M}/s) \, ds \\
&\le C \frac{1}{\log^\tau(\mathcal{M}/|E|)} + \frac{1}{\log^{N-1}(\mathcal{M}/|E|)} \int_0^{|E|} f^*(s) \log^{N-1}(\mathcal{M}/s) \, ds.
\end{aligned}$$

Moreover, we have

$$\|A(x, u_n, \nabla u_n)\|_{L^{p'}(E)} \le |E|^{1-(p-1)/q} \left(\int_E |\nabla u_n|^q \, dx \right)^{\frac{p-1}{q}}$$

By Vitali's theorem, we can pass to the limit in (5.10). This proves that u is a solution obtained as a limit of approximations to (1.4). □

5.3 The case $p - 1 + \frac{p}{N} \le q \le p$

Once again we begin this section by proving a priori estimates.

Theorem 5.7 *Let us suppose that* (1.5)–(1.8) *hold true with* $1 < p < N$ *and*

$$p - 1 + \frac{p}{N} \le q \le p. \tag{5.40}$$

Let $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a weak solution to the problem* (1.4) *with* $f \in L^\infty(\Omega)$. *If the norm of* f *in* $M^\gamma(\Omega)$, *with* $\gamma = \frac{N(q-p+1)}{q}$ *is small enough, that is it satisfies* (4.18), *then*

$$\|\nabla u\|_{L^p} \leq C\,, \tag{5.41}$$

Moreover, if $p-1+\frac{p}{N} \leq q < p$,

$$\|u\|_{M^{\frac{q\gamma}{p-q}}} \leq C\,, \tag{5.42}$$

while, if $q = p$,

$$u^*(s) \leq \frac{X_0^{\frac{1}{p-1}}}{(N\omega_N^{1/N})^{\frac{p}{p-1}}} \log\left(\frac{|\Omega|}{s}\right), \qquad s \in (0, |\Omega|)\,, \tag{5.43}$$

where X_0 *is the smallest nonnegative solution to the Eq.* (4.21). *Here C is a positive constant which depends only on* $p,\ q,\ N,\ |\Omega|,\ \beta$ *and by the norm of the datum* f.

Proof In the case $q = p$ we have already proved the pointwise estimate in Theorem 4.2, while for the gradient estimate (5.41) we refer to [30]. So, we will consider only the case $q < p$, and we observe that in view of Theorem 4.2, we have just to prove (5.41). To this aim consider the function

$$\varphi(x) = \operatorname{sign}(u(x)) \int_0^{|u(x)|} \frac{1}{[\mu(t)]^\alpha}\, dt, \tag{5.44}$$

with

$$0 < \alpha < 1 - \frac{p}{N(q-p+1)}.$$

Since $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and therefore, it is a test function for the problem (1.4).

By assumptions (1.5) and (1.8), we get

$$\int_\Omega \frac{|\nabla u|^p}{[\mu(|u(x)|)]^\alpha}\, dx \leq \beta \int_\Omega |\nabla u|^q |\varphi|\, dx + \int_\Omega |f\varphi|\, dx. \tag{5.45}$$

We begin by evaluating the first integral on the right-hand side. By Hölder inequality, we get

$$\int_\Omega |\nabla u|^q |\varphi(x)|\, dx \leq \left(\int_\Omega \frac{|\nabla u|^p}{[\mu(|u(x)|)]^\alpha}\, dx\right)^{\frac{q}{p}} \left(\int_\Omega |\varphi(x)|^{\frac{p}{p-q}} [\mu(|u(x)|)]^{\frac{\alpha q}{p-q}}\, dx\right)^{1-\frac{q}{p}}. \tag{5.46}$$

By co-area formula and classical properties of rearrangements, we deduce

$$\int_\Omega |\varphi|^{\frac{p}{p-q}} [\mu(|u(x)|)]^{\frac{\alpha q}{p-q}}\, dx \leq \int_0^{|\Omega|} u^*(s)^{\frac{p}{p-q}} s^{-\alpha}\, ds.$$

and, by Theorem 4.2, since $\alpha < 1 - \frac{p}{N(q-p+1)}$,

$$\int_0^{|\Omega|} u^*(s)^{\frac{p}{p-q}} s^{-\alpha}\, ds \leq c \int_0^{|\Omega|} s^{-\frac{p}{N(q-p-1)}-\alpha}\, ds = c.$$

By (5.46), we conclude

$$\int_\Omega |\nabla u|^q |\varphi|\, dx \le c \left(\int_\Omega \frac{|\nabla u|^p}{[\mu(|u(x)|)]^\alpha}\, dx \right)^{\frac{q}{p}}. \tag{5.47}$$

Now, we evaluate the integral

$$\int_\Omega |f\varphi|\, dx.$$

By definition of φ, it follows

$$|\varphi(x)| \le \frac{|u(x)|}{(\mu(|u(x)|))^\alpha}.$$

Therefore, by Hardy inequality and Theorem 4.2, we deduce

$$\begin{aligned}\int_\Omega |f\varphi|\, dx &\le \int_\Omega \frac{|f(x)||u(x)|}{(\mu(|u(x)|))^\alpha}\, dx \le \int_0^{|\Omega|} \frac{f^*(s)u^*(s)}{s^\alpha}\, ds \\ &\le c \int_0^{|\Omega|} s^{-\frac{p}{N(q-p+1)}-\alpha}\, ds = c.\end{aligned} \tag{5.48}$$

Finally, by (5.45), (5.47), and (5.48), we obtain the following estimate

$$\int_\Omega \frac{|\nabla u|^p}{[\mu(u(x))]^\alpha}\, dx \le c\,.$$

By Hölder inequality, this yields the a priori estimate in $W_0^{1,p}(\Omega)$. □

As in the previous cases, the above a priori estimates allow to prove an existence results for weak solutions to the problem (1.4). We just state the following result (some details can be found, for example, in [32] and [28]).

Theorem 5.8 *Assume* (1.5)–(1.8) *hold true with* $1 < p < N$ *and* (5.40). *If* $f \in M^\gamma(\Omega)$, *with* $\gamma = \frac{N(q-p+1)}{q}$, *satisfies* (4.18), *then there exists at least a solution obtained as a limit of approximations to the problem* (1.4). *It satisfies either* (5.41) *and* (5.42), *if* $p-1+\frac{p}{N} \le q < p$ *or* (5.41) *and* (5.43), *if* $q = p$.

Remark 5.1 Let us remark that, if $p - 1 + \frac{p}{N} < q \le p$, the solution u given by the above theorem is a weak solution to problem (1.4), in view of the fact that f is an element of $W^{-1,p'}(\Omega)$. In the limit case $q = p - 1 + \frac{p}{N}$, u is a weak solution to problem (1.4) if we assume that f belongs to the smaller Lorentz space $L^{\frac{Np}{Np-N+p},p'}(\Omega)$, which is included in $W^{-1,p'}(\Omega)$.

Remark 5.2 We finally remark that uniqueness results for solutions to (1.4) are proved in [8,46] and [11] when

$$q \le p - 1 + \frac{p}{N}. \tag{5.49}$$

In [11], the uniqueness is proved under the sharp assumptions on the size of the norm of f decribed above.

The uniqueness when $q > p - 1 + \frac{p}{N}$ is still open; some results are contained in [6] and [7].

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References

1. Alaa, N., Pierre, M.: Weak solutions of some quasilinear elliptic equations with data measures. SIAM J. Math. Anal. **24**, 23–35 (1993)
2. Allegretto, W., Huang, Y.X.: A Picone's identity for the p-Laplacian and applications. Nonlinear Anal. **32**, 819–830 (1998)
3. Alvino, A., Ferone, V., Trombetti, G.: Estimates for the gradient of solutions of nonlinear elliptic equations with L^1 data. Ann. Mat. Pura Appl. (4) **178**, 129–142 (2000)
4. Alvino, A., Mercaldo, A.: Nonlinear elliptic equations with lower order terms and symmetrization methods. Boll. Un. Mat. Ital. (9) **1**, 645–661 (2008)
5. Alvino, A.: Sharp a priori estimates for some nonlinear elliptic problems. Boll. Accademia Gioenia di Scienze Naturali in Catania **46**, 2–11 (2013)
6. Arcoya, D., De Coster, C., Jeanjean, L., Tanaka, K.: Continuum of solutions for an elliptic problem with critical growth in the gradient. ArXiv
7. Barles, G., Murat, F.: Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions. Arch. Ration. Mech. Anal. **133**, 77–101 (1995)
8. Barles, G., Porretta, A.: Uniqueness for unbounded solutions to stationary viscous Hamilton-Jacobi equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5**, 107–136 (2006)
9. Bénilan, Ph, Boccardo, L., Gallouët, Th, Gariepy, R., Pierre, M., Vázquez, J.L.: An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **22**, 241–273 (1995)
10. Bennett, C., Rudnick, K.: On Lorentz–Zygmund spaces. Diss. Math. (Rozprawy Mat.) **175**, 1–67 (1980)
11. Betta, M.F., Di Nardo, R., Mercaldo, A., Perrotta, A.: Gradient estimates and comparison principle for some nonlinear elliptic equations. in preparation
12. Betta, M.F., Mercaldo, A., Murat, F., Porzio, M.M.: Existence and uniqueness results for nonlinear elliptic problems with a lower order term and measure datum. C. R. Math. Acad. Sci. Paris **334**, 757–762 (2002)
13. Betta, M.F., Mercaldo, A., Murat, F., Porzio, M.M.: Existence of renormalized solutions to nonlinear elliptic equations with a lower-order term and right-hand side a measure. J. Math. Pures Appl. (9) **82**, 90–124 (2003)
14. Bidaut-Véron, M.F., Hamid, H.A.: Correlation between two quasilinear elliptic problems with a source terms of order 0 or 1. Commun. Contemp. Math. **12**, 727–788 (2010)
15. Boccardo, L.: Hardy potentials and quasi-linear elliptic problems having natural growth terms. In: Bandle, C., Berestycki, H., Brighi, B., Brillard, A., Chipot, M., Coron, J.-M., Sbordone, C., Shafrir, I., Valente, V., Vergara Caffarelli, G. (Eds.) Elliptic and Parabolic Problems: a Special Tribute to the Work of Haim Brezis, Progress in Nonlinear Differential Equations and their Applications, **63**, Birkhäuser, Boston, 67–82 (2005)
16. Boccardo, L., Gallouët, T.: Nonlinear elliptic equations with right-hand side measure. Commun. Partial Differ. Equ. **17**, 641–655 (1992)
17. Boccardo, L., Murat, F.: Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. Nonlinear Anal. **19**, 581–597 (1992)
18. Boccardo, L., Murat, F., Puel, J.-P.: L^∞ estimate for some nonlinear elliptic partial differential equations and application to an existence result. SIAM J. Math. Anal. **23**, 326–333 (1992)
19. Capuzzo Dolcetta, I., Leoni, F., Porretta, A.: Hölder estimates for degenerate elliptic equations with coercive Hamiltonians. Trans. Am. Math. Soc. **362**, 4511–4536 (2010)
20. Chiacchio, F.: An existence result for a nonlinear problem in a limit case. Matematiche (Catania) **53**(1998), 293–304 (1999)
21. Chiacchio, F.: Regularity for solutions of nonlinear elliptic equations with natural growth in the gradient. Bull. Sci. Math. **124**, 57–74 (2000)

22. Dal Maso, G., Malusa, A.: Some properties of reachable solutions of nonlinear elliptic equations with measure data. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25**, 375–396 (1997)
23. Dal Maso, G., Murat, F., Orsina, L., Prignet, A.: Renormalized solutions of elliptic equations with general measure data. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28**, 741–808 (1999)
24. Dall'Aglio, A.: Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations. Ann. Mat. Pura Appl. (4) **170**, 207–240 (1996)
25. Della Pietra, F.: Existence results for nonuniformly elliptic equations with general growth in the gradient. Differ. Integral Equ. **21**, 821–836 (2008)
26. Della Pietra, F., Gavitone, N.: Anisotropic elliptic equations with general growth in the gradient and Hardy-type potentials. J. Differ. Equ. **255**, 3788–3810 (2013)
27. Ferone, V., Posteraro, M.R.: On a class of quasilinear elliptic equations with quadratic growth in the gradient. Nonlinear Anal. **20**, 703–711 (1993)
28. Ferone, V., Murat, F.: Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small. Nonlinear Anal. **42**, 1309–1326 (2000)
29. Ferone, V., Messano, B.: Comparison and existence results for classes of nonlinear elliptic equations with general growth in the gradient. Adv. Nonlinear Stud. **7**, 31–46 (2007)
30. Ferone, V., Murat, F.: Nonlinear elliptic equations with natural growth in the gradient and source terms in Lorentz spaces. J. Differ. Equ. **256**, 577–608 (2014)
31. Garcia Azorero, J.P., Peral Alonso, I.: Hardy inequalities and some critical elliptic and parabolic problems. J. Differ. Equ. **144**, 441–476 (1998)
32. Grenon, N., Murat, F., Porretta, A.: A priori estimates and existence for elliptic equations with gradient dependent term. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **13**(5):137–205 (2014)
33. Hansson, K., Maz'ya, V., Verbisky, I.: Criteria of solvability for multidimensional Riccati's equation. Arkiv för Mat. **37**, 247–276 (1999)
34. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, Cambridge (1964)
35. Hunt, R.: On $L(p, q)$ spaces. Enseignement Math. **12**, 249–276 (1966)
36. Kazdan, J.L., Kramer, R.J.: Invariant criteria for existence of solutions to second-order quasilinear elliptic equations. Commun. Pure Appl. Math. **31**, 619–645 (1978)
37. Leray, J., Lions, J.-L.: Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder. Bull. Soc. Math. France **93**, 97–107 (1965)
38. Lions, P.-L., Murat, F.: Sur les solutions renormalisées d'equations elliptiques non linéaires, manuscript
39. Lions, J.-L.: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod et Gauthier-Villars, Paris (1969)
40. Maderna, C., Salsa, S.: Dirichlet problem for elliptic equations with nonlinear first order terms: a comparison result. Ann. Mat. Pura Appl. **148**, 277–288 (1987)
41. Maderna, C., Pagani, C.D., Salsa, S.: Quasilinear elliptic equations with quadratic growth in the gradient. J. Differ. Equ. **97**, 54–70 (1992)
42. Maz'ya, V.G.: On weak solutions of the Dirichlet and Neumann problems. Trans. Moscow Math. Soc. **20**, 135–172 (1969)
43. Mercaldo, A.: A priori estimates and comparison principle for some nonlinear elliptic equations. Geometric properties for parabolics and elliptic PDE's. Springer INdAM Series **2**, 223–235 (2013)
44. Murat, F.: Soluciones renormalizadas de EDP elipticas no lineales, Preprint 93023, Laboratoire d'Analyse Numérique de l'Université Paris VI (1993)
45. O'Neil, R.: Integral transform and tensor products on Orlicz spaces and $L(p, q)$ spaces. J. Anal. Math. **21**, 1–276 (1968)
46. Porretta, A.: On the comparison principle for p-laplace operators with first order terms. In: Quaderni di Matematica **23**, Department of Mathematics, Seconda Università di Napoli, Caserta (2008)
47. Talenti, G.: Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces. Ann. Mat. Pura Appl. **120**, 160–184 (1979)
48. Talenti, G.: Linear elliptic P.D.E'.s: Level sets, rearrangements and a priori estimates of solutions. Boll. Un. Mat. **4–B**, 917–949 (1985)