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Sharp bounds for partition dimension of generalized Möbius ladders

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Abstract: The concept of minimal resolving partition and resolving set plays a pivotal role in diverse areas such as robot navigation, networking, optimization, mastermind games and coin weighing. It is hard to compute exact values of partition dimension for a graphic metric space, (G, d_G) and networks. In this article, we give the sharp upper bounds and lower bounds for the partition dimension of generalized Möbius ladders, $M_{m,n}$, for all $n \geq 3$ and $m \geq 2$.

Keywords: Metric dimension, Partition dimension, Generalized Möbius Ladder

MSC: 05C12, 05C15, 05C78

1 Introduction

Computer networks can be modeled on the grounds of graphs, where hosts, servers or hubs can be considered as vertices and edges – as connecting medium between them. Vertex is actually a possible location to find a fault or some damaged devices in a computer network. This idea somehow urged Slater and independently Harary and Meletr in [1] to uniquely recognize each vertex of a graph in a network so that a fault could be controlled in an efficient way. Thus, the basis for notion of locating sets and locating number of graphs came into existence. Since then, the resolving sets have been investigated a lot [1]. The resolving set contributes in various areas such as connected joins in graphs [2], network discovery [3–5], strategies for the mastermind games [3, 4], applications of pattern recognition, combinatorial optimization, image processing [6], pharmaceutical chemistry and game theory.

Consider a simple, connected graph G, and metric $d_G : V(G) \times V(G) \to \mathbb{N} \cup 0$, where $\mathbb{N}$ is the set of positive integers and $d_G(x, y)$ is the minimum number of edges in any path between x and y. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation $r(v|W)$ of v with respect to W is the k–tuple $(d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. If distinct vertices of G have distinct representation with respect to W, then W is called a resolving set of G, see [1]. Such resolving set with minimum cardinality is a basis of G and metric dimension of G, denoted by $dim(G)$ is its cardinality, [7, 8].

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Buczkowski et al. established metric dimension of wheel W_n to be $\lfloor \frac{2n+2}{5} \rfloor$ for $n \geq 7$ [9], Caceres et. al. [10] found that the metric dimension of fan is $\lfloor \frac{2n+2}{5} \rfloor$ for $n \geq 7$ and Tomescu et. al. [11] determined the dimension of Jahangir graphs J_{2n} to be $\lfloor \frac{2n}{3} \rfloor$ for all $n \geq 4$.

A particular metric-feature of the family of graphs is independence of metric dimension on the particular element of the family. A connected graph has constant metric dimension if $dim(G) = k$ where $k \in \mathbf{Z}^+$. In [8] Chartrand et. al. proved that a graph has constant metric dimension 1 iff it is a path. In [12] the authors discussed some families of constant meric dimensions. The authors computed metric dimension of wheels in [13] and uni-cyclic graphs in [14]. The authors in [15] computed metric dimension of alpha boron nanotubes. Javaid et. al. computed metric dimension of $P(n, 3)$ and established new results on metric dimension of rotationally-symmetric graph. Murtaza et. al. computed partial results of metric dimension of Möbius ladder in [16] whereas Munir et. al. computed exact and complete results for metric dimension of Möbius Ladders in [17].

A variant of metric dimension of a connected graph is a partition dimension of graph introduced in [19–23] given as : Let G be a connected graph, a subset $S \subset V(G)$ and a vertex v, distance $d(v, S) = min\{d(v, x) : x \in S\}$. If $\Pi = \{S_1, ...S_t\}$ is an ordered t-partition of $V(G)$, then $r(v|\Pi) = \{d(v, S_1), ..., d(v, S_t)\}$ is the t-tuple representation of v with respect to Π. If this t-tuple representation of v, $r(v|\Pi)$ for all $v \in V(G)$ being all distinct, then this Π is called a resolving partition and the minimum cardinality of such resolving partition is a partition dimension, represented as $pd(G)$.

A natural question may be asked: are partition dimension and metric dimension related in some way? In [20, 21], Cartrand et. al. proved that $pd(G) \leq \beta(G) + 1$ for a non-trivial connected graph G. But in [22, 23], Tomescu et. al. proved that it can be much smaller than the metric dimension. In fact, the authors completed the list of all 23 examples of connected graphs of order n having partition dimensions 2, $n - 1$ or n. They also gave an example of graphs with finite partition dimension but those which have infinite metric dimension. Recently, Hernando et. al. has proved that there are only 15 families of such type. Tomescu et. al. computed the bounds for the partition dimension of wheel graph in [23]. In [24], the authors computed some bounds for metric and partition dimension of a connected graph. In [25], the authors obtained some sharp bounds for the partition dimension of unicyclic graphs.

Chartrand et al. proved in [22] that if G is a connected graph of order $n \geq 2$ then $pd(G) = 2$ if and only if G is a path, $pd(G) = n$ if and only if $G = K_n$and for $n \geq 5$ $pd(G) = n - 1$ if and only if G is one of the graphs $K_{1,n-1}$,$K_n - e$, $K_1 + (K1 \cup K_{n+2})$. In [22] Tomescu and Imran studied infinite regular graphs which are generated by tailings of the plane by regular triangles and hexagons. They proved that these graphs have no finite metric bases but their partition dimension is finite and they evaluated this dimension in some cases. In [23], they computed a partition dimension and a connected partition dimension of wheel graphs and showed that $n \geq 4$, $\lceil (2n)^{\frac{1}{3}} \rceil \leq pd(G) \leq 2\lceil n^{\frac{1}{2}} \rceil + 1$. The following lemma gives a general upper bounds for the partition dimension of a graph of size n.

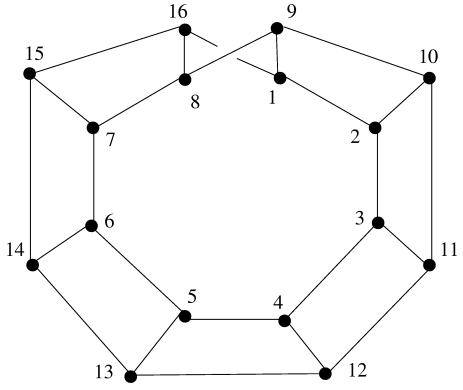
Lemma 1.1. *If* $| G | \geq 3$, *then* $pd(G) \leq n - diam(G) + 1$

In this article we want to compute sharp bounds for partition dimension of Generalized Möbius ladders.

2 Generalized Möbius ladders

The classical *Möbius ladder* M_n is a cubic circulant graph with an even number of vertices, formed from an n-cycle by adding edges connecting opposite pair of vertices in the cycle, except with two pairs which are connected with a twist, as you can see in the figure:

Fig. 1. Möbius ladder M_{16}



This graph has been an active area of research. For instance, [16, 17] give complete results for its metric dimension. In [26] the authors computed a distance labeling of this graph and also introduced its generalization referred to as Möbius ladder. In [27], the authors not only redefined this generalization in a novel way but also computed metric dimension of $M_{m,n}$. They also obtained the results of [16, 17] as easy consequences of the results in [27]. Consider the Cartesian product $P_m \times P_n$ of paths P_m and P_n with vertices $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, u_n$, respectively. Take a 180^o twist and identify the vertices $(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_n)$ with the vertices $(u_m, v_n), (u_m, v_{n-1}), \ldots, (u_m, v_1)$, respectively, and identify the edge $\big((u_1, i), (u_1, i+1)\big)$ with the edge $\big((u_m, v_{n+1-i}), (u_m, v_{n-i})\big)$, where $1 \le i \le n-1$. What we receive is the generalized Möbius ladder $M_{m,n}$. You may observe that we receive the usual Möbius ladder for $n = 2$ and for any odd integer $m \ge 4$. You can see $M_{7,3}$ in the following figure.

Fig. 2. P_7

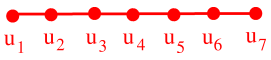
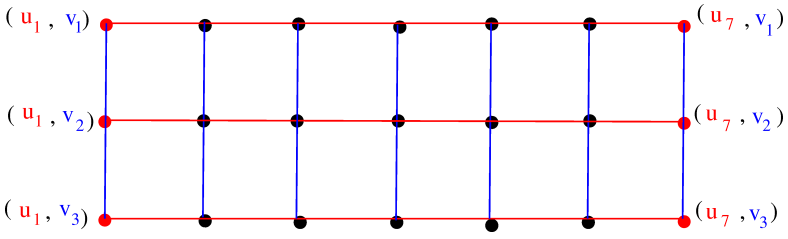


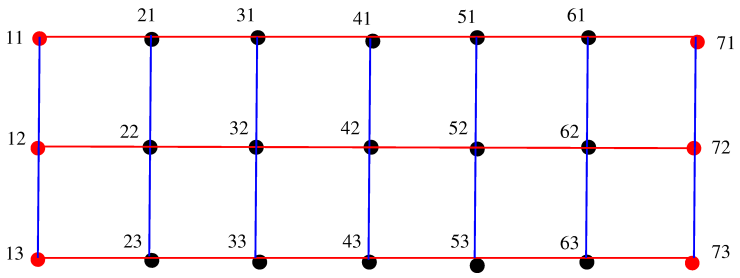
Fig. 3. P_3

Fig. 4. $P_7 \times P_3$



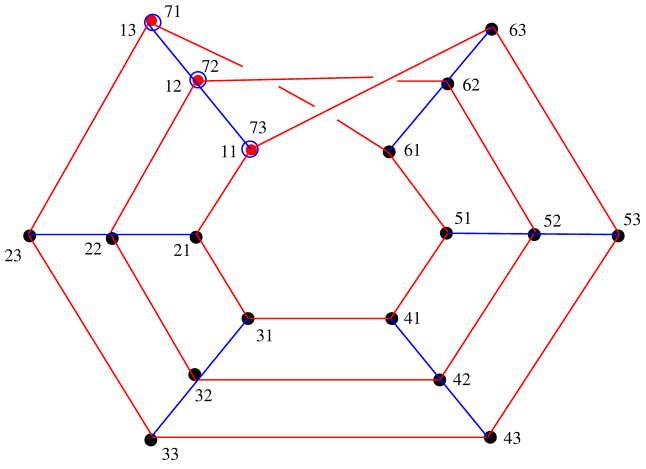
For brevity we shall use the symbol v_{ij} (or simply ij) to represent the vertex (u_i, v_j) of $M_{m,n}$, as you can see in the figure:

Fig. 5. $P_7 \times P_3$ with complete simple labels



The generalized Möbius ladder obtained from $P_7 \times P_3$ is:

Fig. 6. $M_{7,3}$



So the generalized Möbius ladder $M_{m,n}$ is a non-regular simple connected graph on $n(m-1)$ vertices. This article deals with the computation of sharp upper bounds and lower bounds for partition and metric dimensions of $M_{m,n}$.

3 Main results and discussions

In this part we give our main results. We begin with the sharp upper bounds for the partition dimension of $M_{m,n}$. Then we move towards the lower bounds.

Theorem 3.1. *For $m \geq 3$ and $n \geq 2$*

$$3 \leq pd(M_{m,n}) \leq \begin{cases} 5, \text{ when } n \equiv 1(mod2) \text{and } m \equiv 1(mod2),\ m-n \geq 4 \\ 4, \text{ when } n \equiv 0(mod2) \text{and } m \equiv 1(mod2) \\ 4 \text{ when } n \equiv 1(mod2) \text{and } m \equiv 0(mod2) \\ 5 \text{ when } n \equiv 0(mod2) \text{and } m \equiv 0(mod2),\ m-n \geq 4 \end{cases}$$

At first we compute the upper bounds. We construct a general resolving partition on a case by case basis.

3.1 Upper bound

Proof. We divide the proof in two cases on the basis of parities of m and n.

Case I: When m and n are of opposite parity

Let $\Pi = \{S_1, S_2, S_3, S_4\}$ Where $S_1 = \{V_{1,1}\}$, $S_2 = \{V_{1,n}\}$, $S_3 = \{V_{1,2}, V_{1,3}, \ldots, V_{1,n-1}, V_{2,1}, V_{2,2}, \ldots\ldots, V_{2,n}, \ldots\ldots, V_{m-2,1}, V_{m-2,2}, \ldots, V_{m-2,n}, V_{m-1,2}, V_{m-1,3}, \ldots, V_{m-1,n}\}$ $S_4 = \{V_{m-1,1}\}$. We prove that Π is a resolving partition for $M_{m,n}$. To find distance vectors we use two parameters q, i and depending on their different values we divide the entries of distance vectors into four steps.

Step I: Distances of S_1 with all vertices of $M_{m,n}$.

In this case for each value of $q \in \{1, 2, \ldots, n\}$ the parameter i varies from 1 to $m-1$. The entries of different vectors are

$$d(S_1, V_{i,q}) = \begin{cases} i+q-2, 1 \le i \le \frac{1}{2}(m+n-2q+1) \\ m+n-q-i, \frac{1}{2}(m+n-2q+3) \le i \le m-1 \end{cases}$$

Step II: Distances of S_2 with all vertices of $M_{m,n}$.

For each value of $q \in \{1, 2, \ldots, n\}$ the parameter i varies from 1 to m - 1 and we get $d(S_2, V_{i,q}) = d(S_1, V_{i,n+1-q})$.

Step III : Distances of S_3 with all vertices of $M_{m,n}$.

Here for each value of $q \in \{1, 2, \ldots, n\}$ the parameter i varies from 1 to m - 1 and we have

$$d(S_3, V_{i,q}) = \begin{cases} 1, if \text{ i = 1,q = 1,q = n} \\ 1, if i = m-1, q = n \\ 0, otherwise \end{cases}$$

Step IV: Distances of S_4 with all vertices of $M_{m,n}$.

Here we have two parts

a) For q = 1 , we have

$$d(S_4, V_{i,q}) = \begin{cases} i+n-1, 1 \le i \le \frac{1}{2}(m-n-1) \\ m-1-i, \frac{1}{2}(m-n+1) \le i \le m-1 \end{cases}$$

b) For each value of $q \in \{2, \ldots, n\}$ the parameter i varies from 1 to m - 1 and we have $d(S_4, V_{i,q}) = d(S_1, V_{i,n+2-q})$.

These representations are distinct in at least one coordinate. So Π is a resolving partition for $M_{m,n}$ so clearly $pd(M_{m,n}) \le 4$. □

Example. *Clearly $pd(M_{9,4}) \le 4$ as the resolving partition for $M_{9,4}$ is $\Pi = \{S_1, S_2, S_3, S_4\}$ where $S_1 = \{V_{1,1}\}$, $S_2 = \{V_{1,4}\}$, $S_3 = \{V_{1,2}, V_{1,3}, V_{2,1}, V_{2,2}, V_{2,3}, V_{2,4}$, ..., $V_{7,1}, V_{7,2}, V_{7,4}, V_{8,2}, V_{8,3}, V_{8,4}\}$, $S_4 = \{V_{8,1}\}$.*

The representations of different vertices of $M_{9,4}$ with respect to Π are

$V_{1,1}(0,3,1,4), V_{1,2}(1,2,0,3), V_{1,3}(2,1,0,2), V_{1,4}(3,0,1,1),$
$V_{2,1}(1,4,0,5), V_{2,2}(2,3,0,4), V_{2,3}(3,2,0,3), V_{2,4}(4,1,0,2),$
$V_{3,1}(2,5,0,5), V_{3,2}(3,4,0,5), V_{3,3}(4,3,0,4), V_{3,4}(5,2,0,3),$
$V_{4,1}(3,5,0,4), V_{4,2}(4,5,0,5), V_{4,3}(5,4,0,5), V_{4,4}(5,3,0,4),$
$V_{5,1}(4,4,0,3), V_{5,2}(5,5,0,4), V_{5,3}(5,5,0,5), V_{5,4}(4,4,0,5),$
$V_{6,1}(5,3,0,2), V_{6,2}(5,4,0,3), V_{6,3}(4,5,0,4), V_{6,4}(3,5,0,5),$
$V_{7,1}(5,2,0,1), V_{7,2}(4,3,0,2), V_{7,3}(3,4,0,3), V_{7,4}(2,5,0,4),$
$V_{8,1}(4,1,1,0), V_{8,2}(3,2,0,1), V_{8,3}(2,3,0,2), V_{8,4}(1,4,0,3)$

Case II: when m and n are of same parity: We want to prove that $pd(M_{m,n}) \le 5$ by constructing a general resolving partition of size 5, for $m - n \ge 4$ and m, n are of same parity.

Proof. Let $\Pi = \{S_1, S_2, S_3, S_4, S_5\}$ where $S_1 = \{V_{1,1}\}$, $S_2 = \{V_{1,n}\}$, $S_3 = \{V_{1,2}, V_{1,3}, \ldots, V_{1,n-1}, V_{2,1}, V_{2,2}, \ldots\ldots, V_{2,n}, \ldots\ldots, V_{m-2,1}, V_{m-2,2}, \ldots, V_{m-2,n}, V_{m-1,2}, V_{m-1,3}, \ldots, V_{m-1,n-1}\}$, $S_4 = \{V_{m-1,1}\}$, $S_5 = \{V_{m-1,n}\}$.

We prove that Π is a resolving partition for $M_{m,n}$. To find distance vectors we use two parameters q , i and depending on their different values we divide the entries of distance vectors into five steps.

Step I: Distances of S_1 with all vertices of $M_{m,n}$.

In this case for each value of $q \in \{1, 2, \ldots, n\}$ the parameter i varies from 1 to m - 1. The entries of different vectors are

$$d(S_1, V_{i,q}) = \begin{cases} i + q - 2, 1 \le i \le \frac{1}{2}(m + n - 2q + 2) \\ m + n - q - i, \frac{1}{2}(m + n - 2q + 4) \le i \le m - 1 \end{cases}$$

Step II: Distances of S_2 with all vertices of $M_{m,n}$.

For each value of $q \in \{1, 2, \ldots, n\}$ the parameter i varies from 1 to m - 1 and we get $d(S_2, V_{i,q}) = d(S_1, V_{i,n+1-q})$.

Step III : Distances of S_3 with all vertices of $M_{m,n}$.

Here for each value of $q \in \{1, 2, \ldots, n\}$ the parameter i varies from 1 to m - 1 and we have

$$d(S_3, V_{i,q}) = \begin{cases} 1, if \text{ i = 1,q = 1,q = n} \\ 1, if i = m - 1, q = 1, q = n \\ 0, otherwise \end{cases}$$

Step IV : Distances of S_4 with all vertices of $M_{m,n}$. Here we have two parts

a) For q = 1 , we have

$$d(S_4, V_{i,q}) = \begin{cases} i + n - 1, 1 \le i \le \frac{1}{2}(m - n) \\ m - 1 - i, \frac{1}{2}(m - n + 2) \le i \le m - 1 \end{cases}$$

b) For each value of $q \in \{2, \ldots, n\}$ the parameter i varies from 1 to m - 1 and we have $d(S_4, V_{i,q}) = d(S_1, V_{i,n+2-q})$

Step V : Distances of S_5 with all vertices of $M_{m,n}$.

Here for each value of $\in \{1, 2, \ldots, n\}$ the parameter i varies from 1 to m - 1 and we have $d(S_5, V_{i,q}) = d(S_4, V_{i,n+1-q})$.

These representations are distinct in at least one coordinate. So Π is a resolving partition for $M_{m,n}$. Since there is no 4 resolving partition for $M_{m,n}$, hence Π is a minimal resolving partition for $M_{m,n}$. So partition dimension of $M_{m,n}$ is 5. □

Example. *The partition dimension of $M_{9,3}$ is 5. The resolving partition for $M_{9,3}$ is $\Pi = \{S_1, S_2, S_3, S_4, S_5\}$.Where*

$S_1 = \{V_{1,1}\}$

$S_2 = \{V_{1,3}\}$

$S_3 = \{V_{1,2}, V_{2,1}, V_{2,2}, V_{2,3}, \ldots\ldots, V_{7,1}, V_{7,2}, V_{7,3}, V_{8,2}, V_{8,3}\}$

$S_4 = \{V_{8,1}\}$

$S_5 = \{V_{8,3}\}$

The representations of different vertices of $M_{9,3}$ with respect to Π are

$V_{1,1}(0, 2, 1, 3, 1), V_{1,2}(1, 1, 0, 2, 2), V_{1,3}(2, 0, 1, 1, 2),$

$V_{2,1}(1, 3, 0, 4, 2), V_{2,2}(2, 2, 0, 3, 3), V_{2,3}(3, 1, 0, 2, 4),$

$V_{3,1}(2, 4, 0, 5, 3), V_{3,2}(3, 3, 0, 4, 4), V_{3,3}(4, 2, 0, 3, 5),$

$V_{4,1}(3, 5, 0, 4, 4), V_{4,2}(4, 4, 0, 5, 5), V_{4,3}(5, 3, 0, 4, 4),$

$V_{5,1}(4, 4, 0, 3, 5), V_{5,2}(5, 5, 0, 4, 4), V_{5,3}(4, 4, 0, 5, 3),$

$V_{6,1}(5, 3, 0, 2, 4), V_{6,2}(4, 4, 0, 3, 3), V_{6,3}(3, 5, 0, 4, 2),$

$V_{7,1}(4, 2, 0, 1, 3), V_{7,2}(3, 3, 0, 2, 2), V_{7,3}(2, 4, 0, 3, 1),$

$V_{8,1}(3, 1, 1, 0, 2), V_{8,2}(2, 2, 0, 1, 1), V_{8,3}(1, 3, 1, 2, 0,)$

3.2 Lower bound

Proof. It is clear that $2 < pd(M_{m,n})$ as it is not a path, [8]. So it is obvious that $3 \leq pd(M_{n,m})$. □

Theorem 3.2. *For $m \geq 3$ and $n \geq 2$*

$$2 \leq \beta(M_{m,n}) \leq \begin{cases} 4, \text{ when } n \equiv 1(mod2) \text{and } m \equiv 1(mod2),\ m-n \geq 4 \\ 3, \text{ when } n \equiv 0(mod2) \text{and } m \equiv 1(mod2) \\ 3 \text{ when } n \equiv 1(mod2) \text{and } m \equiv 0(mod2) \\ 4 \text{ when } n \equiv 0(mod2) \text{and } m \equiv 0(mod2),\ m-n \geq 4 \end{cases}$$

Proof. Proof is just straightforward after taking into account the fundamental inequality between metric and patrtition dimensions. □

4 Conclusions and open problems

In this article we have computed sharp upper bounds for the partition dimension of the generalized Möbius ladders and arrive at the following results

Theorem 4.1. *For $m \geq 3$ and $n \geq 2$*

$$3 \leq pd(M_{m,n}) \leq \begin{cases} 5, \text{ when } n \equiv 1(mod2) \text{and } m \equiv 1(mod2),\ m-n \geq 4 \\ 4, \text{ when } n \equiv 0(mod2) \text{and } m \equiv 1(mod2) \\ 4 \text{ when } n \equiv 1(mod2) \text{and } m \equiv 0(mod2) \\ 5 \text{ when } n \equiv 0(mod2) \text{and } m \equiv 0(mod2),\ m-n \geq 4 \end{cases}$$

and

Theorem 4.2. *For $m \geq 3$ and $n \geq 2$*

$$2 \leq \beta(M_{m,n}) \leq \begin{cases} 4, \text{ when } n \equiv 1(mod2) \text{and } m \equiv 1(mod2),\ m-n \geq 4 \\ 3, \text{ when } n \equiv 0(mod2) \text{and } m \equiv 1(mod2) \\ 3 \text{ when } n \equiv 1(mod2) \text{and } m \equiv 0(mod2) \\ 4 \text{ when } n \equiv 0(mod2) \text{and } m \equiv 0(mod2),\ m-n \geq 4 \end{cases}$$

At the same time we pose natural open problems regarding the exact values of partition dimension, $pd(M_{m,n})$ and $\beta(M_{m,n})$, and sharp lower bounds for this new family of graphs. For further problems about the dimensions of graphs please see [28, 29].

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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