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SHARP BOUNDS ON CASTELNUOVO-MUMFORD REGULARITY

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ABSTRACT. The Castelnuovo-Mumford regularity is one of the most important invariants in studying the minimal free resolution of the defining ideals of the projective varieties. There are some bounds on the Castelnuovo-Mumford regularity of the projective variety in terms of the other basic measures such as dimension, codimension and degree.

In this paper we consider an upper bound on the regularity $\text{reg}(X)$ of a nondegenerate projective variety X, $\text{reg}(X) \leq \lceil (\deg(X)-1)/\operatorname{codim}(X)\rceil + k\cdot \dim(X)$, provided X is k-Buchsbaum for $k \geq 1$, and investigate the projective variety with its Castelnuovo-Mumford regularity having such an upper bound.

1. Introduction

Let X be a projective scheme of $\mathbb{P}^N_K$ over a field K. Let $S = K[x_0, \cdots, x_N]$ be the polynomial ring and $\mathfrak{m} = (x_0, \cdots, x_N)$ be the irrelevant ideal. Then we put $\mathbb{P}^N_K = \text{Proj}(S)$. We denote by $\mathcal{I}_X$ the ideal sheaf of X. Let m be an integer. Then X is said to be m-regular if $\text{H}^i(\mathbb{P}^N_K, \mathcal{I}_X(m-i)) = 0$ for all $i \geq 1$. The Castelnuovo-Mumford regularity of $X \subseteq \mathbb{P}^N_K$, introduced by Mumford by generalizing ideas of Castelnuovo, is the least such integer m and is denoted by $\text{reg}(X)$. The interest in this concept stems partly from the well-known fact that X is m-regular if and only if for every $p \geq 0$ the minimal generators of the p-th syzygy module of the defining ideal I of $X \subseteq \mathbb{P}^N_K$ occur in degree $\leq m+p$, see, e.g., [4], [5], [6].

Let k be a nonnegative integer. Then X is called k-Buchsbaum if the graded S-module $\text{M}^i(X) = \bigoplus_{\ell\in\mathbb{Z}} \text{H}^i(\mathbb{P}^N_K, \mathcal{I}_X(\ell))$, called the deficiency module of X, is annihilated by $\mathfrak{m}^k$ for $1 \leq i \leq \dim(X)$, see, e.g., [17], [18]. Further, we call the minimal nonnegative integer k, if it exists, such that X is k-Buchsbaum, as the Ellia-Migliore-Miró Roig number of X and denote it by $k(X)$. In case X is not k-Buchsbaum for all $k \geq 0$, we put $k(X) = \infty$. It is known that the numbers $k(X)$ are invariant in a liaison class, see, e.g., [17], [24]. Note that $k(X) < \infty$ if and only if X is locally Cohen-Macaulay and equi-dimensional.

In what follows, for a rational number $\ell \in \mathbb{Q}$, we write $\lceil \ell \rceil$ for the minimal integer which is larger than or equal to ℓ, and $\lfloor \ell \rfloor$ for the maximal integer which is smaller than or equal to ℓ.

In recent years upper bounds on the Castelnuovo-Mumford regularity of a projective variety X have been given by several authors in terms of $\dim(X)$, $\deg(X)$,

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codim(X) and $k(X)$, see, e.g., [13], [14], [15], [19], [22], [23]. The following bound, first obtained in [23], is the most optimal among the known results. Even so, whether such a bound is sharp is still a question.

Proposition 1.1 (see [19], [23]). *Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}^N_K$ over an algebraically closed field K of characteristic zero. Then we have*

$$\operatorname{reg}(X) \leq \left\lceil \frac{\deg(X)-1}{\operatorname{codim}(X)} \right\rceil + \max\{k(X)\dim(X), 1\}.$$

The purpose of this paper is to study sharp examples which attain the upper bounds of the inequality in Proposition 1.1 and to show that a projective variety having such property must be a curve on a surface of minimal degree if its degree is large enough.

Theorem 1.2. *Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}^N_K$ over an algebraically closed field K of characteristic zero. Assume that $k(X) \geq 1$, $\deg(X) \geq 2\operatorname{codim}(X)^2 + \operatorname{codim}(X) + 2$ and*

$$\operatorname{reg}(X) = \left\lceil \frac{\deg(X)-1}{\operatorname{codim}(X)} \right\rceil + k(X)\dim(X).$$

Then $\dim(X) = 1$ and X is a curve on a rational ruled surface Y.

The results related to Theorem 1.2 are obtained in [20], [26] for arithmetically Cohen-Macaulay varieties, that is, $k(X) = 0$, especially [20] for the positive characteristic case, and in [28] for arithmetically Buchsbaum curves, that is, $k(X) = 1$ and $\dim(X) = 1$; also see [21] for arithmetically Buchsbaum varieties.

More precisely, we obtain the following classification of the projective variety with its Castelnuovo-Mumford regularity having such upper bound.

Theorem 1.3. *Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}^N_K$ satisfying the assumptions of (1.2). Then X is a divisor on a rational ruled surface Y constructed as follows:*

Let $\pi : Y = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1_K$ be a projective bundle, see, e.g., [11, (V.2)], *where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_K} \oplus \mathcal{O}_{\mathbb{P}^1_K}(-e)$ for some $e \geq 0$. Let Z be a minimal section of π corresponding to the natural map $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1_K}(-e)$ and F be a fibre corresponding to $\pi^*\mathcal{O}_{\mathbb{P}^1_K}(1)$. We have an embedding of Y in $\mathbb{P}^N_K$ by a very ample sheaf corresponding to a divisor $H = Z + n \cdot F$ $(n > e)$, where $N = 2n - e + 1$. Then X is a divisor on Y linearly equivalent to $a \cdot Z + b \cdot F$ such that $a \geq 1$ and $an + 2 \leq b \leq (a+2)n - e + 1$.*

In this case, $\operatorname{codim}(X) = 2n - e$, $\deg(X) = a(n-e) + b$, $k(X) = \lfloor (b - an - 2)/(n-e) \rfloor + 1$ *and* $\operatorname{reg}(X) = \lfloor (b - an - 2)/(n-e) \rfloor + a + 2$.

This result indicates that the inequality

$$\operatorname{reg}(X) \leq \left\lceil \frac{\deg(X)-1}{\operatorname{codim}(X)} \right\rceil + \max\{k(X), 1\}$$

is sharp for a nondegenerate irreducible reduced projective curve X in $\mathbb{P}^N_K$ over an algebraically closed field K of characteristic zero. In fact, for positive integers c and t with $2 \leq c \leq t-2$, we take the integers q and r satisfying that $t - 2 = cq + r$ and $0 \leq r \leq c-1$. Then we define a non-empty set

$$\mathfrak{S}(c,t) = \{1 + \lfloor 2r/\ell \rfloor \mid \ell \in \mathbb{Z},\, 2 \leq \ell \leq c\}.$$

Note that every element $k \in \mathfrak{S}(c,t)$ satisfies $1 \leq k \leq r + 1 (\leq c)$.

Theorem 1.4. *Let c, t and k be positive integers with* $2 \leq c \leq t-2$. *Let us put a subset* $\mathfrak{S}(c,t)$ *of* $\mathbb{Z}$ *as above. Let K be an algebraically closed field.*

(i) *If* $k \in \mathfrak{S}(c,t)$, *then there exists a nondegenerate irreducible smooth projective curve X in* $\mathbb{P}^{c+1}_K$ *with* $\deg(X) = t$, $k(X) = k$ *and* $\operatorname{reg}(X) = \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + k(X)$.

(ii) *Assume that* $t \geq 2c^2 + c + 2$ *and* $\operatorname{char}(K) = 0$. *If there exists a nondegenerate irreducible reduced projective curve X in* $\mathbb{P}^{c+1}_K$ *with* $\deg(X) = t$, $k(X) = k$ *and* $\operatorname{reg}(X) = \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + k(X)$, *then* $k \in \mathfrak{S}(c,t)$.

Theorem 1.5. *Let K be an algebraically closed field. For any given positive integers c and k with* $c \geq k$, *there exists a nondegenerate irreducible smooth projective curve X in* $\mathbb{P}^{c+1}_K$ *with* $k(X) = k$ *and* $\operatorname{reg}(X) = \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + k(X)$.

These results motivate us to state the following problem.

Problem 1.6. *Does the inequality* $\operatorname{reg}(X) \leq \lceil (\deg(X)-1)/\operatorname{codim}(X) \rceil + k(X)$ *hold for a nondegenerate irreducible reduced projective variety X with* $k(X) \geq 1$ *over an algebraically closed field K?*

For the case $\dim(X) = 1$ and $\operatorname{char}(K) = 0$, Proposition 1.1 and the theorems in this paper are answers to this problem and show that the inequality is best possible. The theorems give a classification of projective varieties with the regularity bound under the assumption $\deg(X) \gg 0$. However, the assumption is indispensable. In fact, the canonical embedding of a non-hyperelliptic curve C in $\mathbb{P}^{g-1}_K$ with the genus of $g \geq 5$, gives the upper bound of $\operatorname{reg}(C)$, while not contained in any surface of minimal degree, see [28]. On the other hand, you can find how scarce the curves are which achieve the bound. If C is a space curve with the degree bound and the regularity bound, then C is a divisor of either type $(a, a+2)$ or type $(a, a+3)$ on a smooth quadric surface from Theorem 1.3. Accordingly we describe the following problem arising from our consideration.

Problem 1.7. *Classify all the nondegenerate irreducible reduced projective curves C with* $\operatorname{reg}(C) = \lceil (\deg(C) - 1)/\operatorname{codim}(C) \rceil + \max\{k(C), 1\}$. *Or describe the best possible condition that the curve C having the equality above is contained in a surface of minimal degree.*

Finally, we conclude this section by stating Hoa's conjectures.

Conjecture 1.8 ([12])**.** *Let X be a nondegenerate irreducible reduced projective variety in* $\mathbb{P}^N_K$ *over an algebraically closed field K. Let k be a positive integer. Assume that, for all* $r \geq 0$, *the variety* $X \cap L$ *has the Ellia-Migliore-Miró Roig number* $k(X \cap L) \leq k$ *for any* $(N-r)$*-plane L with* $\dim(X \cap L) = \dim(X) - r$, *in other words, X is* $(k, \dim(X))$*-Buchsbaum by using the terminology of* [13], [15]. *Then we have*

$$\operatorname{reg}(X) \leq \left\lceil \frac{\deg(X) - 1}{\operatorname{codim}(X)} \right\rceil + k.$$

Furthermore, assume that $\deg(X)$ *is large enough. Then the equality holds only if X is a divisor on a variety of minimal degree.*

Throughout this paper we only consider the characteristic zero case. However, if you apply some results of [2], [3], you might partially have the corresponding results in positive characteristic case.

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2. Bounds on Castelnuovo-Mumford regularity

This section is devoted to the proof of the theorems stated in §1.

First, we describe a sketch of a proof of the upper bound of the Castelnuovo-Mumford regularity, following, e.g., [19, Section 4], in order to make clear what the sharp examples should be.

Let $R = K[R_1]$ be a finitely generated graded algebra over a field K. We denote by $\mathfrak{m}$ the irrelevant ideal of R. Let M be a finitely generated graded R-module with $\dim(M) = d+1 > 0$. We write $[M]_n$ for the n-th graded piece of M, and $M(p)$ for the graded module with $[M(p)]_n = [M]_{p+n}$. Then, for $i = 0, \cdots, d+1$, we set

$$a_i(M) = \max\{n \mid [\mathrm{H}^i_{\mathfrak{m}}(M)]_n \neq 0\}$$

if the max exists, and $a_i(M) = -\infty$ otherwise. In particular, we set $a(M) = a_{d+1}(M)$. The Castelnuovo-Mumford regularity of M is defined as follows:

$$\mathrm{reg}(M) = \max\{a_i(M) + i \mid i = 0, \cdots, d+1\}.$$

For an integer $k \geq 0$, the graded R-module M is called k-Buchsbaum if $\mathfrak{m}^k \mathrm{H}^i_{\mathfrak{m}}(M) = 0$ for all $i = 0, \cdots, d$. The following result is an easy consequence of the proof of [19, (2.7.2)].

Proposition 2.1. *Let M be a finitely generated graded R-module with* $\dim(M) = d+1 > 0$. *Let k be a positive integer. Assume that M is k-Buchsbaum. Then*

$$a_i(M) \leq \max\{a_j(M) + j - i + k \mid j = i+1, \cdots, d+1\}$$

for $i = 1, \cdots, d-1$, and

$$a_d(M) \leq a(M/hM) + k - 1$$

for any linear parameter $h \in R_1$ for the graded R-module M. Furthermore, the equalities hold only if $a_i(M) \neq -\infty$ and

$$\left[\mathrm{H}^i_{\mathfrak{m}}(R)/h\mathrm{H}^i_{\mathfrak{m}}(R)\right]_\ell = 0, \qquad \ell \geq a_i(R) - k + 2,$$

for $i = 1, \cdots, d$. Consequently, for any integer $1 \leq i \leq d$, we have

$$a_i(M) + i \leq a(M/hM) + d + k(d+1-i) - 1,$$

for any linear parameter $h \in R_1$ for the R-module M.

Let X be a projective scheme in $\mathbb{P}^N_K = \mathrm{Proj}(S)$, where S is the polynomial ring $K[x_0, \cdots, x_N]$. Let I be the defining ideal $\bigoplus_{\ell \in \mathbb{Z}} \Gamma(\mathbb{P}^N_K, \mathcal{I}_X(\ell))$ of X and R be the coordinate ring S/I of X. Then we see that $\mathrm{reg}(X) = \mathrm{reg}(I) = \mathrm{reg}(R) + 1$. By taking $M = R$ in the above proposition, we have the following bound by using the Ellia-Migliore-Miró Roig number $k(X)$.

Lemma 2.2. *Let X be a projective scheme in $\mathbb{P}^N_K$. Let R be the coordinate ring of X. Then*

$$\mathrm{reg}(X) \leq a(R/hR) + \dim(X) + \max\{k(X)\dim(X), 1\}$$

for any linear parameter $h \in R_1$.

Now we state a well-known fact, see, e.g., [23, (4.6.b)].

Lemma 2.3. *Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}^N_K$ with $\dim(X) = d$ over an algebraically closed field K of characteristic zero. Let R be the coordinate ring of X. Then*

$$a(R/h_1R) + d \leq \cdots \leq a(R/(h_1, \cdots, h_d)R) + 1 \leq \left\lceil \frac{\deg(X) - 1}{\operatorname{codim}(X)} \right\rceil$$

for any part of linear system of parameters $h_1, \cdots, h_d$ of the graded ring R.

In this way we obtained Proposition 1.1 from Lemma 2.2 and Lemma 2.3, see [19]. Furthermore, the following result has an important role in studying the projective variety having an upper bound on the Castelnuovo-Mumford regularity in the inequality of Proposition 1.1.

Proposition 2.4. *Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}^N_K$ with $\dim(X) = d$ over an algebraically closed field K of characteristic zero. Let R be the coordinate ring of X. Assume that $k(X) \geq 1$ and the equality in Proposition 1.1 holds, that is, $\operatorname{reg}(X) = \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + k(X)d$. Let $h_1, \cdots, h_d$ be a part of linear system of parameters of the graded ring R.*

(i) $a_i(R) = a_{i+1}(R) + k(X) + 1$ *for* $1 \leq i \leq d - 1$.
(ii) $a_d(R) = a_d(R/h_1R) + k(X) - 1$ *and* $a(R) + 1 \leq a_d(R/h_1R) \leq a(R) + 2$.
(iii) $[\mathrm{H}^i_{\mathfrak{m}}(R)/h_1\mathrm{H}^i_{\mathfrak{m}}(R)]_\ell = 0$ *for* $1 \leq i \leq d$ *and* $\ell \geq a_i(R) - k(X) + 2$.
(iv) $a(R/h_1R) + d = \cdots = a(R/(h_1, \cdots, h_d)R) + 1 = \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil$.

Proof. It follows immediately from (2.1), (2.2) and (2.3). ☐

Now let us describe a refined result of [16] and [28] on the relationship between a zero-dimensional scheme with uniform position and its h-vectors.

Lemma 2.5. *Let X be a zero-dimensional scheme in uniform position in $\mathbb{P}^N_K$ over an algebraically closed field K. Let R be the coordinate ring of X. Assume that*

$$\deg(X) \geq N^2 + 2N + 2 \qquad \text{and} \qquad a(R) + 1 = \left\lceil \frac{\deg(X) - 1}{N} \right\rceil.$$

Then X lies on a rational normal curve.

Proof. Let $(h_0, \cdots, h_s)$ be the h-vector of the one-dimensional graded ring R. In other words, we write $h_i = \dim_K(R_i) - \dim_K(R_{i-1})$ for all nonnegative integers i, and s for the maximal integer such that $h_s \neq 0$. Note that $h_0 = 1$, $h_1 = N$, $s = a(R)+1$ and $\deg(X) = h_0 + \cdots + h_s$. Suppose that X does not lie on a rational normal curve. By [27, (2.3)], we have that $h_i \geq h_1 + 1$ for all $2 \leq i \leq s - 2$, and $h_{s-1} \geq h_1$. Thus we have

$$\begin{aligned}
\frac{\deg(X) - 1}{N} &= \frac{h_1 + \cdots + h_s}{h_1} \\
&\geq 1 + \overbrace{\frac{N+1}{N} + \cdots + \frac{N+1}{N}}^{s-3} + 1 + \frac{h_s}{N} \\
&= a(R) + \frac{a(R) - 2 + h_s}{N} \\
&\geq a(R) + \frac{a(R) - 1}{N}.
\end{aligned}$$

Since $a(R)+1 \geq (\deg(X)-1)/N$, we see that $a(R) \leq N+1$. Hence we have

$$\deg(X) - 1 \leq N(a(R)+1) \leq N(N+2),$$

which contradicts the hypothesis. □

Remark 2.6. There is a counterexample in case $\deg(X) = N^2 + 2N + 1$, namely, a complete intersection of type $(2,2,4)$ in $\mathbb{P}^3_K$, which is pointed out by the referee. So we really need the strong condition on the degree.

Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}^N_K$ over an algebraically closed field K. It is well-known that $\deg(X) \geq \operatorname{codim}(X)+1$, and that if the equality holds, then X is either (i) a smooth hyperquadric, (ii) the Veronese surface in $\mathbb{P}^5_K$, (iii) a rational normal scroll, or their cone, see [10, (3.10)] or [7]. In these cases, X is called a variety of minimal degree. Of course, a rational normal curve is a curve of minimal degree. The next lemma yields an application of Lemma 2.5 to higher dimensional cases through hyperplane section method.

Lemma 2.7. *Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}^N_K$ with $\dim(X) \geq 1$ over an algebraically closed field K. Assume that X is linearly normal, that is, $\mathrm{H}^1(\mathbb{P}^N_K, \mathcal{I}_X(1)) = 0$. If, for infinitely many general hyperplanes H, its hyperplane section $X_0 = X \cap H$ is a divisor on a variety Y_0 of minimal degree with $\Gamma(Y_0, \mathcal{I}_{X_0/Y_0}(2)) = 0$, then X is a divisor on a variety of minimal degree.*

Proof. The defining ideal of the projective variety Y_0 in $H \cong \mathbb{P}^{N-1}_K$ is generated by quadric polynomials. Since X is nondegenerate and linearly normal, we have $\Gamma(\mathbb{P}^N_K, \mathcal{I}_X(2)) \cong \Gamma(\mathbb{P}^{N-1}_K, \mathcal{I}_{X_0}(2))$. On the other hand, $\Gamma(Y_0, \mathcal{I}_{X_0/Y_0}(2)) = 0$ gives an isomorphism $\Gamma(\mathbb{P}^{N-1}_K, \mathcal{I}_{X_0}(2)) \cong \Gamma(\mathbb{P}^{N-1}_K, \mathcal{I}_{Y_0}(2))$. So the defining equations $f_1, \cdots, f_r$ of Y_0 can be lifted to polynomials $g_1, \cdots, g_r$ with $\varphi(f_1) = g_1, \cdots, \varphi(f_r) = g_r$ in $\Gamma(\mathbb{P}^N_K, \mathcal{O}_{\mathbb{P}^N_K}(2))$ through the isomorphism $\varphi : \Gamma(\mathbb{P}^{N-1}_K, \mathcal{I}_{Y_0}(2)) \cong \Gamma(\mathbb{P}^N_K, \mathcal{I}_X(2))$. Let Y be a projective scheme defined by the polynomials $g_1, \cdots, g_r$ in $\mathbb{P}^N_K$. Then Y is the intersection of the quadric hypersurfaces containing X. Note that $\dim(Y) = \dim(X)+1$. Then there exists an irreducible component Y' of Y such that Y' is a variety of minimal degree with $Y' \cap H = Y_0$, and in fact $Y' = Y$ by showing $\Gamma(\mathbb{P}^N_K, \mathcal{I}_Y(2)) = \Gamma(\mathbb{P}^N_K, \mathcal{I}_{Y'}(2))$. Hence X is a divisor on the projective variety Y of minimal degree, and in this case $Y \cap H = Y_0$. □

In the following we show a useful lemma for the proof of a criterion of the linear normality.

Lemma 2.8. *Let R be a graded ring with $\dim(R) = d+1 \geq 1$ over a field K, and $\mathfrak{m}$ be the irrelevant ideal of R. Let h be a linear parameter of R. Then*

$$a(R/hR) = \max\{a(R)+1, n\},$$

where $n = \max\{\ell \mid [H^d_{\mathfrak{m}}(R)/hH^d_{\mathfrak{m}}(R)]_\ell \neq 0\}$.

Proof. It immediately follows from the exact sequence:

$$0 \to H^d_{\mathfrak{m}}(R)/hH^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R/hR) \to H^{d+1}_{\mathfrak{m}}(R)[-1] \xrightarrow{\cdot h} H^{d+1}_{\mathfrak{m}}(R).$$

□

Now let us show a criterion of the linear normality which is applied to give a proof of (2.10) on the dimensional induction by combining (2.7) and (2.11).

Lemma 2.9. *Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}_K^N$ over an algebraically closed field K of characteristic zero. Assume that*

$$\operatorname{reg}(X) = \left\lceil \frac{\deg(X) - 1}{\operatorname{codim}(X)} \right\rceil + \max\{k(X)\dim(X), 1\}$$

and

$$\deg(X) \geq 2\operatorname{codim}(X)^2 + \operatorname{codim}(X) + 2.$$

Then X is linearly normal, that is, $\mathrm{H}^1(\mathbb{P}_K^N, \mathcal{I}_X(1)) = 0$.

Proof. If $k(X) = 0$, then X is, of course, linearly normal. So we may assume that $k(X) \geq 1$. We put $\mathbb{P}_K^N = \operatorname{Proj}(S)$, where S is the polynomial ring and $\mathfrak{m}$ is the irrelevant ideal of S. Suppose that X is not linearly normal. Then there is a nondegenerate projective variety X' in $\mathbb{P}_K^{N+1}$ such that X' is isomorphic to X in $\mathbb{P}_K^N$ by a linear projection. Let R and R' be the coordinate rings of X and X' respectively.

Then we have only to prove that

$$\left\lceil \frac{\deg(X) - 1}{\operatorname{codim}(X)} \right\rceil \leq \left\lceil \frac{\deg(X') - 1}{\operatorname{codim}(X')} \right\rceil + 1.$$

In fact, this inequality yields $(t-1)/c \leq (t-1)/(c+1) + 2 - 1/(c+1)$, where $t = \deg(X) = \deg(X')$ and $c = \operatorname{codim}(X) = \operatorname{codim}(X') - 1$. Therefore $t \leq 2c^2 + c + 1$, which contradicts the hypothesis.

For the proof of $\lceil (t-1)/c \rceil \leq \lceil (t-1)/(c+1) \rceil + 1$, we have only to show that

$$a(R/hR) \leq a(R'/hR') + 1,$$

where h is a linear parameter for R and R', because $a(R'/hR') + \dim(X') \leq \lceil (t-1)/(c+1) \rceil$ by (2.3) and $a(R/hR) + \dim(X) = \lceil (t-1)/c \rceil$ by (2.4), (iv).

Note that $\mathrm{H}^i_{\mathfrak{m}}(R) \cong \mathrm{H}^i_{\mathfrak{m}}(R')$ and $\mathrm{H}^i_{\mathfrak{m}}(R/hR) \cong \mathrm{H}^i_{\mathfrak{m}}(R'/hR')$ for $i \geq 2$ since R' is a finite R-algebra. In particular, we have $a(R/hR) = a(R'/hR')$ in case $\dim(X) \geq 2$. Hence the assertion is proved for the case $\dim(X) \geq 2$.

Now we may assume that $\dim(X) = 1$. Since $\mathrm{H}^1_{\mathfrak{m}}(R')$ is a homomorphic image of $\mathrm{H}^1_{\mathfrak{m}}(R)$, we see

$$\left[\mathrm{H}^1_{\mathfrak{m}}(R)/h\mathrm{H}^1_{\mathfrak{m}}(R)\right]_\ell = \left[\mathrm{H}^1_{\mathfrak{m}}(R')/h\mathrm{H}^1_{\mathfrak{m}}(R')\right]_\ell = 0$$

for $\ell \geq a_1(R) - k(X) + 2$ by (2.4), (iii). Therefore, by using Lemma 2.8, we have

$$a(R/hR) = a(R'/hR') \quad (= a(R) + 1)$$

in case $a(R) = a_1(R) - k(X)$, and

$$a(R/hR) = a(R'/hR') \quad \text{or} \quad a(R'/hR') + 1 \quad (= a(R) + 2)$$

in case $a(R) = a_1(R) - k(X) - 1$, see (2.4), (ii). Hence the assertion is proved. □

Proposition 2.10. *Let X be a nondegenerate irreducible reduced projective variety in $\mathbb{P}_K^N$ over an algebraically closed field K of characteristic zero. If*

$$\operatorname{reg}(X) = \left\lceil \frac{\deg(X) - 1}{\operatorname{codim}(X)} \right\rceil + \max\{k(X)\dim(X), 1\}$$

and

$$\deg(X) \geq 2\operatorname{codim}(X)^2 + \operatorname{codim}(X) + 2,$$

then X is a divisor on a variety of minimal degree.

Proof. It follows immediately from (2.5), (2.7), (2.9) and (2.11) by induction on $\dim(X)$. Lemma 2.11 is proved later. □

By Proposition 2.10 we need to study a divisor X of a variety Y of minimal degree in order to give a classification of the projective varieties having an equality in Theorem 1.2. In case Y is a cone over the projective variety Z either (i), (ii) or (iii) described in the paragraph before (2.7), the divisor X on Y is linearly equivalent to the cone over a divisor X_0 on Z, see, e.g., [11, (II.Exercise 6.3)]. Since $\operatorname{codim}(X) = \operatorname{codim}(X_0)$, $\deg(X) = \deg(X_0)$, $\operatorname{reg}(X) = \operatorname{reg}(X_0)$ and $k(X) = k(X_0)$, the projective variety X cannot be an extremal case. In case Y is a smooth hyperquadric, X is a complete intersection of Y and a hypersurface and so $k(X) = 0$, except the case Y a smooth quadric surface, see, e.g., [11, (II.Exercise 6.5)]. In case Y is the Veronese surface, we see $k(X) = 0$. Since we have only to consider the case $k(X) \geq 1$, the projective variety Y can be assumed to be a rational normal scroll.

Let C be the projective line $\mathbb{P}^1_K$. Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_K} \oplus \mathcal{O}_{\mathbb{P}^1_K}(-e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}(-e_r)$. Let $\pi : Y = \mathbb{P}(\mathcal{E}) \to C$ be a projective bundle. Let Z be the divisor corresponding to the natural map $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1_K}(-e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}(-e_r)$. Then we see $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \cong \mathcal{O}_Y(Z)$ and $\operatorname{Pic}(Y)$ is a free Abelian group of rank 2 generated by Z and F, where F is a fibre corresponding to $\pi^*\mathcal{O}_{\mathbb{P}^1_K}(1)$. Then we easily have intersection numbers $Z^{r+1} = -e_1 - \cdots - e_r$, $Z^r \cdot F = 1$ and $Z^i \cdot F^{r+1-i} = 0$ for $0 \leq i \leq r-1$. We consider an embedding of Y in $\mathbb{P}^N_K$ by a very ample divisor $H = Z + n \cdot F$ $(n > e_r)$, where $N = rn + r + n - e_1 - \cdots - e_r$. Then Y is called a rational normal scroll.

Let X be an irreducible reduced effective divisor on Y linearly equivalent to $a \cdot Z + b \cdot F$. Since X is nondegenerate, in other words,

$$\Gamma(Y, \mathcal{I}_{X/Y}(1)) = \Gamma(Y, \mathcal{O}_Y((1-a)\cdot Z + (n-b)\cdot F)) = 0,$$

we may assume that $a = 1$ and $b \geq n+1$, or $a \geq 2$ and $b \geq 1$. Thus X is a nondegenerate projective variety in $\mathbb{P}^N_K$, where $N = rn + r + n - e_1 - \cdots - e_r$. Also, we have $\operatorname{codim}(X) = rn + n - e_1 - \cdots - e_r$ and $\deg(X) = (a\cdot Z + b\cdot F)\cdot(Z + n\cdot F)^r = a(rn - e_1 - \cdots - e_r) + b$.

Now let us show the following lemma to finish the proof of Proposition 2.10.

Lemma 2.11. *Let X be an effective divisor of a rational normal scroll Y with the ideal sheaf $\mathcal{I}_{X/Y}$ as the notation above.*

(i) $\Gamma(Y, \mathcal{I}_{X/Y}(2)) \neq 0$ *if and only if* $a \leq 2$ *and* $b \leq 2n$.
(ii) *If* $\deg(X) \geq 2\operatorname{codim}(X) + 1$, *then* $\Gamma(Y, \mathcal{I}_{X/Y}(2)) = 0$.

Proof. Part (i) follows from isomorphisms

$$\begin{aligned}\Gamma(Y, \mathcal{I}_{X/Y}(2)) &\cong \Gamma(Y, \mathcal{O}_Y((2-a)\cdot Z + (2n-b)\cdot F)) \\ &\cong \Gamma(C, \pi_*\mathcal{O}_Y((2-a)\cdot Z + (2n-b)\cdot F)) \\ &\cong \Gamma(C, \operatorname{Sym}^{2-a}(\mathcal{E}) \otimes \mathcal{O}_C(2n-b)).\end{aligned}$$

Part (ii) is an easy consequence of (i). □

Now we are in the position to get the Castelnuovo-Mumford regularity and the Ellia-Migliore-Miró Roig number of the projective variety. Let S be the polynomial ring $K[x_0, \cdots, x_N]$ and $\mathfrak{m}$ be the irrelevant ideal $(x_0, \cdots, x_N)$. Then we put $\mathbb{P}^N_K = \operatorname{Proj}(S)$. Since Y is arithmetically Cohen-Macaulay, the deficiency module $\mathrm{M}^i(X)$

of X in $\mathbb{P}^N_K$, $1 \le i \le r$, is isomorphic to $\bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^i(Y, \mathcal{I}_{X/Y}(\ell))$ as graded S-modules. Thus we have

$$\mathrm{M}^i(X) \cong \bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^i(Y, \mathcal{O}_Y((-a+\ell)\cdot Z + (-b+\ell n)\cdot F))$$

for $1 \le i \le r$. In Lemma 2.12 and Lemma 2.13 we calculate the intermediate cohomologies $\mathrm{H}^i(Y, \mathcal{O}_Y((-a+\ell)\cdot Z+(-b+\ell n)\cdot F))$, $1 \le i \le r$, and get the number $k(X)$ by considering the structure of the graded S-module $\mathrm{M}^i(X)$.

Lemma 2.12. *Under the above condition, assume that $r = 1$.*

(i) $\mathrm{H}^1(Y, \mathcal{O}_Y(\alpha\cdot Z + \beta\cdot F)) \ne 0$ *if and only if either* $\alpha \ge 0$ *and* $\beta \le e_1\alpha - 2$, *or* $\alpha \le -2$ *and* $\beta \ge e_1\alpha + e_1$.

(ii) *X is arithmetically Cohen-Macaulay, that is,* $k(X) = 0$ *if and only if* $an - 2n + e_1 < b < an + 2$.

(iii) *If* $b \ge an + 2$, *then* $k(X) = \lfloor (b - an - 2)/(n - e_1)\rfloor + 1$.

(iv) *If* $b \le an - 2n + e_1$, *then* $k(X) = \lfloor (an - 2n + e_1 - b)/(n - e_1)\rfloor + 1$.

Proof. In case $\alpha \ge 0$, by isomorphisms

$$\begin{aligned}
\mathrm{H}^1(Y, \mathcal{O}_Y(\alpha\cdot Z + \beta\cdot F)) \quad &\cong \quad \mathrm{H}^1(C, \pi_*\mathcal{O}_Y(\alpha\cdot Z + \beta\cdot F)) \\
\cong \quad & \mathrm{H}^1(C, \mathrm{Sym}^\alpha(\mathcal{E}) \otimes \mathcal{O}_C(\beta)) \\
\cong \quad & \mathrm{H}^1(\mathbb{P}^1_K, \mathcal{O}_{\mathbb{P}^1_K}(\beta) \oplus \mathcal{O}_{\mathbb{P}^1_K}(-e_1+\beta) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}(-\alpha e_1 + \beta)),
\end{aligned}$$

we see that $\mathrm{H}^1(Y, \mathcal{O}_Y(\alpha\cdot Z+\beta\cdot F)) \ne 0$ if and only if $\beta \le e_1\alpha - 2$. In case $\alpha \le -2$, by isomorphisms

$$\begin{aligned}
\mathrm{H}^1(Y, \mathcal{O}_Y(\alpha\cdot Z + \beta\cdot F)) \quad &\cong \quad \mathrm{H}^0(C, \mathrm{R}^1\pi_*\mathcal{O}_Y(\alpha\cdot Z + \beta\cdot F)) \\
\cong \quad & \mathrm{H}^0(C, (\mathrm{Sym}^{-\alpha-2}(\mathcal{E}))' \otimes \mathcal{O}_C(e_1) \otimes \mathcal{O}_C(\beta)) \\
\cong \quad & \mathrm{H}^0(\mathbb{P}^1_K, \mathcal{O}_{\mathbb{P}^1_K}(e_1+\beta) \oplus \mathcal{O}_{\mathbb{P}^1_K}(2e_1+\beta) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}((-\alpha-1)e_1 + \beta)),
\end{aligned}$$

we see that $\mathrm{H}^1(Y, \mathcal{O}_Y(\alpha\cdot Z + \beta\cdot F)) \ne 0$ if and only if $\beta \ge e_1\alpha + e_1$. Similarly, we have $\mathrm{H}^1(Y, \mathcal{O}_Y(-Z + \beta\cdot F)) = 0$ for all β. Thus we proved part (i). Part (ii) is an easy consequence of (i). By virtue of these results, the rest of the assertion, (iii) and (iv), immediately follows from a study of the structure of the graded S-module $\bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^1(Y, \mathcal{O}_Y((-a+\ell)\cdot Z + (-b+\ell n)\cdot F))$. In fact, through the surjective homomorphism $S \cong \bigoplus_{\ell\in\mathbb{Z}} \Gamma(\mathbb{P}^N_K, \mathcal{O}_{\mathbb{P}^N_K}(\ell)) \to \bigoplus_{\ell\in\mathbb{Z}} \Gamma(Y, \mathcal{O}_Y(\ell\cdot Z + \ell n\cdot F))$, the structure of $\mathrm{M}^1(X)$ as graded S-module, that is, $S_1 \otimes \mathrm{M}^1(X)_\ell \to \mathrm{M}^1(X)_{\ell+1}$ is given by the natural map

$$\begin{aligned}
&\Gamma(Y, \mathcal{O}_Y(Z + n\cdot F)) \otimes_K \mathrm{H}^1(Y, \mathcal{O}_Y((-a+\ell)\cdot Z + (-b+\ell n)\cdot F)) \\
&\quad \to \quad \mathrm{H}^1(Y, \mathcal{O}_Y((-a+\ell+1)\cdot Z + (-b+(\ell+1)n)\cdot F)).
\end{aligned}$$

This K-linear map is a zero map if and only if either of the cohomologies vanishes, by considering the isomorphisms

$$\mathrm{H}^1(Y, \mathcal{O}_Y(\alpha\cdot Z + \beta\cdot F)) \cong \mathrm{H}^1(C, \mathrm{Sym}^\alpha(\mathcal{E}) \otimes \mathcal{O}_C(\beta))$$

for $\alpha \ge 0$ and $\mathrm{H}^1(Y, \mathcal{O}_Y(\alpha\cdot Z+\beta\cdot F)) \cong \mathrm{H}^0(C, (\mathrm{Sym}^{-\alpha-2}(\mathcal{E}))' \otimes \mathcal{O}_C(e_1) \otimes \mathcal{O}_C(\beta))$ for $\alpha \le -2$. In other words, $k(X)$ equals the diameter of $\mathrm{M}^1(X)$ in this case, see, e.g., [17] for the definition. Thus, by using (i), we have (iii) and (iv). Therefore the assertion is proved $\square$

The proof of (2.12) shows that $k(X)$ equals the diameter of $\mathrm{M}^1(X)$ for a divisor X on a rational normal scroll, while the corresponding results were shown for a curve on a smooth quadric surface in [18] and for a curve on a smooth cubic surface in [9], although there are lots of curves X with $k(X) < \mathrm{diam}(\mathrm{M}^1(X))$ constructed, say, by liaison addition, see [8], [17].

Lemma 2.13. *Under the above condition, assume that* $r > 1$.

(i) $\mathrm{H}^1(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \neq 0$ *if and only if* $\alpha \geq 0$ *and* $\beta \leq e_r\alpha - 2$.
(ii) $\mathrm{H}^i(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) = 0$ *for* $1 < i < r$.
(iii) $\mathrm{H}^r(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \neq 0$ *if and only if* $\alpha \leq -r-1$ *and* $\beta \geq e_r\alpha + re_r - e_1 - \cdots - e_{r-1}$.

Consequently, $a_i(R) = -\infty$ *for* $1 \leq i \leq r$ *unless either* $i = 1$ *and* $b \geq an + 2$, *or* $i = r$ *and* $b \leq an - (r+1)n + e_1 + \cdots + e_r$, *where* R *is the coordinate ring of* X.

Proof. First, we note $\mathrm{R}^i\pi_*\mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F) = 0$ for $i \neq 0, r$ and

$$\mathrm{H}^j(C, \mathrm{R}^i\pi_*\mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) = 0 \quad \text{for } j \geq 2.$$

Thus we obtain (ii). In order to prove (i), we have isomorphisms

$$\begin{aligned}\mathrm{H}^1(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) &\cong \mathrm{H}^1(C, \pi_*\mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \\ &\cong \mathrm{H}^1(C, \mathrm{Sym}^\alpha(\mathcal{E}) \otimes \mathcal{O}_C(\beta)).\end{aligned}$$

Hence we obtain (i) from an isomorphism $\mathrm{Sym}^\alpha(\mathcal{E}) \otimes \mathcal{O}_C(\beta) \cong \mathcal{O}_{\mathbb{P}^1_K}(\beta) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}(-\alpha e_r + \beta)$ for $\alpha \geq 0$. Finally, for the proof of (iii), we have isomorphisms

$$\begin{aligned}\mathrm{H}^r(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) &\cong \mathrm{H}^0(C, \mathrm{R}^r\pi_*\mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \\ &\cong \mathrm{H}^0(C, (\mathrm{Sym}^{-\alpha-r-1}(\mathcal{E}))' \otimes \mathcal{O}_C(e_1 + \cdots + e_r) \otimes \mathcal{O}_C(\beta)).\end{aligned}$$

Hence we obtain (iii) from an isomorphism $(\mathrm{Sym}^{-\alpha-r-1}(\mathcal{E}))' \otimes \mathcal{O}_C(e_1 + \cdots + e_r) \otimes \mathcal{O}_C(\beta) \cong \mathcal{O}_{\mathbb{P}^1_K}(e_1 + \cdots + e_r + \beta) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_K}(e_1 + \cdots + e_{r-1} + (-\alpha - r)e_r + \beta)$ for $\alpha \leq -r-1$. Therefore the assertion is proved. □

Furthermore, we need the following lemma to get the Castelnuovo-Mumford regularity of the divisor X on the rational normal scroll Y in $\mathbb{P}^N_K$.

Lemma 2.14. *Under the above condition,* $\mathrm{H}^{r+1}(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \neq 0$ *if and only if* $\alpha \leq -r-1$ *and* $\beta \leq -2 - e_1 - \cdots - e_{r-1}$.

Proof. Since $\mathrm{R}^{r+1}\pi_*\mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F) = 0$ and $\mathrm{H}^i(C, \mathrm{R}^{r+1-i}\pi_*\mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) = 0$ for $i \geq 2$, we have an isomorphism

$$\mathrm{H}^{r+1}(Y, \mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)) \cong \mathrm{H}^1(C, \mathrm{R}^r\pi_*\mathcal{O}_Y(\alpha \cdot Z + \beta \cdot F)).$$

Hence we have the assertion. □

Now let us prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.10, X is a divisor on a rational normal scroll Y. If we assume that $\dim X = r \geq 2$, then $a_i(R) = -\infty$ for some $1 \leq i \leq r$ by Lemma 2.13, which contradicts Proposition 2.4. Thus we see that X is one-dimensional. Hence the assertion is proved. □

Accordingly, by Theorem 1.2, we may assume that X is one-dimensional, that is, $r = 1$, and put $e_1 = e$ to finish the proofs of the theorems in §1.

Then we state the following lemmas, (2.15) and (2.16), which are immediate consequences of Lemma 2.12 and Lemma 2.14.

Lemma 2.15. *Under the above condition, assume that $b \geq an+2$. Then we have* $\operatorname{reg}(X) = \lfloor (b-an-2)/(n-e) \rfloor + a + 2$ *and* $k(X) = \lfloor (b-an-2)/(n-e) \rfloor + 1$.

Lemma 2.16. *Under the above condition, assume that $b \leq an - 2n + e$. Then we have $a_1(R) = a_2(R)$, where R is the coordinate ring of X.*

We also need the following lemma.

Lemma 2.17. *Under the above condition, assume that $b \geq an+2$. Then* $\operatorname{reg}(X) = \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + k(X)$ *if and only if* $an + 2 \leq b \leq (a+2)n - e + 1$.

Proof. By Lemma 2.15, $\operatorname{reg}(X) = \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + k(X)$ if and only if $a + 1 = \lceil (a(n-e) + b - 1)/(2n-e) \rceil$. Since $(a(n-e) + b - 1)/(2n-e) = a + (b - an - 1)/(2n-e)$, we have the assertion. □

Now let us prove Theorem 1.3, Theorem 1.4 and Theorem 1.5.

Proof of Theorem 1.3. By virtue of Theorem 1.2, as in the notation above, X is a divisor linearly equivalent to $a \cdot Z + b \cdot F$ on a rational ruled surface $Y = \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_K} \oplus \mathcal{O}_{\mathbb{P}^1_K}(-e)$ on $\mathbb{P}^1_K$. Then we see $b \geq an+2$. In fact, if $an - 2n - e < b < an + 2$, then $k(X) = 0$ by (2.12). If $b \leq an - 2n - e$, then $a(R/hR) = a_1(R) + 1$ by (2.8) and (2.16), which contradicts (2.4), (ii), where R is the coordinate ring of X and h is a linear parameter of R. So we exclude both cases and have only to consider the case $b \geq an+2$. By Lemma 2.15 and Lemma 2.17, we have $a \geq 1$ and $an + 2 \leq b \leq (a+2)n - e + 1$. Hence the assertion is proved. □

Proof of Theorem 1.4. We have only to consider a curve on a rational ruled surface by Theorem 1.2, and follow the notation in Theorem 1.3. By putting $c = 2n - e$ and $t = a(n-e) + b$, we have $n = (c+e)/2$ and $b = t - a(c-e)/2$. By substituting them, we have $ac + 2 \leq t \leq ac + c + 1$ and $e \leq c - 2$ from $an + 2 \leq b \leq (a+2)n - e + 1$ and $n \geq e + 1$. In particular, $a = \lfloor (t-2)/c \rfloor$. In order to prove (i), we take the integers q and r such that $t - 2 = qc + r$ and $0 \leq r \leq c - 1$ for given integers c and t. Note that q must be equal to a. Then we can take an integer e such that $k = 1 + \lfloor 2(t - 2 - ac)/(c-e) \rfloor = 1 + \lfloor 2r/(c-e) \rfloor$ if k is an element of $\mathfrak{S}(c,t)$. On the other hand, the linear system $|a \cdot Z + b \cdot F|$ on Y contains an irreducible smooth curve for $a \geq 1$ and $b \geq an+2$ by [11, (V.2.18)]. Thus there exists a nondegenerate smooth projective curve X with $\operatorname{codim}(X) = c$, $\deg(X) = t$ and $k(X) = k$ such that $\operatorname{reg}(X) = \lceil (\deg(X) - 1)/\operatorname{codim}(X) \rceil + k(X)$. Hence we proved (i). The proof of (ii) is similar to that of (i) and is left to the readers. □

Proof of Theorem 1.5. For given positive integers c and k with $c \geq k$, we take $e = c - 2$, $n = c - 1$, $a = 1$ and $b = c + k$ and construct a nondegenerate smooth projective curve X as a divisor linearly equivalent to $a \cdot Z + b \cdot F$ on a rational ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1_K} \oplus \mathcal{O}_{\mathbb{P}^1_K}(-e))$ embedded by a very ample divisor $Z + n \cdot F$ to the projective space, as in the notation of Theorem 1.3. Then we have $\operatorname{codim}(X) = c$, $\deg(X) = c + 1 + k$, $k(X) = k$ and $\operatorname{reg}(X) = k + 2$. Hence the assertion is proved. □

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