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Research Article

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Sharp conditions for the convergence of greedy expansions with prescribed coefficients

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Abstract: Greedy expansions with prescribed coefficients were introduced by V. N. Temlyakov in a general case of Banach spaces. In contrast to Fourier series expansions, in greedy expansions with prescribed coefficients, a sequence of coefficients $\{c_n\}_{n=1}^{\infty}$ is fixed in advance and does not depend on an expanded element. During the expansion, only expanding elements are constructed (or, more precisely, selected from a predefined set – a dictionary). For symmetric dictionaries, V. N. Temlyakov obtained conditions on a sequence of coefficients sufficient for a convergence of a greedy expansion with these coefficients to an expanded element. In case of a Hilbert space these conditions take the form $\sum_{n=1}^{\infty} c_n = \infty$ and $\sum_{n=1}^{\infty} c_n^2 < \infty$. In this paper, we study a possibility of relaxing the latter condition. More specifically, we show that the convergence is guaranteed for $c_n = o\left(\frac{1}{\sqrt{n}}\right)$, but can be violated if $c_n \asymp \frac{1}{\sqrt{n}}$.

Keywords: greedy expansion, prescribed coefficients, Hilbert space, greedy approximation, convergence

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1 Introduction

We consider greedy expansions with prescribed coefficients in Hilbert spaces. This type of greedy expansion was initially introduced by V. N. Temlyakov [1,2] (see also [3]) in a more general case of Banach spaces. In case of Hilbert spaces, the definition of greedy expansions with prescribed coefficients (further — just greedy expansions) takes the following form.

Definition 1.1. Let H be a Hilbert space over $\mathbb{R}$, D be a symmetric unit-normed dictionary in H (i.e., $\overline{\operatorname{span} D} = H$, all elements in D have a unit norm, and if $g \in D$, then $-g \in D$). Additionally, let $\{t_n\}_{n=1}^{\infty} \subset (0, 1]$ be a weakness sequence and $\{c_n\}_{n=1}^{\infty} \subset (0, +\infty)$ be a sequence of expansion coefficients. For an expanded element $f \in H$ remainders $\{r_n\}_{n=0}^{\infty} \subset H$ and expanding elements $\{e_n\}_{n=1}^{\infty} \subset D$ are defined as follows.

First, r_0 is set to f. Then, if $r_{n-1} \subset H$ ($n \in \mathbb{N}$) has already been defined, an (arbitrary) element which satisfies the condition $(r_{n-1}, e_n) \geq t_n \sup_{e \in D}(r_{n-1}, e)$ is selected as e_n, and r_n is defined as $r_{n-1} - c_n e_n$.

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The series $\sum_{n=1}^{\infty} c_n e_n(f)$ is called a *greedy expansion* of f in the dictionary D with the prescribed coefficients $\{c_n\}_{n=1}^{\infty}$ and the weakness sequence $\{t_n\}_{n=1}^{\infty}$.

It immediately follows from the definition of a greedy expansion that

$$r_N = f - \sum_{n=1}^{N} c_n e_n(f) \quad (N \in \mathbb{N}),$$

and hence the convergence of the expansion to an expanded element is equivalent to the convergence of remainders r_n to zero.

As a selection of an expanding element e_n is potentially not unique, there may exist different realizations of a greedy expansion for a given expanded element f and a given dictionary D. Furthermore, if $t_n = 1$ for at least one $n \in \mathbb{N}$, greedy expansion may turn out to be nonrealizable due to the absence of an element $e \in D$ which provides $\sup\limits_{e\in D}(r_{n-1}, e)$.

In this paper, we consider only symmetric unit-normed dictionaries and the weakness sequence $t_n = 1$ $(n \in \mathbb{N})$.

Earlier we have shown [4, Theorem 2] (see also [1] or [2] for the case of Banach spaces) that if $\sum_{n=1}^{\infty} c_n = \infty$ and $\sum_{n=1}^{\infty} c_n^2 < \infty$, then a greedy expansion of f converges to f (in other words, all realizations of this expansion converge to f).

Also, we have constructed an example [4, Theorem 3] which shows that a convergence can be violated for a coefficient sequence $\{c_n\}_{n=1}^{\infty}$ with $c_n \le \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} c_n = \infty$. However, the absence of convergence has been shown only for one of the possible realizations of this expansion.

In this paper, we present improvements for both the negative and the positive results.

2 Main results

An improvement of the negative result can be stated as follows.

Theorem 2.1. *There exist a Hilbert space H, a symmetric unit-normed dictionary $D \subset H$, an element $f \in H$ and a monotonic sequence $c_n \asymp \frac{1}{\sqrt{n}}$ such that a greedy expansion of f in the dictionary D with the prescribed coefficients $\{c_n\}_{n=1}^{\infty}$ has a unique realization and it does not converge to f.*

An improvement of the positive result can be stated as follows.

Theorem 2.2. *Let H be a Hilbert space and D be a symmetric unit-normed dictionary in H. Let a sequence $\{c_n\}_{n=1}^{\infty}$ satisfy the conditions*

(1) $c_n = o\left(\frac{1}{\sqrt{n}}\right) \quad (n \to \infty)$,

(2) $\sum_{n=1}^{\infty} c_n = \infty$.

Then for every element f all realizations of its greedy expansion in the dictionary D with the prescribed coefficients $\{c_n\}_{n=1}^{\infty}$ converge to f.

3 Proof of Theorem 2.1

Let $r_0 \in H$ be an arbitrary vector with $\|r_0\| = \frac{1}{2}$.

Let us consider the inequality

$$\frac{c_{n+1}-\frac{c_{n+2}}{2}}{c_n-\frac{c_{n+1}}{2}} > 1-\frac{c_n c_{n+1}}{2\|r_0\|^2}. \tag{1}$$

For $c_n = \frac{1}{\sqrt{n}}$ the left part of (1) equals

$$1-\frac{1}{2n}+O\left(\frac{1}{n^2}\right) \quad (n\to\infty),$$

and the right part equals

$$1-\frac{1}{2n\|r_0\|^2}+O\left(\frac{1}{n^2}\right) \quad (n\to\infty).$$

Hence, as $\|r_0\| = \frac{1}{2}$, there exists a number N, that for all $n > N$ the inequality (1) holds.

Now, let us consider the inequality

$$\left(c_n-\frac{c_{n+1}}{2}\right)^2 > \left(c_{n+1}-\frac{c_{n+2}}{2}\right)^2\left(1+\frac{8}{n^2}\right). \tag{2}$$

For $c_n = \frac{1}{\sqrt{n}}$ the left part of (2) equals

$$\frac{1}{4n}+\frac{1}{4n^2}+O\left(\frac{1}{n^3}\right) \quad (n\to\infty),$$

and the right part equals

$$\frac{1}{4n}+O\left(\frac{1}{n^3}\right) \quad (n\to\infty).$$

Hence, there exists a number M, that for all $n > M$ the inequality (2) holds. Thus, we can reenumerate (shift) a sequence:

$$c_n := c_{n+\max\{N,M,9\}}, \tag{3}$$

and as a result obtain a sequence for which the inequalities (1) and (2) hold for all $n \in \mathbb{N}$.

Let α_n denote $\arccos\frac{(r_{n-1},e_n)}{\|r_{n-1}\|}$ (i.e., an angle between r_{n-1} and e_n), and h_n denote $\|r_{n-1}\|\sin\alpha_n$. Let e_{-1}, e_0 be vectors (which will be included in the dictionary D), such that r_0, e_{-1}, e_0 lie on the same plane, $(e_0, r_0) = (e_{-1}, r_0)$ and the angle between r_0 and e_0 is greater than $\arccos\frac{c_1-\frac{c_2}{2}}{\|r_0\|}$ (the inequality $\frac{c_{n+1}-\frac{c_{n+2}}{2}}{\|r_n\|} < 1$ for every $n = 0$, 1, 2,... will be proven further).

We construct the example inductively. First, as e_1 we take an (arbitrary) element of H which satisfies the following conditions:

(1) the angle between e_1 and r_0 equals $\alpha_1 = \arccos\frac{c_1-\frac{c_2}{2}}{\|r_0\|}$;

(2) the projection of e_1 on the plane $\langle e_{-1}, e_0\rangle$ is collinear to r_0.

We then set r_1 to $r_0 - c_1e_1$.

Let vectors $e_{-1}, e_0, e_1, \ldots, e_n$ and $r_0, r_1, r_2, \ldots, r_n$ have already been constructed. Then as e_{n+1} we take an (arbitrary) element of H which satisfies the following conditions:

(1) the angle between e_{n+1} and r_n equals $\alpha_{n+1} = \arccos\frac{c_{n+1}-\frac{c_{n+2}}{2}}{\|r_n\|}$;

(2) the projection of e_{n+1} on the subspace $\langle e_{-1}, e_0, \ldots, e_n\rangle$ is collinear to r_n;

and define r_{n+1} as $r_n - c_{n+1}e_{n+1}$.

We note that in this construction the angle α_{n+1} is equal to the angle between e_{n+1} and the subspace $\langle e_{-1}, e_0, \ldots, e_n\rangle$.

Let D be defined as $\{e_n\}_{n=-1}^{\infty} \cup \{-e_n\}_{n=-1}^{\infty}$. It is sufficient to show that for each $n = 0, 1, 2, \ldots$ the element e_{n+1} is the only vector from D that can be selected as an expanding element at the step n, and that $\|r_n\| \not\to 0$ as $n \to \infty$.

We split the proof of these assertions into the following steps. First, we show that $\|r_n\| \not\to 0$. Second, we show that at the step $(n+1)$ the vector $-e_n$ cannot be selected as an expanding element. Third, we show the same for vectors e_k and $-e_k$, $k > n+1$. As $\widehat{(-e_{n+1}, r_n)}$ and $\widehat{(e_n, r_n)}$ are obtuse (where $\widehat{(a, b)}$ is the angle between vectors a and b), vectors $-e_{n+1}$ and e_n also cannot be selected as expanding elements at the step $(n+1)$. And finally, we show that it also holds for vectors e_k and $-e_k$, $k < n$.

1. Due to the law of cosines

$$\begin{aligned} \|r_n\|^2 &= \|r_{n-1}\|^2 + c_n^2 - 2\|r_{n-1}\|c_n \cos \alpha_n \\ &= \|r_{n-1}\|^2 + c_n^2 - 2c_n\left(c_n - \frac{c_{n+1}}{2}\right) \\ &= \|r_{n-1}\|^2 - c_n^2 + c_n c_{n+1} > \|r_{n-1}\|^2 - c_n^2 + c_{n+1}^2 \end{aligned} \tag{4}$$

(the last inequality holds due to monotonicity of the sequence $\{c_n\}_{n=1}^{\infty}$). Hence, using (3), which directly implies that $c_1^2 < \frac{1}{9}$, we get the inequality

$$\|r_n\|^2 > \|r_0\|^2 - c_1^2 > \frac{1}{4} - \frac{1}{9} = \frac{5}{36}.$$

We see that $\frac{c_{n+1} - \frac{c_{n+2}}{2}}{\|r_n\|} < \frac{c_1}{\|r_n\|} < \frac{1}{3} \cdot \frac{6}{\sqrt{5}} < 1$ and, additionally, that the sequence $\{\|r_n\|\}_{n=0}^{\infty}$ is monotonically decreasing. Thus, $\|r_n\| \not\to 0$, which completes the first part of the proof.

2. Next, we note that $\alpha_{n+1} < \gamma_n$ where $\gamma_n = \arccos\frac{(r_n, -e_n)}{\|r_n\|}$ (i.e., an angle between r_n and $-e_n$, see Figure 1), which implies that a vector $-e_n$ is not selected at the step $(n+1)$ of the algorithm. Indeed, it immediately follows from the inequality

$$c_{n+1} - \frac{c_{n+2}}{2} > \frac{c_{n+1}}{2},$$

which is a direct corollary of the monotonicity of the $\{c_n\}_{n=1}^{\infty}$.

3. Here we show that vectors e_{n+2}, $-e_{n+2}$ are not selected at the step $(n+1)$. As α_{n+1} is the angle between the vector e_{n+1} and the subspace $\langle e_{-1}, e_0, \ldots, e_n \rangle$, it is sufficient to show that $\alpha_{n+1} < \alpha_{n+2}$, i.e.,

$$\frac{c_{n+1} - \frac{c_{n+2}}{2}}{\|r_n\|} > \frac{c_{n+2} - \frac{c_{n+3}}{2}}{\|r_{n+1}\|}. \tag{5}$$

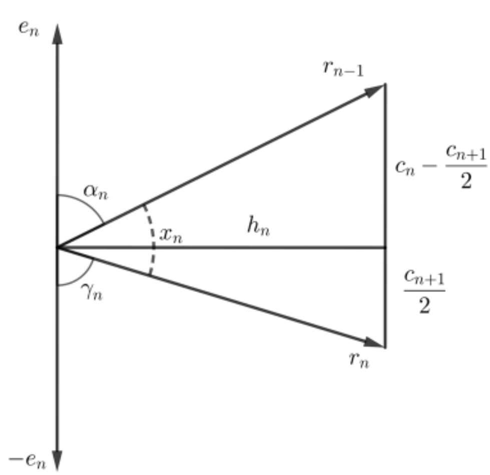

Figure 1: Selection of e_n.

Using (2), (3) and (4) we can note that the following inequalities hold:

$$\begin{aligned}\left(c_{n+1}-\tfrac{c_{n+2}}{2}\right)^2 &> \left(c_{n+2}-\tfrac{c_{n+3}}{2}\right)^2\left(1+\frac{8}{(n+1)^2}\right)\\ &> \left(c_{n+2}-\tfrac{c_{n+3}}{2}\right)^2\left(1+\frac{c_{n+1}^2-c_{n+2}^2}{\frac{5}{36}}\right)\\ &> \left(c_{n+2}-\tfrac{c_{n+3}}{2}\right)^2\left(1+\frac{c_{n+1}^2-c_{n+2}^2}{\|r_{n+1}\|^2}\right)\\ &> \left(c_{n+2}-\tfrac{c_{n+3}}{2}\right)^2\frac{\|r_n\|^2}{\|r_{n+1}\|^2},\end{aligned}$$

which is equivalent to (5). Similarly, vectors e_k and $-e_k (k > n+2)$ are not selected at this step.

4. Now, it remains to prove that if $k < n$, then the angles $\widehat{(-e_k, r_n)}$ and $\widehat{(e_k, r_n)}$ exceed α_n. We prove it inductively by n. The induction base holds as the angle between e_0 and r_0 exceeds α_1. Let β_n^k be the $\min\{\widehat{(r_n, e_k)}, \widehat{(r_n, -e_k)}\}$, and let x_n be the angle between r_n and r_{n-1} (Figure 1). Without loss of generality let $\beta_{n+1}^k = \widehat{(r_{n+1}, -e_k)}$. Due to the spherical law of cosines [5, Chapter 12] we get

$$\cos\beta_{n+1}^k = \cos\gamma_n^k \cos x_n,$$

where $\gamma_n^k = \widehat{(r_n, -e_k)}$. As $\|r_n\|^2 > \frac{5}{36}$, $\|r_{n+1}\|^2 > \frac{5}{36}$ and $c_n^2 < c_1^2 < \frac{1}{9}$, then $\|r_n\| > c_n$ and $\|r_{n+1}\| > c_n$. Therefore, the angle x_n is acute. Thus, the sign of $\cos\beta_{n+1}^k$ coincides with the sign of $\cos\gamma_n^k$, therefore, $\gamma_n^k = \beta_n^k$.

Due to the induction assumption, the inequality $\cos\alpha_n > \cos\beta_n^k$ holds (in the case of $k = n-1$ this inequality holds due to the point 2 of the proof). Hence, it is sufficient to show that

$$\cos\alpha_{n+1} > \cos\alpha_n \cos x_n. \tag{6}$$

Let us prove the inequality (6). The following series of inequalities holds:

$$\begin{aligned}\cos x_n &= \cos(\pi-\alpha_n-\gamma_n) = -\cos(\alpha_n+\gamma_n)\\ &= -\cos\alpha_n\cos\gamma_n + \sin\alpha_n\sin\gamma_n\\ &= \frac{h_n}{\|r_{n-1}\|}\cdot\frac{h_n}{\|r_n\|} - \frac{c_n-\frac{c_{n+1}}{2}}{\|r_{n-1}\|}\cdot\frac{\frac{c_{n+1}}{2}}{\|r_n\|}\\ &= \frac{1}{\|r_n\|\|r_{n-1}\|}\left(\|r_n\|^2-\left(\tfrac{c_{n+1}}{2}\right)^2-c_n\frac{c_{n+1}}{2}+\left(\tfrac{c_{n+1}}{2}\right)^2\right)\\ &= \frac{1}{\|r_n\|\|r_{n-1}\|}\left(\|r_n\|^2-c_n\frac{c_{n+1}}{2}\right).\end{aligned}\tag{7}$$

Combining (7) and (6), we see that it is sufficient to prove that

$$\frac{c_{n+1}-\frac{c_{n+2}}{2}}{\|r_n\|} > \frac{c_n-\frac{c_{n+1}}{2}}{\|r_{n-1}\|}\cdot\frac{1}{\|r_n\|\|r_{n-1}\|}\left(\|r_n\|^2-c_n\frac{c_{n+1}}{2}\right),$$

which is equivalent to the inequality

$$\frac{c_{n+1}-\frac{c_{n+2}}{2}}{c_n-\frac{c_{n+1}}{2}} > \frac{1}{\|r_{n-1}\|^2}\left(\|r_n\|^2-c_n\frac{c_{n+1}}{2}\right).$$

We have already shown (see step 1 of the proof) that the sequence $\{\|r_n\|\}_{n=0}^{\infty}$ is monotonically decreasing. Therefore, it is sufficient to show that

$$\frac{c_{n+1}-\frac{c_{n+2}}{2}}{c_n-\frac{c_{n+1}}{2}} > \frac{1}{\|r_{n-1}\|^2}\left(\|r_{n-1}\|^2-c_n\frac{c_{n+1}}{2}\right),$$

or, equivalently

$$\frac{c_{n+1}-\frac{c_{n+2}}{2}}{c_n-\frac{c_{n+1}}{2}} > 1-\frac{c_nc_{n+1}}{2\|r_{n-1}\|^2}.$$

And due to the monotonicity of the sequence $\{\|r_n\|\}_{n=0}^{\infty}$, it is sufficient to prove that

$$\frac{c_{n+1} - \frac{c_{n+2}}{2}}{c_n - \frac{c_{n+1}}{2}} > 1 - \frac{c_n c_{n+1}}{2\|r_0\|^2},$$

which coincides with (1). It completes the proof of this step and the theorem in general.

4 Proof of Theorem 2.2

For every non-zero element $f \in H$, let $F_f(g) = \frac{(f,g)}{\|f\|}$ and let $r_D(f) = \sup_{g \in D} F_f(g) = \frac{\sup_{g\in D}(f,g)}{\|f\|}$.

We split the proof of Theorem 2.2 into two parts.

1. First, we show that

$$\liminf_{n\to\infty} \|r_n\| = 0.$$

Let us assume the contrary, i.e.,

$$\liminf_{n\to\infty} \|r_n\| > 0.$$

If there exists a number $k > 0$ such that $\|r_k\| = 0$, then it is obvious that a greedy expansion converges to an expanded element. Otherwise, there exists a number $r > 0$, such that for every $k \in \mathbb{N}$

$$\|r_k\| \geq r. \tag{8}$$

Let S_k be the kth partial sum of the sequence $\{c_n\}_{n=1}^{\infty}$, i.e., $S_k = \sum_{j=1}^{k} c_j$. Then the following lemma holds.

Lemma 4.1. *Let the conditions of* Theorem 2.2 *hold together with the inequality* $\|r_k\| \geq r > 0$ $(k \in \mathbb{N})$. *Then*

$$\liminf_{n\to\infty} S_n r_D(r_{n-1}) = 0.$$

Proof. Assume the contrary. In this case, there exists a number $c > 0$, such that

$$r_D(r_{k-1}) S_k \geq c \quad (k \geq 1). \tag{9}$$

We note that

$$\begin{aligned}\|r_n\|^2 &= (r_n, r_n) = (r_{n-1} - c_n e_n, r_{n-1} - c_n e_n)\\ &= \|r_{n-1}\|^2 - 2c_n(r_{n-1}, e_n) + c_n^2 = \dots\\ &= \|r_0\|^2 - 2\sum_{k=1}^{n} c_k(r_{k-1}, e_k) + \sum_{k=1}^{n} c_k^2\\ &= \|r_0\|^2 - 2\sum_{k=1}^{n} \frac{c_k}{S_k}\|r_{k-1}\| r_D(r_{k-1}) S_k + \sum_{k=1}^{n} c_k^2.\end{aligned}$$

Taking into account formulas (8) and (9), we get that

$$\|r_n\|^2 \leq \|r_0\|^2 - 2cr\sum_{k=1}^{n} \frac{c_k}{S_k} + \sum_{k=1}^{n} c_k^2. \tag{10}$$

It is known (see [6, Chapter 11]) that if $\sum_{k=1}^{\infty} c_k = \infty$, then we have $\sum_{k=1}^{\infty} \frac{c_k}{S_k} = \infty$. Hence, (10) implies that $\sum_{k=1}^{\infty} c_k^2 = \infty$.

Next, as there exists a number $a > 0$ such that $c_k < \frac{a}{\sqrt{k}}$ for all $k \in \mathbb{N}$, there exists a number $A > 0$, such that $S_k < A\sqrt{k}$. As $c_k = o\left(\frac{1}{\sqrt{k}}\right)$, there exists a function $f(k)$, such that $c_k = \frac{1}{\sqrt{k f(k)}}$ and $f(k) \to \infty$ as $k \to \infty$. Therefore,

$$\sum_{k=1}^{n} \frac{c_k}{S_k} = \sum_{k=1}^{n} \frac{1}{\sqrt{k f(k)} S_k} \geq \sum_{k=1}^{n} \frac{1}{A k f(k)}. \tag{11}$$

We note that the equality

$$\sum_{k=1}^{\infty} c_k^2 = \sum_{k=1}^{\infty} \frac{1}{kf^2(k)} = \infty$$

implies that

$$\sum_{k=1}^{\infty} \frac{1}{kf(k)} = \infty. \tag{12}$$

Combining (10), (11) and (12) results in

$$0 < \|r_n\|^2 \le \|r_0\|^2 - 2cr\sum_{k=1}^{n} \frac{1}{Akf(k)} + \sum_{k=1}^{n} \frac{1}{kf^2(k)} = \|r_0\|^2 - \sum_{k=1}^{n} \frac{1}{kf(k)}\left(\frac{2cr}{A} - \frac{1}{f(k)}\right) \to -\infty.$$

It contradicts our assumption. Thus, the proof of Lemma 4.1 is complete. □

Now, let us proceed to the proof of Theorem 2.2.

Lemma 4.1 and monotonicity of S_n imply that

$$\liminf_{n\to\infty} S_n r_D(r_n) = 0.$$

Therefore, there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{n\to\infty} S_{n_k} r_D\big(r_{n_k}\big) = 0.$$

Let us consider a sequence of functionals $\{F_{r_{n_k}}\}_{k=1}^{\infty}$. The norm of each functional equals 1. As a unit sphere is a weak compact (according to the Banach–Alaoglu theorem), there exists a weakly converging subsequence $\{F_{r_{n_{k_i}}}\}_{i=1}^{\infty}$. For simplicity we denote $F_{r_{n_{k_i}}}$ as $\mathbf{F}_i$. As noted above, there exists a weak limit

$$F := \lim_{i\to\infty} \mathbf{F}_i.$$

Due to the fact that the dictionary D is symmetric, for all sufficiently large i the following inequality holds:

$$\mathbf{F}_i(f) = \mathbf{F}_i\left(r_{n_{k_i}} + \sum_{j=1}^{n_{k_i}} c_j e_j\right) = \|r_{n_{k_i}}\| + \sum_{j=1}^{n_{k_i}} c_j \mathbf{F}_i(e_j) \ge r - S_{n_{k_i}} r_D\big(r_{n_{k_i}}\big). \tag{13}$$

Passage to the limit results in an estimate $F(f) \ge r$, which implies that $F \ne 0$.

On the other hand, for every g from the dictionary D

$$\begin{aligned} F(g) &= \lim_{i\to\infty} \mathbf{F}_i(g) \le \lim_{i\to\infty} r_D\big(r_{n_{k_i}}\big) = 0, \\ F(-g) &= \lim_{i\to\infty} \mathbf{F}_i(-g) \le \lim_{i\to\infty} r_D\big(r_{n_{k_i}}\big) = 0. \end{aligned} \tag{14}$$

Hence, $F(g)$ equals 0 for all $g \in D$ and, due to completeness of D, we get that $F = 0$. We have come to a contradiction, which proved the equality

$$\liminf_{n\to\infty} \|r_n\| = 0.$$

Thus, we completed the first part of the proof of Theorem 2.2.

2. Now it remains to prove the following two lemmas.

Lemma 4.2. *Let H be a Hilbert space, D be a symmetric unit-normed dictionary, $f \in H$ be an expanded element. For a greedy algorithm let α_n denote* $\arccos\frac{(r_{n-1}, e_n)}{\|r_{n-1}\|}$ *and h_n denote* $\|r_{n-1}\| \sin\alpha_n$ (Figure 1). *Then* $\{h_n\}_{n=1}^{\infty}$ *is a non-increasing sequence.*

Lemma 4.3. *Let H be a Hilbert space, D be a symmetric unit-normed dictionary* and $f \in H$ *be an expanded element. Let a sequence $\{c_n\}_{n=1}^{\infty}$ converge to zero. Then the equality*

$$\liminf_{n\to\infty} \|r_n\| = 0$$

(where $\{r_n\}_{n=0}^{\infty}$ is a sequence of remainders of the greedy expansion of f in D with coefficients $\{c_n\}_{n=0}^{\infty}$) implies that $r_n \to 0$ $(n \to \infty)$.

Proof of Lemma 4.2. Let β_n denote $\min\{\widehat{(r_n, e_n)}, \widehat{(r_n, -e_n)}\}$. Then $h_n = \|r_n\| \sin \beta_n$. Since an expansion is greedy with $t_n = 1$, we have that $\alpha_{n+1} \le \beta_n$, and thus

$$h_{n+1} = \|r_n\| \sin \alpha_{n+1} \le \|r_n\| \sin \beta_n = h_n. \qquad \square$$

Proof of Lemma 4.3. Similar to Lemma 4.2, let h_n denote $\|r_{n-1}\| \sin \alpha_n$. Due to Lemma 4.2 a sequence $\{h_n\}_{n=1}^{\infty}$ is non-increasing.

Let us fix an arbitrary $\varepsilon > 0$. Then there exists a number $m \in \mathbb{N}$, such that for all $n > m$

$$c_n < \frac{\varepsilon}{\sqrt{2}}. \tag{15}$$

If $\liminf_{n\to\infty} \|r_n\| = 0$, there exists a number $k > m$ such that

$$\|r_{k-1}\| < \frac{\varepsilon}{\sqrt{2}}.$$

But then for all $p \ge k$ we have that

$$h_p < \frac{\varepsilon}{\sqrt{2}}. \tag{16}$$

Furthermore, by induction we show that the estimate $\|r_p\| \le \varepsilon$ holds for all $p \ge k$. Indeed, for $p \ge k$ either

$$\|r_p\| \le \|r_{p-1}\| \le \varepsilon$$

or (due to (15) and (16))

$$\|r_p\|^2 \le h_p^2 + c_p^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2.$$

Hence, $\lim_{n\to\infty} r_n = 0$. $\square$

Thus, we have proved both Lemmas 4.2 and 4.3, and therefore, we completed the proof of Theorem 2.2.

Remark 4.4. The statement of Theorem 2.2 does not cover the statement of [4, Theorem 2] entirely.

Proof. Indeed, let us construct a sequence $\{c_n\}_{n=1}^{\infty}$ in the following way. Let

$$K = \{k^2\}_{k=1}^{\infty},$$
$$c_n = \begin{cases} \dfrac{1}{\sqrt{n}}, & n \in K, \\ \dfrac{1}{2^n}, & n \notin K. \end{cases}$$

Then the sequence $\{c_n\}_{n=1}^{\infty}$ satisfies the assumptions of Theorem [4, Theorem 2], but does not satisfy the assumptions of Theorem 2.2. $\square$

5 Generalization

One of the disadvantages of a greedy expansion with the prescribed coefficients in case of $t_n \equiv 1$ is that there might be no greedy expansion for an expanded element. Another disadvantage is that (irrespective of t_n) in a general case selection of expanding element is not constructive.

We consider the following generalization of the greedy algorithm that eliminates these undesirable properties. Let symmetric sets $\{D_n\}_{n=1}^{\infty}$ be an exhaustion of a dictionary D, i.e., $D_n \subset D_{n+1}$ for all $n \in \mathbb{N}$, and $\bigcup_{n=1}^{\infty} D_n = D$.

At the step n of a greedy algorithm we select an element e_n from D_n such that $(r_{n-1}, e_n) = \sup_{e \in D_n}(r_{n-1}, e)$. In other words, at the step n we consider D_n instead of D. If D_n is finite, then e_n always exists and can be constructively selected by a simple exhaustive search. Let us call this modification of a greedy expansion "a greedy expansion with prescribed coefficients in the exhaustion $\{D_n\}_{n=1}^{\infty}$."

Let us note that for this modification the analogue of Theorem 2.2 holds. It can be stated in the following way.

Theorem 5.1. *Let H be a Hilbert space, D be a symmetric unit-normed dictionary and $\{D_n\}_{n=1}^{\infty}$ be its exhaustion. If a sequence $\{c_n\}_{n=1}^{\infty}$ satisfies conditions*

(1) $c_n = o\left(\frac{1}{\sqrt{n}}\right) \quad (n \to \infty)$,

(2) $\sum_{n=1}^{\infty} c_n = \infty$,

then for every element f all realizations of its greedy expansion in the exhaustion $\{D_n\}_{n=1}^{\infty}$ with the prescribed coefficients $\{c_n\}_{n=1}^{\infty}$ converge to f.

Proof. The proof almost repeats the proof of Theorem 2.2. For clarity, we specify which parts of this proof require adaptation.

1. We should change the statement of Lemma 4.1 in the following way:

$$\liminf_{n\to\infty} S_n r_{D_n}(r_{n-1}) = 0.$$

The proof remains identical.

2. Next, we have that

$$\liminf_{n\to\infty} S_n r_{D_n}(r_n) = 0,$$

and we select a subsequence $\{n_k\}_{k=1}^{\infty}$ in such a way that

$$\lim_{n\to\infty} S_{n_k} r_{D_{n_k}}(r_{n_k}) = 0.$$

3. Nestedness of sets D_n allows rewriting inequality (13) as follows:

$$\mathbf{F}_i(f) \geq r - S_{n_{k_i}} r_{D_{n_{k_i}}}(r_{n_{k_i}}).$$

4. Now, we show that inequalities (14) hold. Let $g \in D$. Then there exists D_p such that $g \in D_p$. Hence, for all $n_{k_i} > p$ we have that $\mathbf{F}_i(g) \leq r_{D_{n_{k_i}}}(r_{n_{k_i}})$. Thus, the inequalities can be rewritten as follows:

$$F(g) = \lim_{i\to\infty} \mathbf{F}_i(g) \leq \lim_{i\to\infty} r_{D_{n_{k_i}}}\left(r_{n_{k_i}}\right) = 0,$$
$$F(-g) = \lim_{i\to\infty} \mathbf{F}_i(-g) \leq \lim_{i\to\infty} r_{D_{n_{k_i}}}\left(r_{n_{k_i}}\right) = 0.$$

All other steps of this proof repeat the corresponding steps of the proof of Theorem 2.2. □

6 Conclusion

In this paper, we presented improvements for both negative and positive results on the convergence of greedy expansions with prescribed coefficients. These improvements, in particular, allowed to remove completely a gap between positive and negative results.

We expect that the technique we used in this research is applicable also to greedy expansions with errors in coefficient calculation [7], where a gap between negative and positive results still exists, and we are going to present results for greedy expansions with errors in coefficient calculation in our subsequent publications. We are also going to study a possibility of generalizing positive results to the case of $t_n \equiv t \in (0, 1)$.

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